Lipschitz estimates for the Berezin transform

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Abstract. We consider the generalized Fock space $\mathcal{A}^2(\mu_m)$, where μ_m is the measure with weight $e^{-|z|^m}$, $m > 0$, with respect to the Lebesgue measure on $\mathbb{C}^n$. Improving upon a recent result of L. Coburn and J. Xia, we show that for any bounded operator X on $\mathcal{A}^2(\mu_m)$, the Berezin transform of X satisfies Lipschitz estimates.

1. Introduction and statement of the main results

We consider the Fock space $\mathcal{A}^2(\mu_m)$ of those holomorphic functions which are square integrable on $\mathbb{C}^n$ with respect to the measure $d\mu_m(z) = e^{-|z|^m} dV(z)$, where $dV(z)$ is the Lebesgue measure and $m > 0$ is a positive parameter. Let K_m be the Bergman kernel of the Hilbert space $\mathcal{A}^2(\mu_m)$ and denote by k_a the corresponding normalized kernel. It is given by

$$k_a(z) := K(z, a) / \sqrt{K(a, a)}$$

and is a unit vector of $\mathcal{A}^2(\mu_m)$. The Berezin transform of a bounded linear operator X on $\mathcal{A}^2(\mu_m)$ is the function defined by $\tilde{X}(a) = \langle Xk_a, k_a \rangle$. Recently, Coburn [4], [5] has shown that in the classical case $m = 2$,

corresponding to the Segal-Bargmann space endowed with the Gaussian measure on $\mathbb{C}^n$, $\tilde{X}$ is a Lipschitz function, satisfying

$$|\tilde{X}(a) - \tilde{X}(b)| \leq 2 \|X\| |a - b|,$$

where $\|X\|$ is the norm of X .

The main goal of this paper is to investigate the general case of positive parameters $m > 0$. Namely, we will show that the Berezin transform is locally Lipschitz and when $n = 1$ and $m \leq 1$, the Lipschitz estimate is uniform. In the proof of our results we also establish estimates of the quantity

$$\rho_m(a, b) := \sqrt{1 - |\langle k_a, k_b \rangle|^2}.$$

If A and B are two quantities, we use the symbol $A \approx B$ whenever $A \leq C_1 B$ and $B \leq C_2 A$, where C_1 and C_2 are positive constants independent of the varying parameters. Our main result is the following

Theorem 1. *If X is a bounded operator on $\mathcal{A}^2(\mu_m)$, then its Berezin transform $\tilde{X}$ is locally Lipschitz. In addition, there exist positive constants A, C , depending on n and m only, such that for any $a \in \mathbb{C}^n$ with $|a| > A$, there is a neighbourhood U of a that satisfies*

$$|\tilde{X}(x) - \tilde{X}(a)| \leq C \|X\| |a|^{\frac{m}{2}-1} |x - a|, \text{ for } x \in U.$$

The proof of Theorem 1 gives a more accurate estimate of $\rho_m(a, b)$ when $n = 1$.

Theorem 2. *On $\mathbb{C}$, the quantity $\rho_m(a, b)$ satisfies a Lipschitz estimate. More precisely, there exists a positive constant A , depending on m only, such that for any $a \in \mathbb{C}^n$ with $|a| > A$, there is a neighbourhood U of a that satisfies*

$$\rho_m(a, x) \approx |a|^{\frac{m}{2}-1} |x - a|, \text{ for } x \in U.$$

When m is an even integer, we have the following consequence of Theorem 2:

Corollary 3. *Suppose $n = 1$ and m an even positive integer. There exists a positive constant A , depending on n and m only, such that, for any $a \in \mathbb{C}$ with $|a| > A$, there is a neighbourhood U of a that satisfies*

$$\rho_m(a, x) \approx |a^{m/2} - x^{m/2}| \text{ for all } x \in U.$$

A careful study of $\rho_m(a, x)$ when $m \leq 1$ shows that the Lipschitz condition is uniform.

Corollary 4. *Assume $n = 1$ and $m \leq 1$ and let X be a bounded operator on $\mathcal{A}^2(\mu_m)$. Then $\tilde{X}$ is Lipschitz. More precisely, there are positive constants A, C, δ , depending on n and m only, such that for any $a \in \mathbb{C}^n$ with $|a| > A$,*

$$|\tilde{X}(x) - \tilde{X}(a)| \leq C \|X\| |x - a|, \text{ for all } x \text{ with } |x - a| < \delta.$$

2. Preparatory results

We will use a theorem shown by Coburn in [4], which is valid for any reproducing kernel.

Lemma 1. *For a bounded operator X on $\mathcal{A}^2(\mu_m)$ and a, b in $\mathbb{C}^n$,*

$$|\tilde{X}(b) - \tilde{X}(a)| \leq 2 \|X\| \rho_m(a, b).$$

We recall

$$\rho_m(a, b)^2 = \frac{1}{K_m(a, a)K_m(b, b)} [K_m(a, a)K_m(b, b) - |K_m(a, b)|^2],$$

where K_m is the Bergman kernel of $\mathcal{A}^2(\mu_m)$. It follows from [1] that the Bergman kernel K_m can be expressed in terms of the generalized Mittag-Leffler function

$$E_{\frac{2}{m}, \frac{2}{m}}(\lambda) := \sum_{d=0}^{+\infty} \frac{\lambda^d}{\Gamma(\frac{2d}{m} + \frac{2}{m})}$$

by the equation

$$K_m(z, w) = \frac{m}{(n-1)!} E_{\frac{2}{m}, \frac{2}{m}}^{(n-1)}(\langle z, w \rangle)$$

Setting $\alpha := 2/m$ and $F := E_{\alpha, \alpha}^{(n-1)}$, we see that for a, u in $\mathbb{C}^n$,

$$\rho_m(a, a+u)^2 = \frac{1}{F(|a|^2)F(|a+u|^2)} \{F(|a|^2)F(|a+u|^2) - |F(\langle a, a+u \rangle)|^2\}.$$

In order to study $\rho_m(a, a+u)^2$ near a , we will use a Taylor's expansion formula. Thus we need to estimate F and its higher order derivatives.

3. Asymptotic expansion of the derivatives of the Mittag-Leffler function $E_{\alpha,\alpha}$

Following the method developed by Wong and Zhao in [10], we consider several subcases, corresponding to different values of α .

Assume p is a non-negative integer. For $z \in \mathbb{C} \setminus \{0\}$, we suppose the branch of the argument is chosen such that $\arg z \in [-\pi, \pi[$. If s is an integer, let

$$Z_s := |z|^{\frac{1}{\alpha}} \exp \left[\frac{i}{\alpha} (\arg z + 2\pi s) \right]$$

and denote by $\mathcal{S}$ the set of those integers s satisfying $|\arg Z_s| < \pi$.

Lemma 2. *If $\alpha \in \mathbb{N}_0 \setminus \{0\}$, then there exists a polynomial P_{p+1} of degree $p+1$ such that for any $z \neq 0$,*

$$E_{\alpha,\alpha}^{(p)}(z) = \sum_{s=0}^{\alpha-1} e^{Z_s} Z_s^{-\alpha(p+1)} P_{p+1}(Z_s).$$

Proof. We recall the Hankel representation of the entire function $1/\Gamma$

$$\frac{1}{\Gamma(w)} = \frac{1}{2\pi i} \int_{\gamma_{a,\phi_0}} e^t t^{-w} dt,$$

where $\frac{\pi}{2} < \phi_0 < \pi$, $|z| < a^\alpha$ and γ_{a,ϕ_0} is the Hankel contour which starts at $\infty e^{-i\phi_0}$, follows the half-ray $\arg t = -\phi_0$, encircles the origin once counter-clockwise on the circular part $|t| = a$ and returns to $\infty e^{i\phi_0}$ on the half-ray $\arg t = \phi_0$.

Fix $z \in \mathbb{C} \setminus \{0\}$, and if d is an integer, set $I_d := \frac{z^d}{\Gamma(\alpha d + \alpha)}$. Then from the formula above we have

$$I_d = \frac{1}{2\pi i} \int_{\gamma_{a,\phi_0}} e^t t^{-\alpha} \left(\frac{z}{t^\alpha} \right)^d dt.$$

We write $I_d = I_{d,lin} + I_{d,circ}$, where $I_{d,lin}$ and $I_{d,circ}$ are respectively the integrals on the linear part and circular part of γ_{a,ϕ_0} .

On the half rays of γ_{a,ϕ_0} we have

$$|I_{d,lin}| \leq 2 \frac{1}{2\pi} \int_a^{+\infty} e^{\tau \cos \phi_0} \tau^{-\alpha} \left| \frac{z}{\tau^\alpha} \right|^d d\tau.$$

Therefore

$$\sum_{d=0}^{+\infty} |I_{d,lin}| \leq \frac{1}{\pi} \int_a^{+\infty} e^{\tau \cos \phi_0} \tau^{-\alpha} \frac{1}{1 - \frac{|z|}{\tau^\alpha}} d\tau = \frac{1}{\pi} \int_a^{+\infty} \frac{e^{\tau \cos \phi_0}}{\tau^\alpha - |z|} d\tau < +\infty.$$

On the circular part of γ_{a,ϕ_0} we have

$$|I_{d,circ}| \leq \frac{1}{2\pi} \int_{-\phi_0}^{\phi_0} e^{a \cos \phi} a^{-\alpha} \left(\frac{|z|}{a^\alpha} \right)^d d\phi.$$

Then

$$\sum_{d=0}^{+\infty} |I_{d,circ}| \leq 2 \frac{1}{2\pi} \phi_0 e^{a \cos \phi_0} \frac{1}{a^\alpha - |z|}.$$

Thus we can write

$$E_{\alpha,\alpha}(z) = \frac{1}{2\pi i} \int_{\gamma_{a,\phi_0}} e^t t^{-\alpha} \sum_{d=0}^{+\infty} \left(\frac{z}{t^\alpha} \right)^d dt = \frac{1}{2\pi i} \int_{\gamma_{a,\phi_0}} e^t \frac{dt}{t^\alpha - z}.$$

Let p be a non-negative integer. On the linear part of γ_{a,ϕ_0} , $t = \tau e^{\pm i\phi_0}$ with $\tau \geq a$, then $\left| \frac{e^t}{(t^\alpha - z)^{p+1}} \right| \leq \frac{e^{\tau \cos \phi_0}}{(\tau^\alpha - |z|)^{p+1}}$, and the integral

$\int_a^{+\infty} \frac{e^{\tau \cos \phi_0}}{(\tau^\alpha - |z|)^{p+1}} d\tau$ is convergent.

By differentiation we get

$$E_{\alpha,\alpha}^{(p)}(z) = \frac{1}{2\pi i} \int_{\gamma_{a,\phi_0}} f_p(t) dt,$$

where

$$f_p(t) := p! \frac{e^t}{(t^\alpha - z)^{p+1}}.$$

Now we will get another expression of this integral by deforming the contour γ_{a,ϕ_0} . Choose r, R, ϕ_1 such that $0 < r < a < R$, $r^\alpha < |z| < a^\alpha$, and $0 < \phi_1 < \pi - \phi_0$. Consider the counter-clockwise loop which starts at $Re^{i(-\pi+\phi_1)}$, follows the circular path $|t| = R$ to $Re^{-i\phi_0}$, a straight line to $ae^{-i\phi_0}$, a part of the circle $|t| = a$ to $ae^{i\phi_0}$, a straight line to $Re^{i\phi_0}$, the circle $|t| = R$ to $Re^{i(\pi-\phi_1)}$, a straight horizontal line to $re^{i\frac{\pi}{2}}$, the semi-circle $|t| = r$ to $re^{-i\frac{\pi}{2}}$, then a straight horizontal line to the starting point $Re^{i(-\pi+\phi_1)}$.

For any integer s we put

$$Z_s := |z|^{\frac{1}{\alpha}} \exp \left[\frac{i}{\alpha} (\arg z + 2\pi s) \right]$$

and

$$\theta_s := \frac{1}{\alpha}(\arg z + 2\pi s).$$

Define the set $\mathcal{S} := \{s \in \mathbb{Z}, |\arg Z_s| < \pi\}$. If we choose r small enough for the poles Z_s (such that $|\arg Z_s| < \pi$) of $f_p(t)$ to be inside $\gamma_{a,\phi_0,R,r}$, the theorem of residues leads to

$$\frac{p!}{2\pi i} \int_{\gamma_{a,\phi_0,R,r}} e^t \frac{dt}{(t^\alpha - z)^{p+1}} = \sum_{s \in \mathcal{S}} \text{Res}(f_p; Z_s).$$

On the big circular part of $\gamma_{a,\phi_0,R,r}$, we have

$$\left| \int_{\phi_0}^{\pi-\phi_1} \frac{e^{Re^{i\phi}}}{(R^\alpha e^{i\alpha\phi} - z)^{p+1}} i e^{i\phi} R d\phi \right| \leq \pi R \frac{e^{R \cos \phi_0}}{(R^\alpha - |z|)^{p+1}}$$

and

$$\left| \int_{-\pi+\phi_1}^{-\phi_0} \frac{e^{Re^{i\phi}}}{(R^\alpha e^{i\alpha\phi} - z)^{p+1}} i e^{i\phi} R d\phi \right| \leq \pi R \frac{e^{R \cos \phi_0}}{(R^\alpha - |z|)^{p+1}}$$

Since $\cos \phi_0 < 0$, we see that

$$\lim_{R \rightarrow +\infty} \frac{R e^{R \cos \phi_0}}{(R^\alpha - |z|)^{p+1}} = 0.$$

When R tends to $+\infty$, $\gamma_{a,\phi_0,R,r}$ is deformed in two paths, namely γ_{a,ϕ_0} and γ'_r , where γ'_r starts at $-ir - \infty$, follows a straight horizontal line to $r e^{-i\frac{\pi}{2}}$, encircles the origin once counter-clockwise on the semi-circle $|t| = r$, and follows a straight horizontal line to $ir - \infty$.

On the rectilinear part of γ'_r we have $\left| \frac{e^t}{(t^\alpha - z)^{p+1}} \right| \leq \frac{e^{\tau \cos \phi_0}}{(|\tau|^\alpha - |z|)^{p+1}}$,

thus the integral $\int_{\gamma'_r} e^t \frac{dt}{(t^\alpha - z)^{p+1}}$ is absolutely convergent.

Therefore

$$\sum_{s \in \mathcal{S}} \text{Res}(f_p; Z_s) = \frac{p!}{2\pi i} \int_{\gamma_{a,\phi_0}} e^t \frac{dt}{(t^\alpha - z)^{p+1}} - \frac{p!}{2\pi i} \int_{\gamma'_r} e^t \frac{dt}{(t^\alpha - z)^{p+1}},$$

so that

$$E_{\alpha,\alpha}^{(p)}(z) = \sum_{s \in \mathcal{S}} \text{Res}(f_p; Z_s) + \frac{1}{2\pi i} \int_{\gamma'_r} f_p(t) dt.$$

Next, for $s \in \mathcal{S}$, we shall compute $\text{Res}(f_p; Z_s)$. Let $\mathcal{C}_s$ be a circle centered at Z_s of radius η and $\mathcal{C}'_s$ the image of $\mathcal{C}_s$ by the function power α , where we choose η small enough for $\mathcal{C}_s$ and $\mathcal{C}'_s$ to be included in $\mathbb{C} \setminus \mathbb{R}_-$ and the function power α is determined by $u = t^\alpha = \exp[\alpha |\ln t| +$

$i\alpha \arg t]$, $|\arg t| < \pi$. In a neighbourhood of $\mathcal{C}'_s$, the function power $\frac{1}{\alpha}$ is defined by

$$u^{\frac{1}{\alpha}} = \exp\left[\frac{1}{\alpha} \ln |u| + \frac{i}{\alpha}(\arg u + 2\pi s)\right], \quad |\arg u| < \pi.$$

The change of variables $u = t^\alpha$ shows that

$$\text{Res}(f_p; Z_s) = \frac{1}{2\pi i} p! \int_{\mathcal{C}'_s} \frac{e^{u^{\frac{1}{\alpha}}}}{(u-z)^{p+1}} \frac{1}{\alpha} u^{\frac{1}{\alpha}-1} du.$$

We get $\text{Res}(f_p; Z_s) = \frac{d^{p+1}}{du^{p+1}} \left[e^{u^{\frac{1}{\alpha}}} \right]_{u=z}$.

Now, we have to compute the higher order derivatives of $h(u) := e^{u^{\frac{1}{\alpha}}}$. By induction we see that, for any $p \in \mathbb{N}_0$, there exists a polynomial P_p of degree p such that $h^{(p)}(u) = \frac{e^{u^{\frac{1}{\alpha}}}}{u^p} P_p(u^{\frac{1}{\alpha}})$. By differentiating the last equality, we get

$$h^{(p+1)}(u) = \left[\frac{e^{u^{\frac{1}{\alpha}}}}{u^p} P_p(u^{\frac{1}{\alpha}}) \right]' = \frac{e^{u^{\frac{1}{\alpha}}}}{u^{p+1}} \left[\frac{1}{\alpha} u^{\frac{1}{\alpha}} P_p(u^{\frac{1}{\alpha}}) + \frac{1}{\alpha} u^{\frac{1}{\alpha}} P'_p(u^{\frac{1}{\alpha}}) - p P_p(u^{\frac{1}{\alpha}}) \right].$$

It follows that the polynomials $(P_p)_{p \in \mathbb{N}}$ satisfy the relation:

$$P_{p+1}(t) = \frac{1}{\alpha} t P_p(t) + \frac{1}{\alpha} t P'_p(t) - p P_p(t).$$

Since $P_0(t) = t$, P_p has real coefficients, and the degree of P_p is p .

Thus $\text{Res}(f_p; Z_s) = \frac{e^{Z_s}}{Z_s^{(p+1)\alpha}} P_{p+1}(Z_s)$.

We write

$$E_{\alpha, \alpha}^{(p)}(z) = \sum_{s \in \mathcal{S}} e^{Z_s} Z_s^{-(p+1)\alpha} P_{p+1}(Z_s) + \frac{p!}{2\pi i} \int_{\gamma'_r} e^t \frac{dt}{(t^\alpha - z)^{p+1}}.$$

Next we estimate the second term on the right-hand in the above equality. Observe that for any nonnegative integer N we have

$$\begin{aligned} \frac{p!}{(1-u)^{p+1}} &= \frac{d^p}{du^p} \left(\frac{1}{1-u} \right) \\ &= \sum_{k=p}^N \frac{k!}{(k-p)!} u^{k-p} + \sum_{l=0}^p p! \binom{N+1}{p-l} u^{N+1-p+l} \frac{1}{(1-u)^{l+1}} \end{aligned}$$

This shows that

$$\begin{aligned} \frac{p!}{(t^\alpha - z)^{p+1}} &= (-1)^{p+1} \sum_{k=0}^{N-p} \frac{(k+p)!}{k!} \frac{t^{k\alpha}}{z^{k+p+1}} \\ &\quad + \sum_{l=0}^p p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}}. \end{aligned}$$

Integrating on γ'_r yields

$$\frac{p!}{2\pi i} \int_{\gamma'_r} \frac{e^t}{(t^\alpha - z)^{p+1}} dt = (-1)^{p+1} \sum_{k=0}^{N-p} \frac{(k+p)!}{k!} \frac{1}{z^{k+p+1}} \frac{1}{2\pi i} \int_{\gamma'_r} e^t t^{k\alpha} dt + \mathcal{R}_N^{(p)}(z),$$

where

$$\begin{aligned} \mathcal{R}_N^{(p)}(z) &= \sum_{l=0}^p \mathcal{R}_{l,N}^{(p)}(z), \text{ with } \mathcal{R}_{l,N}^{(p)}(z) \\ &= p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} \frac{1}{2\pi i} \int_{\gamma'_r} e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}} dt. \end{aligned}$$

Hence

$$\frac{p!}{2\pi i} \int_{\gamma'_r} \frac{e^t}{(t^\alpha - z)^{p+1}} dt = (-1)^{p+1} \sum_{k=0}^{N-p} \frac{(k+p)!}{k!} \frac{1}{z^{k+p+1}} \frac{1}{\Gamma(-k\alpha)} + \mathcal{R}_N^{(p)}(z).$$

Since α is a positive integer, each term in the sum above is zero. First assume that for any integer s , $|\arg Z_s| \neq \pi$. When r tends to 0, if we choose N such that $(N+1-p+l)\alpha > -1$, we can write, for $l \in \{0, \dots, p\}$, $\mathcal{R}_{l,N}^{(p)}(z) = \mathcal{L}_{l,N}^{(p)}(z) + \mathcal{U}_{l,N}^{(p)}(z)$, where $\mathcal{L}_{l,N}^{(p)}(z)$ and $\mathcal{U}_{l,N}^{(p)}(z)$ are the integrals given by

$$\begin{aligned} \mathcal{L}_{l,N}^{(p)}(z) &= p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} \\ &\quad \times \frac{1}{2\pi i} \int_{+\infty}^0 e^{-v} \frac{e^{-i\alpha\pi(N+1-p+l)} v^{(N+1-p+l)\alpha}}{(e^{-i\alpha\pi} v^\alpha - z)^{l+1}} (-dv), \end{aligned}$$

$$\begin{aligned} \mathcal{U}_{l,N}^{(p)}(z) &= p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} \\ &\quad \times \frac{1}{2\pi i} \int_0^{+\infty} e^{-v} \frac{e^{i\alpha\pi(N+1-p+l)} v^{(N+1-p+l)\alpha}}{(e^{i\alpha\pi} v^\alpha - z)^{l+1}} (-dv) \end{aligned}$$

which can be rewritten

$$\mathcal{L}_{l,N}^{(p)}(z) = p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} e^{-i\alpha\pi(N-p)} \frac{1}{2\pi i} \int_0^{+\infty} \frac{e^{-v} v^{(N+1-p+l)\alpha}}{(v^\alpha - z e^{i\alpha\pi})^{l+1}} dv$$

and

$$\mathcal{U}_{l,N}^{(p)}(z) = -p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} e^{i\alpha\pi(N-p)} \frac{1}{2\pi i} \int_0^{+\infty} \frac{e^{-v} v^{(N+1-p+l)\alpha}}{(v^\alpha - z e^{-i\alpha\pi})^{l+1}} dv.$$

Since α is a positive integer, we see that $\mathcal{S} = \{0, \dots, \alpha - 1\}$ and $\mathcal{L}_{l,N}^{(p)}(z) = -\mathcal{U}_{l,N}^{(p)}(z)$ so that $\mathcal{R}_{l,N}^{(p)}(z) = 0$ for $l \in \{0, \dots, p\}$. We have the exact formula

$$E_{\alpha,\alpha}^{(p)}(z) = \sum_{s \in \mathcal{S}} e^{Z_s} Z_s^{-(p+1)\alpha} P_{p+1}(Z_s).$$

If there exists an integer s_0 such that $|\arg Z_{s_0}| = \pi$, the same identity holds by analytic continuation. $\square$

Lemma 3. *Suppose $\alpha \in]0, 1[$ and ϵ is a positive number with $0 < \sin \epsilon < |\sin(\alpha\pi)|$. When $|\arg z| \leq \alpha\pi - \epsilon$, $\mathcal{S} = \{0\}$; when $\alpha\pi + \epsilon \leq |\arg z| \leq \pi$, $\mathcal{S} = \emptyset$. In both cases, there exists a polynomial P_{p+1} of degree $p+1$ defined in Lemma 2 and a non-negative integer N such that for any $z \neq 0$,*

$$\begin{aligned} E_{\alpha,\alpha}^{(p)}(z) &= \sum_{s \in \mathcal{S}} e^{Z_s} Z_s^{-\alpha(p+1)} P_{p+1}(Z_s) \\ &\quad + \sum_{k=0}^{N-p} (-1)^{p+1} \frac{(k+p)!}{p!} \frac{1}{\Gamma(-k\alpha)} \frac{1}{z^{p+1+k}} + \mathcal{R}_N^{(p)}(z) \end{aligned}$$

and there exists a constant $C > 0$, depending on α, p, N, ϵ only such that for $|z| \geq 1$ $|\mathcal{R}_N^{(p)}(z)| \leq \frac{C}{|z|^{N+2}}$.

Proof. Using the notations of Lemma 2 we obtain

$$E_{\alpha,\alpha}^{(p)}(z) = \sum_{s \in \mathcal{S}} \text{Res}(f_p; Z_s) + \frac{1}{2\pi i} \int_{\gamma'_r} f_p(t) dt$$

and

$$\frac{p!}{2\pi i} \int_{\gamma'_r} \frac{e^t}{(t^\alpha - z)^{p+1}} dt = (-1)^{p+1} \sum_{k=0}^{N-p} \frac{(k+p)!}{k!} \frac{1}{z^{k+p+1}} \frac{1}{\Gamma(-k\alpha)} + \mathcal{R}_N^{(p)}(z),$$

where

$$\begin{aligned}\mathcal{R}_N^{(p)}(z) &= \sum_{l=0}^p \mathcal{R}_{l,N}^{(p)}(z) \text{ with } \mathcal{R}_{l,N}^{(p)}(z) \\ &= p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} \frac{1}{2\pi i} \int_{\gamma_r'} e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}} dt.\end{aligned}$$

First assume that for any integer s , $|\arg Z_s| \neq \pi$. When r tends to 0, if we choose N such that $(N+1-p+l)\alpha > -1$, we can write, for $l \in \{0, \dots, p\}$, $\mathcal{R}_{l,N}^{(p)}(z) = \mathcal{L}_{l,N}^{(p)}(z) + \mathcal{U}_{l,N}^{(p)}(z)$, where $\mathcal{L}_{l,N}^{(p)}(z)$ and $\mathcal{U}_{l,N}^{(p)}(z)$ are the following integrals:

$$\mathcal{L}_{l,N}^{(p)}(z) = p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} e^{-i\alpha\pi(N-p)} \frac{1}{2\pi i} \int_0^{+\infty} \frac{e^{-v} v^{(N+1-p+l)\alpha}}{(v^\alpha - ze^{i\alpha\pi})^{l+1}} dv,$$

$$\mathcal{U}_{l,N}^{(p)}(z) = -p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} e^{i\alpha\pi(N-p)} \frac{1}{2\pi i} \int_0^{+\infty} \frac{e^{-v} v^{(N+1-p+l)\alpha}}{(v^\alpha - ze^{-i\alpha\pi})^{l+1}} dv$$

We distinguish the three cases corresponding to different values of $\arg z$.

First suppose $-\pi \leq \arg z \leq -\alpha\pi - \epsilon$. We determine the set $\mathcal{S}$. If s is an integer put $\theta_s := \frac{1}{\alpha}(\arg z + 2\pi s)$. We have $\frac{\pi}{\alpha}(-1+2s) \leq \theta_s \leq -\pi - \frac{\epsilon}{\alpha} + \frac{2\pi s}{\alpha}$. If $s \leq 0$, then $\theta_s \leq -\pi - \frac{\epsilon}{\alpha} < -\pi$, and hence $s \notin \mathcal{S}$. Otherwise, $s \geq 1$ and thus $\theta_s \geq \frac{\pi}{\alpha}$; it follows that $s \notin \mathcal{S}$. This shows $\mathcal{S} = \emptyset$. Since $-\pi + \alpha\pi \leq \arg(ze^{i\alpha\pi}) \leq -\epsilon$, we see that $|ze^{i\alpha\pi} - v^\alpha| \geq \sin \epsilon |z|$ for any real positive number v .

In the same way, for the case $-\pi - \alpha\pi \leq \arg(ze^{-i\alpha\pi}) \leq -2\alpha\pi - \epsilon$, we obtain that for any v positive number

$$(3.1) \quad |ze^{-i\alpha\pi} - v^\alpha| \geq \sin(\alpha\pi)|z| > \sin \epsilon |z|.$$

Similar arguments show that for the remaining cases $-\alpha\pi + \epsilon \leq \arg z \leq \alpha\pi - \epsilon$, we have $\mathcal{S} = \{0\}$ and $\alpha\pi + \epsilon \leq \arg z \leq \pi$, we have $\mathcal{S} = \emptyset$. Moreover, (3.1) holds in all cases.

Hence for $l \in \{0, \dots, p\}$ both $\mathcal{L}_{l,N}^{(p)}(z)$ and $\mathcal{U}_{l,N}^{(p)}(z)$ satisfy the same inequality

$$|\mathcal{L}_{l,N}^{(p)}(z)| \leq \frac{p! \binom{N+1}{p-l}}{(\sin \epsilon)^{l+1}} \frac{\Gamma((N+1-p+l)\alpha + 1)}{2\pi} \frac{1}{|z|^{N+2+l}},$$

$$|\mathcal{U}_{l,N}^{(p)}(z)| \leq \frac{p! \binom{N+1}{p-l}}{(\sin \epsilon)^{l+1}} \frac{\Gamma((N+1-p+l)\alpha + 1)}{2\pi} \frac{1}{|z|^{N+2+l}},$$

and

$$|\mathcal{R}_{l,N}^{(p)}(z)| \leq \frac{p! \binom{N+1}{p-l}}{(\sin \epsilon)^{l+1}} \frac{\Gamma((N+1-p+l)\alpha+1)}{\pi} \frac{1}{|z|^{N+2+l}}.$$

The required result follows by taking the sum over $l \in \{0, \dots, p\}$. Now assume that there exists an integer s_0 such that $|\arg Z_{s_0}| = \pi$. We choose r sufficiently small so that the poles Z_s with $s \in \mathcal{S}$ are to the right of γ'_r .

Recall that $E_{\alpha,\alpha}^{(p)}(z) = \sum_{s \in \mathcal{S}} e^{Z_s} Z_s^{-(p+1)\alpha} P_{p+1}(Z_s) + \frac{p!}{2\pi i} \int_{\gamma'_r} e^t \frac{dt}{(t^\alpha - z)^{p+1}}$.

We deform γ'_r in another Hankel contour $\gamma'_{r,a'}$ surrounding the singularity Z_{s_0} from above and below by circular parts of radius a' such that the poles Z_s with $s \in \mathcal{S}$ are to the right of $\gamma'_{r,a'}$. We have

$$\frac{p!}{2\pi i} \int_{\gamma'_r} e^t \frac{dt}{(t^\alpha - z)^{p+1}} = \frac{p!}{2\pi i} \int_{\gamma'_{r,a'}} e^t \frac{dt}{(t^\alpha - z)^{p+1}}.$$

Following the same method as above we get

$$E_{\alpha,\alpha}^{(p)}(z) = \sum_{s \in \mathcal{S}} e^{Z_s} Z_s^{-(p+1)\alpha} P_{p+1}(Z_s) + \frac{p!}{2\pi i} \int_{\gamma'_{r,a'}} e^t \frac{dt}{(t^\alpha - z)^{p+1}}$$

and

$$\frac{p!}{2\pi i} \int_{\gamma'_{r,a'}} \frac{e^t}{(t^\alpha - z)^{p+1}} dt = (-1)^{p+1} \sum_{k=0}^{N-p} \frac{(k+p)!}{k!} \frac{1}{z^{k+p+1}} \frac{1}{\Gamma(-k\alpha)} + \mathcal{R}_N^{(p)}(z),$$

where

$$\begin{aligned} \mathcal{R}_N^{(p)}(z) &= \sum_{l=0}^p \mathcal{R}_{l,N}^{(p)}(z) \text{ with } \mathcal{R}_{l,N}^{(p)}(z) \\ &= p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} \frac{1}{2\pi i} \int_{\gamma'_{r,a'}} e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}} dt. \end{aligned}$$

If r tends to 0, $\gamma'_{r,a'}$ is deformed to $\gamma_{0,a'}$ which consists of the following four disconnected lines segments, the first one starts below the negative real axis at $-\infty$ to $Z_{s_0} - a'$, the second one joins $Z_{s_0} + a'$ to 0, the third joins 0 to $Z_{s_0} + a'$ above the negative real axis and the fourth returning from $Z_{s_0} - a'$ to $-\infty$. Then we see that

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{1}{2\pi i} \int_{\gamma'_{r,a'}} e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}} dt &= Res \left(e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}}; t = Z_{s_0} \right) \\ &+ \frac{1}{2\pi i} \int_{\gamma'_{0,a'}} e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}} dt. \end{aligned}$$

Let C_{s_0} be a circle centered at Z_{s_0} of radius η . Then we have

$$Res(e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}}; t = Z_{s_0}) = \frac{1}{2\pi i} \int_{C_{s_0}} e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}} dt.$$

Let be C'_{s_0} the image of C_{s_0} by the function power α , where η is chosen small enough so that C_s and C'_{s_0} are included in $\mathbb{C} \setminus \mathbb{R}_+$ and $u = t^\alpha = |t|^\alpha \exp[i\alpha(\arg t + 2\pi s_0)]$, where $0 < \arg t < 2\pi$. In a neighbourhood of C'_{s_0} , we define $u^{\frac{1}{\alpha}} = |u|^{\frac{1}{\alpha}} \exp[\frac{i}{\alpha}(\arg u + 2\pi s)]$, where $0 < \arg u < 2\pi$. The change of variables $u = t^\alpha$ shows that

$$\begin{aligned} Res(e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}}; t = Z_{s_0}) &= Res \left(\frac{1}{\alpha} e^{u^{\frac{1}{\alpha}}} \frac{u^{N-p+l+\frac{1}{\alpha}}}{(u - z)^{l+1}}; u = z \right) \\ &= \frac{1}{l!} \frac{d^l}{du^l} \left(\frac{1}{\alpha} e^{u^{\frac{1}{\alpha}}} u^{N-p+l+\frac{1}{\alpha}} \right)_{u=z} \\ &= e^{Z_{s_0}} Q_{N-p,l}(Z_{s_0}^\alpha, Z_{s_0}), \end{aligned}$$

for some polynomial $Q_{N-p,l}$. Thus

$$\begin{aligned} \mathcal{R}_{l,N}^{(p)}(z) &= p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} \left[e^{Z_{s_0}} Q_{N-p,l}(Z_{s_0}^\alpha, Z_{s_0}) \right. \\ &\quad \left. + \frac{1}{2\pi i} \int_{\gamma'_{0,a'}} e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}} dt \right]. \end{aligned}$$

Since Z_{s_0} is a negative real number and the integral $\frac{1}{2\pi i} \int_{\gamma'_{0,a'}} e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}} dt$ can be handled in the same way as the integral $\frac{1}{2\pi i} \int_{\gamma'_r} e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}} dt$ previously treated, we see that for any $l \in \{0, \dots, p\}$, $\mathcal{R}_{l,N}^{(p)}(z)$ satisfies a similar estimate. Hence for the ranges $|\arg z| \leq \alpha\pi - \epsilon$ and $\alpha\pi + \epsilon \leq |\arg z| \leq \pi$, there exists a constant C depending on α, p, N only such that when $|z| \geq 1$, $|\mathcal{R}_N^{(p)}(z)| \leq \frac{C}{|z|^{N+2}}$. $\square$

Lemma 4. Suppose $\alpha \in]2L-1, 2L[$, where L is a positive integer, and ϵ is a positive number with $0 < \sin \epsilon < |\sin(\alpha\pi)|$.

- (i) If $-\pi \leq \arg z \leq -(2L-\alpha)\pi - \epsilon$, then $\mathcal{S} = \{-(L-1), \dots, 0, \dots, L\}$.
- (ii) If $|\arg z| \leq (2L-\alpha)\pi - \epsilon$, then $\mathcal{S} = \{-(L-1), \dots, 0, \dots, L-1\}$
- (iii) If $(2L-\alpha)\pi + \epsilon \leq \arg z \leq \pi$, then $\mathcal{S} = \{-L, \dots, 0, \dots, L-1\}$

In those cases, there exists a polynomial P_{p+1} of degree $p+1$ defined in Lemma 2 and a non-negative N such that for any $z \neq 0$,

$$\begin{aligned} E_{\alpha, \alpha}^{(p)}(z) &= \sum_{s \in \mathcal{S}} e^{Z_s} Z_s^{-\alpha(p+1)} P_{p+1}(Z_s) \\ &+ \sum_{k=0}^{N-p} (-1)^{p+1} \frac{(k+p)!}{p!} \frac{1}{\Gamma(-k\alpha)} \frac{1}{z^{p+1+k}} + \mathcal{R}_N^{(p)}(z) \end{aligned}$$

and there exists a constant $C > 0$, depending on α, p, N, ϵ only such that

$$|\mathcal{R}_N^{(p)}(z)| \leq \frac{C}{|z|^{N+2}}, \text{ for } |z| \geq 1.$$

Proof. Arguing as in Lemma 2 we have

$$E_{\alpha, \alpha}^{(p)}(z) = \sum_{s \in \mathcal{S}} \text{Res}(f_p; Z_s) + \frac{1}{2\pi i} \int_{\gamma'_r} f_p(t) dt$$

and

$$\frac{p!}{2\pi i} \int_{\gamma'_r} \frac{e^t}{(t^\alpha - z)^{p+1}} dt = (-1)^{p+1} \sum_{k=0}^{N-p} \frac{(k+p)!}{k!} \frac{1}{z^{k+p+1}} \frac{1}{\Gamma(-k\alpha)} + \mathcal{R}_N^{(p)}(z),$$

where $\mathcal{R}_N^{(p)}(z) = \sum_{l=0}^p \mathcal{R}_{l,N}^{(p)}(z)$ with

$$\mathcal{R}_{l,N}^{(p)}(z) = p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} \frac{1}{2\pi i} \int_{\gamma'_r} e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}} dt.$$

First assume that for any integer s , $|\arg Z_s| \neq \pi$. Choose N such that $(N+1-p+l)\alpha > -1$. We can write, for $l \in \{0, \dots, p\}$, $\mathcal{R}_{l,N}^{(p)}(z) = \mathcal{L}_{l,N}^{(p)}(z) + \mathcal{U}_{l,N}^{(p)}(z)$, where

$$\mathcal{L}_{l,N}^{(p)}(z) = p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} e^{-i\alpha\pi(N-p)} \frac{1}{2\pi i} \int_0^{+\infty} \frac{e^{-v} v^{(N+1-p+l)\alpha}}{(v^\alpha - z e^{i\alpha\pi})^{l+1}} dv$$

$$\mathcal{U}_{l,N}^{(p)}(z) = -p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} e^{i\alpha\pi(N-p)} \frac{1}{2\pi i} \int_0^{+\infty} \frac{e^{-v} v^{(N+1-p+l)\alpha}}{(v^\alpha - z e^{-i\alpha\pi})^{l+1}} dv$$

For the different ranges of $\arg z$, namely $-\pi \leq \arg z \leq -(2L - \alpha)\pi - \epsilon$, $-(2L - \alpha)\pi + \epsilon \leq \arg z \leq (2L - \alpha)\pi - \epsilon$, and $(2L - \alpha)\pi + \epsilon \leq \arg z \leq \pi$, the set $\mathcal{S}$ is easy to determine. In each case, for a positive number v , the inequality (3.1) holds. Therefore we get, for $l \in \{0, \dots, p\}$

$$|\mathcal{L}_{l,N}^{(p)}(z)| \leq \frac{p! \binom{N+1}{p-l}}{(\sin \epsilon)^{l+1}} \frac{\Gamma((N+1-p+l)\alpha+1)}{2\pi} \frac{1}{|z|^{N+2+l}}$$

$$|\mathcal{L}_{l,N}^{(p)}(z)| \leq \frac{p! \binom{N+1}{p-l}}{(\sin \epsilon)^{l+1}} \frac{\Gamma((N+1-p+l)\alpha+1)}{2\pi} \frac{1}{|z|^{N+2+l}},$$

and

$$|\mathcal{R}_{l,N}^{(p)}(z)| \leq \frac{p! \binom{N+1}{p-l}}{(\sin \epsilon)^{l+1}} \frac{\Gamma((N+1-p+l)\alpha+1)}{\pi} \frac{1}{|z|^{N+2+l}}.$$

Taking the sum over $l \in \{0, \dots, p\}$ we get the required result.

Now we consider the results in the case when z is a negative real number. Notice that if $\arg z = -\pi$, then $\arg Z_s = \frac{\pi}{\alpha}(-1 + 2s)$ with $s \in \{-(L-1), \dots, 0, \dots, L\}$ and if $\arg z = \pi$, then $\arg Z_s = \frac{\pi}{\alpha}(1 + 2s)$ with $s \in \{-L, \dots, 0, \dots, L-1\}$. It follows that $\{Z_s, s \in \{-(L-1), \dots, 0, \dots, L\}\} = \{Z_s, s \in \{-L, \dots, 0, \dots, L-1\}\}$. Therefore when z is a negative real number, the formulas obtained in the first and the third subcases coincide.

Finally if there exists an integer s_0 such that $|\arg Z_{s_0}| = \pi$, the proof is similar to that of Lemma 3. $\square$

Lemma 5. Suppose $\alpha \in]2L, 2L+1[$, where L is a positive integer, and ϵ is a positive number with $0 < \sin \epsilon < |\sin(\alpha\pi)|$.

- (i) If $-\pi \leq \arg z \leq -(\alpha-2L)\pi - \epsilon$, then $\mathcal{S} = \{-(L-1), \dots, 0, \dots, L\}$.
- (ii) If $|\arg z| \leq (\alpha-2L)\pi - \epsilon$, then $\mathcal{S} = \{-(L-1), \dots, 0, \dots, (L-1)\}$.
- (iii) If $(\alpha-2L)\pi + \epsilon \leq \arg z \leq \pi$, then $\mathcal{S} = \{-L, \dots, 0, \dots, L-1\}$.

In those cases, there exists a polynomial P_{p+1} of degree $p+1$ defined in Lemma 2 and a non-negative integer N such that for any $z \neq 0$,

$$\begin{aligned} E_{\alpha,\alpha}^{(p)}(z) &= \sum_{s \in \mathcal{S}} e^{Z_s} Z_s^{-\alpha(p+1)} P_{p+1}(Z_s) \\ &\quad + \sum_{k=0}^{N-p} (-1)^{p+1} \frac{(k+p)!}{p!} \frac{1}{\Gamma(-k\alpha)} \frac{1}{z^{p+1+k}} + \mathcal{R}_N^{(p)}(z) \end{aligned}$$

and there exists a constant $C > 0$, depending on α , p , N , ϵ only such that

$$|\mathcal{R}_N^{(p)}(z)| \leq \frac{C}{|z|^{N+2}} \text{ for } |z| \geq 1.$$

Proof. Proceeding as in Lemma 2 we get again

$$E_{\alpha,\alpha}^{(p)}(z) = \sum_{s \in \mathcal{S}} \text{Res}(f_p; Z_s) + \frac{1}{2\pi i} \int_{\gamma'_r} f_p(t) dt$$

and

$$\frac{p!}{2\pi i} \int_{\gamma'_r} \frac{e^t}{(t^\alpha - z)^{p+1}} dt = (-1)^{p+1} \sum_{k=0}^{N-p} \frac{(k+p)!}{k!} \frac{1}{z^{k+p+1}} \frac{1}{\Gamma(-k\alpha)} + \mathcal{R}_N^{(p)}(z),$$

where $\mathcal{R}_N^{(p)}(z) = \sum_{l=0}^p \mathcal{R}_{l,N}^{(p)}(z)$ with

$$\mathcal{R}_{l,N}^{(p)}(z) = p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} \frac{1}{2\pi i} \int_{\gamma'_r} e^t \frac{t^{(N+1-p+l)\alpha}}{(t^\alpha - z)^{l+1}} dt.$$

First we consider the case $|\arg Z_s| \neq \pi$ for all integers s . Choosing N such that $(N+1-p+l)\alpha > -1$, we can write, for $l \in \{0, \dots, p\}$,

$\mathcal{R}_{l,N}^{(p)}(z) = \mathcal{L}_{l,N}^{(p)}(z) + \mathcal{U}_{l,N}^{(p)}(z)$, where

$$\mathcal{L}_{l,N}^{(p)}(z) = p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} e^{-i\alpha\pi(N-p)} \frac{1}{2\pi i} \int_0^{+\infty} \frac{e^{-v} v^{(N+1-p+l)\alpha}}{(v^\alpha - z e^{i\alpha\pi})^{l+1}} dv$$

$$\mathcal{U}_{l,N}^{(p)}(z) = -p! \binom{N+1}{p-l} (-1)^{p+l} \frac{1}{z^{N+1}} e^{i\alpha\pi(N-p)} \frac{1}{2\pi i} \int_0^{+\infty} \frac{e^{-v} v^{(N+1-p+l)\alpha}}{(v^\alpha - z e^{-i\alpha\pi})^{l+1}} dv$$

The set $\mathcal{S}$ can then be determined as in Lemmas 3 and 4 and the estimate for $|\mathcal{R}_N^{(p)}(z)|$ is similar. If z is a negative real number, we observe that

$$\{Z_s, s \in \{-(L-1), \dots, 0, \dots, L\}\} = \{Z_s, s \in \{-L, \dots, 0, \dots, L-1\}\}.$$

Thus the formulas obtained for $-\pi \leq \arg z \leq -(\alpha - 2L)\pi - \epsilon$ and $(\alpha - 2L)\pi + \epsilon \leq \arg z \leq \pi$ coincide.

If $|\arg Z_{s_0}| = \pi$ for some integer s_0 , then the proof is the same as in Lemma 3. $\square$

Recall that

$$\rho_m(a, b) = \frac{N(a, b)}{D(a, b)},$$

where $N(a, b) = F(|a|^2)F(|b|^2) - |F(\langle b, a \rangle)|^2$ and $D(a, b) = F(|a|^2)F(|b|^2)$.

We first estimate the numerator by using a Taylor expansion near a . To do so we need to study F and its higher order derivatives near the positive real axis.

Lemma 6. *Let p be a non-negative integer. There exists a constant $\epsilon > 0$ such that, for any complex z with $|\arg z| \leq \epsilon$, we have*

$$F^{(p)}(z) = e^{Z_0} Z_0^{-(n+p)\alpha} P_{n+p}(Z_0) + F_p(z),$$

where $Z_s = |z|^{\frac{1}{\alpha}} \exp \left[\frac{i}{\alpha} (\arg z + 2\pi s) \right]$, for s integer, the polynomials $(P_p)_{p \in \mathbb{N}}$ are defined by the relation

$$(3.2) \quad \frac{d^p}{du^p} \left[e^{u^{\frac{1}{\alpha}}} \right] = \frac{e^{u^{\frac{1}{\alpha}}}}{u^p} P_p(u^{\frac{1}{\alpha}}),$$

$$\text{and } F_p(z) = \begin{cases} O\left(\frac{1}{|z|}\right)^{p+1} & \text{if } \alpha \in]0, 1], \\ O(|z|^q e^{Z_{\pm 1}}) & \text{for any integer } q \text{ if } \alpha > 1 \end{cases}.$$

Proof. Follows from Lemmas 2, 3, 4, 5 and the fact that when $\arg z$ tends to 0, with s in $\mathcal{S}$ such that $|s| \geq 2$ and q, q', q'' are integers, $e^{Z_s}|z|^{q''}$ is negligible in comparison of $e^{Z_{\pm 1}}|z|^{q'}$ and $e^{Z_{\pm 1}}|z|^{q'}$ is negligible in comparison of $e^{Z_0}|z|^q$. $\square$

We will also need some properties of the polynomials $(P_p)_{p \in \mathbb{N}}$.

Lemma 7. *The sequence $(P_p)_{p \in \mathbb{N}}$ defined by (3.2) satisfies the recurrence relation:*

$$P_{p+1}(t) = \left(\frac{t}{\alpha} - p \right) P_p(t) + \frac{t}{\alpha} P'_p(t).$$

In particular each P_p has real coefficients, and is of degree p . In addition, the polynomial $P_p(t) - \left[\frac{1}{\alpha^p} t^p + \frac{p(p-1)}{2\alpha^{p-1}} \left(\frac{1}{\alpha} - 1 \right) t^{p-1} \right]$ is of degree at most $p-2$ and satisfies

$$(P_p P_{p+2} - P_{p+1}^2)(t) \sim \frac{1}{\alpha^{2p+1}} \left(\frac{1}{\alpha} - 1 \right) t^{2p+1}, \quad t \rightarrow +\infty$$

Proof. The relation between the polynomials (P_p) and the fact that P_p has real coefficients and degree p have been shown in the proof of the Lemma 2. We observe that the term of highest degree term of P_p is $\frac{1}{\alpha^p} t^p$.

Then P_p can be written in the form

$$P_p(t) = \frac{1}{\alpha^p} t^p + b_p t^{p-1} + \tilde{P}_p(t),$$

where $\tilde{P}_p(t)$ has a degree at most $p-2$. We need to compute b_p . By differentiation we have $P'_p(t) = \frac{p}{\alpha^p} t^{p-1} + (p-1)b_p t^{p-2} + \tilde{P}'_p(t)$, and therefore

$$P_{p+1}(t) = \frac{1}{\alpha^{p+1}} t^{p+1} + t^p \left[\frac{b_p}{\alpha} + \frac{p}{\alpha^{p+1}} - \frac{p}{\alpha^p} \right] + \tilde{P}_{p+1}(t).$$

It follows from this that $b_{p+1} = \frac{b_p}{\alpha} + \frac{p}{\alpha^p} \left(\frac{1}{\alpha} - 1 \right)$.

Since $P_1(t) = \frac{1}{\alpha} t$, we see that $b_1 = 0$; an induction process gives

$$b_p = \frac{p(p-1)}{2\alpha^{p-1}} \left(\frac{1}{\alpha} - 1 \right).$$

Using the identities

$$P_q(t) = \frac{1}{\alpha^q} t^q + \frac{q(q-1)}{2\alpha^{q-1}} \left(\frac{1}{\alpha} - 1 \right) t^{q-1} + \tilde{P}_q(t), \quad \text{for } q = p, p+1, p+2,$$

we find that the term of highest degree in $P_{p+2}P_p - P_{p+1}^2$ is $\frac{1}{\alpha^{2p+1}} \left(\frac{1}{\alpha} - 1 \right) t^{2p+1}$. $\square$

4. Preparatory estimates

To study $\rho_m(a, a+u)$, we first focus on the numerator $N(a, a+u)$.

Lemma 8. *There are positive constants A and C such that for any a, u in $\mathbb{C}^n$ with $|a| > A$ and $|u| < 1$, we have*

$$N(a, a+u) = (FF')(|a|^2)|u|^2 + (FF'' - F'^2)(|a|^2)|\langle u, a \rangle|^2 + R(a, u),$$

where

$$|R(a, u)| \leq C e^{2|a| \frac{2}{\alpha}} |a|^{\frac{2}{\alpha}(1-\alpha)2n} \left[|a|^{\frac{4}{\alpha}-3} + |a|^{\frac{6}{\alpha}-3} + |a|^{\frac{8}{\alpha}-4} + |a|^{\frac{10}{\alpha}-5} + |a|^{\frac{12}{\alpha}-6} \right] |u|^3.$$

Proof. The Taylor expansion formula gives

$$\begin{aligned} F(|a+u|^2) &= F(|a|^2) + F'(|a|^2)(\langle u, a \rangle + \langle a, u \rangle + |u|^2) \\ &\quad + \frac{1}{2} F''(|a|^2)(\langle u, a \rangle + \langle a, u \rangle)^2 + R_1(a, u), \end{aligned}$$

and

$$F(\langle a, a+u \rangle) = F(|a|^2) + F'(|a|^2)\langle a, u \rangle + \frac{1}{2}F''(|a|^2)(\langle a, u \rangle)^2 + R_2(a, u),$$

where

$$\begin{aligned} R_1(a, u) &= \frac{1}{2}F''(|a|^2) (|u|^4 + 2|u|^2(\langle u, a \rangle + \langle a, u \rangle)) \\ &+ \frac{1}{2} \int_0^1 (1-t)^2 F^{(3)}(|a|^2 + t(|u|^2 + \langle u, a \rangle + \langle a, u \rangle)) (\langle u, a \rangle + \langle a, u \rangle + |u|^2)^3 dt. \end{aligned}$$

and

$$R_2(a, u) = \frac{1}{2} \int_0^1 (1-t)^2 F^{(3)}(|a|^2 + t\langle a, u \rangle) \langle a, u \rangle^3 dt.$$

We write

$$\begin{aligned} |F(\langle a, a+u \rangle)|^2 &= F^2(|a|^2) + (FF')(|a|^2)(\langle a, u \rangle + \langle u, a \rangle) \\ &+ |\langle a, u \rangle|^2 F'^2(|a|^2) + \frac{1}{2}(\langle a, u \rangle^2 + \langle u, a \rangle^2)(F'F''(|a|^2) + R_3(a, u)), \end{aligned}$$

and let $R(a, u) = F(|a|^2)R_1(a, u) - R_3(a, u)$. We get

$$N(a, a+u) = (FF')(|a|^2)|u|^2 + (FF'' - F'^2)(|a|^2)|\langle a, u \rangle|^2 + R(a, u).$$

We shall estimate the remainder $R(a, u)$. By Lemma 6, there are positive constants A_1, D_1, ϵ such that for any $p = 0, \dots, 3$ and for any $z \in \mathbb{C}$ with $|\arg z| < \epsilon$ and $|z| > A_1$, we have

$$|F^{(p)}(z)| \leq D_1 e^{Z_0} Z_0^{(n+p)(1-\alpha)}, \text{ where } Z_0 = |z|^{\frac{1}{\alpha}} \exp \left[\frac{i}{\alpha} \arg z \right].$$

Notice that

$$\begin{cases} \arg [|a|^2 + t(|u|^2 + 2\Re(\langle u, a \rangle))] &= \arg \left[1 + \frac{t}{|a|^2} (|u|^2 + 2\Re(\langle u, a \rangle)) \right] \\ \arg [|a|^2 + t\langle a, u \rangle] &= \arg \left[1 + \frac{t}{|a|^2} \langle a, u \rangle \right]. \end{cases}$$

Thus there exists a positive constant A_2 such that for any $p = 0, \dots, 3$ and for any a, u in $\mathbb{C}^n$ with $|a| > A_2$ and $|u| < 1$, we have $|\arg [|a|^2 + t(|u|^2 + 2\Re(\langle u, a \rangle))]| < \epsilon$, and $|\arg [|a|^2 + t\langle a, u \rangle]| < \epsilon$.

Therefore

$$\begin{cases} |F^{(p)}(|a|^2)| \leq D_1 e^{|a|^{\frac{2}{\alpha}}} |a|^{\frac{2}{\alpha}(n+p)(1-\alpha)}, \\ |(F^{(3)}(|a|^2 + t(|u|^2 + 2\Re(\langle u, a \rangle)))| \leq D_1 e^{|a|^{\frac{2}{\alpha}}} |a|^{\frac{2}{\alpha}(n+3)(1-\alpha)} \text{ for any } t \in [0, 1], \\ |F^{(3)}(|a|^2 + t\langle a, u \rangle)| \leq D_1 e^{|a|^{\frac{2}{\alpha}}} |a|^{\frac{2}{\alpha}(n+3)(1-\alpha)}. \end{cases}$$

It follows that

$$\begin{aligned} |R(a, u)| &\leq D_1^2 e^{2|a|^{\frac{2}{\alpha}}} |a|^{\frac{2}{\alpha}(1-\alpha)2n} \left[(4|a| + |u|)|a|^{\frac{4}{\alpha}(1-\alpha)} + (2|a| + |u|)^3 |a|^{\frac{6}{\alpha}(1-\alpha)} \right. \\ &\quad + |a|^3 |a|^{\frac{6}{\alpha}(1-\alpha)} + 2|a|^4 |a|^{\frac{8}{\alpha}(1-\alpha)} |u| + 2|a|^3 |a|^{\frac{6}{\alpha}(1-\alpha)} \\ &\quad \left. + |a|^5 |a|^{\frac{10}{\alpha}(1-\alpha)} |u|^2 + |a|^6 |a|^{\frac{12}{\alpha}(1-\alpha)} |u|^3 \right] |u|^3. \end{aligned}$$

Hence there exist positive constants A_3 , C such that for any a , u in $\mathbb{C}^n$ with $|a| > A_3$ and $|u| < 1$, we have

$$|R(a, u)| \leq C e^{2|a|^{\frac{2}{\alpha}}} |a|^{\frac{2}{\alpha}(1-\alpha)(2n)} \left[|a|^{\frac{4}{\alpha}-3} + |a|^{\frac{6}{\alpha}-3} + |a|^{\frac{8}{\alpha}-4} + |a|^{\frac{10}{\alpha}-5} + |a|^{\frac{12}{\alpha}-6} \right] |u|^3,$$

and the proof is complete. $\square$

We next need to study the denominator $D(a, a + u)$ of $\rho_m(a, b)$.

Lemma 9. *There exists a positive constant A such that for any a , u in $\mathbb{C}^n$ with $|a| > A$ and $|u||a|^{\frac{2}{\alpha}-1} < 1$, we have*

$$\frac{F(|a + u|^2)}{e^{|a|^{\frac{2}{\alpha}}} |a|^{\frac{1}{\alpha}(1-\alpha)(2n)}} \approx [1 + |u||a|^{\frac{2}{\alpha}-1}].$$

Proof. A Taylor's expansion formula gives

$$\begin{aligned} F(|a + u|^2) &= F(|a|^2) \\ &\quad + \int_0^1 F'(|a|^2 + t(\langle u, a \rangle + \langle a, u \rangle + |u|^2)) dt \times (\langle u, a \rangle + \langle a, u \rangle + |u|^2) \end{aligned}$$

By Lemma 6, there are positive constants A_4 , C_4 , C_5 such that for a , u in $\mathbb{C}^n$ with $|a| > A_4$ and $|u| < 1$, we have

$$|F(|a + u|^2) - F(|a|^2)| \leq C_4 |u| |a| e^{|a|^{\frac{2}{\alpha}}} |a|^{\frac{2}{\alpha}(1-\alpha)(n+1)},$$

and

$$(1 - C_5) e^{|a|^{\frac{2}{\alpha}}} |a|^{\frac{1}{\alpha}(1-\alpha)2n} \leq |F(|a|^2)| \leq (1 + C_5) e^{|a|^{\frac{2}{\alpha}}} |a|^{\frac{1}{\alpha}(1-\alpha)2n}$$

Therefore

$$1 - C_5 - C_4|u||a||a|^{\frac{2}{\alpha}(1-\alpha)} \leq \frac{F(|a+u|^2)}{e^{|a|^{\frac{2}{\alpha}}} |a|^{\frac{2}{\alpha}(1-\alpha)n}} \leq 1 + C_5 + C_4|u||a||a|^{\frac{2}{\alpha}(1-\alpha)}.$$

This completes the proof. $\square$

5. Proofs of the mains results

Proof of Theorem 1. By Lemma 1, it is sufficient to estimate $\rho_m(a, a+u)$, when a and u are in $\mathbb{C}^n$. Setting $Z_0 = |a|^{\frac{2}{\alpha}}$ and applying Lemma 6, there exist two functions ϵ_1 and ϵ_2 such that

$$(FF')(|a|^2) = e^{2Z_0} Z_0^{-(2n+1)\alpha} P_n(Z_0) P_{n+1}(Z_0) (1 + \epsilon_1(|a|^2))$$

and

$$(FF'' - F'^2)(|a|^2) = e^{2Z_0} Z_0^{-(2n+2)\alpha} (P_n P_{n+2} - P_{n+1}^2)(Z_0) (1 + \epsilon_2(|a|^2)),$$

where the polynomials P_p are defined by (3.2) and

$$\lim_{t \rightarrow +\infty} \epsilon_1(t) = \lim_{t \rightarrow +\infty} \epsilon_2(t) = 0.$$

The properties of the polynomials P_p shown in Lemma 7 imply that

$$(FF'' - F'^2)(|a|^2) = \frac{1}{\alpha^{2n+1}} \left(\frac{1}{\alpha} - 1 \right) e^{2Z_0} Z_0^{-\alpha} Z_0^{(2n+1)(1-\alpha)} (1 + \epsilon_3(|a|^2)),$$

where ϵ_3 is a function satisfying $\lim_{t \rightarrow +\infty} \epsilon_3(t) = 0$. Using the above estimates, Cauchy-Schwarz inequality and Lemma 8, we see that there exist positive constants C_1, A_1 such that for a, u in $\mathbb{C}^n$ with $|a| > A_1$, and $|u| < 1$ we have

$$N(a, a+u) \leq C_1 e^{2|a|^{\frac{2}{\alpha}}} |a|^{\frac{2}{\alpha}(1-\alpha)2n} \left[|a|^{\frac{2}{\alpha}-2} + \left(|a|^{\frac{4}{\alpha}-3} + |a|^{\frac{6}{\alpha}-3} + |a|^{\frac{8}{\alpha}-4} + |a|^{\frac{10}{\alpha}-5} + |a|^{\frac{12}{\alpha}-6} \right) |u| \right] |u|^2$$

By Lemma 9, there exist positive constants C_2, A_2 such that for any a, u in $\mathbb{C}^n$ with $|a| > A$ and $|u||a|^{\frac{2}{\alpha}-1} < 1$, we have

$$\frac{1}{D(a, u)} \leq C_2 \frac{1}{1 - |u||a|^{\frac{2}{\alpha}-1}} e^{-2|a|^{\frac{2}{\alpha}}} |a|^{\frac{-2}{\alpha}(1-\alpha)2n}.$$

Therefore there exist constants C_3 and $\delta(|a|) > 0$ such that if $|u| < \delta(|a|)$, $\rho_m(a, a+u)^2 \leq C_3|a|^{\frac{2}{\alpha}-2}|u|^2$. By Lemma 1, the proof is complete. $\square$

In the case $n = 1$, we can give a more precise estimate for $\rho_m(a, a+u)$.

Proof of Theorem 2. Assume $n = 1$. With the notation of the previous proof, we get

$$(FF')(|a|^2) = \frac{1}{\alpha^3} e^{2Z_0} Z_0^{3(1-\alpha)} (1 + \epsilon_4(|a|^2))$$

and

$$(FF'' - F'^2)(|a|^2) = \frac{1}{\alpha^3} \left(\frac{1}{\alpha} - 1 \right) e^{2Z_0} Z_0^{-\alpha} Z_0^{3(1-\alpha)} (1 + \epsilon_5(|a|^2)),$$

where the functions ϵ_4 and ϵ_5 satisfy $\lim_{t \rightarrow +\infty} \epsilon_4(t) = \lim_{t \rightarrow +\infty} \epsilon_5(t) = 0$.

By Lemma 9 there exist positive constants A_4 , C_2 , C'_2 such that for any a, u in $\mathbb{C}^n$ with $|a| > A_4$ and $|u||a|^{\frac{2}{\alpha}-1} < 1$, we have

$$\begin{aligned} \frac{1}{2} \left[\frac{1}{\alpha^4} e^{2Z_0} Z_0^{3(1-\alpha)} |u|^2 + R(a, u) \right] &\leq N(a, u) \\ &\leq \frac{3}{2} \left[\frac{1}{2\alpha^4} e^{2Z_0} Z_0^{3(1-\alpha)} |u|^2 + R(a, u) \right], \end{aligned}$$

$$\begin{aligned} |R(a, u)| &\leq C_2 e^{2|a|^{\frac{2}{\alpha}}} |a|^{\frac{2}{\alpha}(1-\alpha)^2} \left[|a|^{\frac{4}{\alpha}-3} + |a|^{\frac{6}{\alpha}-3} \right. \\ &\quad \left. + |a|^{\frac{8}{\alpha}-4} + |a|^{\frac{10}{\alpha}-5} + |a|^{\frac{12}{\alpha}-6} \right] |u|^3, \end{aligned}$$

and

$$\begin{aligned} \frac{C'_2}{1 + |u||a|^{\frac{2}{\alpha}-1}} e^{-2|a|^{\frac{2}{\alpha}}} |a|^{\frac{-2}{\alpha}(1-\alpha)^2} &\leq \frac{1}{D(a, u)} \\ &\leq C_2 \frac{1}{1 - |u||a|^{\frac{2}{\alpha}-1}} e^{-2|a|^{\frac{2}{\alpha}}} |a|^{\frac{-2}{\alpha}(1-\alpha)^2}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{N(a, u)}{D(a, u)} &\leq \frac{3}{2} \left[\frac{C_2}{\alpha^4} Z_0^{1-\alpha} + C_2^2 \left(|a|^{\frac{4}{\alpha}-3} + |a|^{\frac{6}{\alpha}-3} + |a|^{\frac{8}{\alpha}-4} \right. \right. \\ &\quad \left. \left. + |a|^{\frac{10}{\alpha}-5} + |a|^{\frac{12}{\alpha}-6} \right) |u| \right] \frac{|u|^2}{1 - |u||a|^{\frac{2}{\alpha}-1}}. \end{aligned}$$

We have also

$$\begin{aligned} \frac{N(a, u)}{D(a, u)} &\geq \frac{1}{2} \left[\frac{C'_2}{\alpha^4} Z_0^{1-\alpha} - C_2'^2 \left(|a|^{\frac{4}{\alpha}-3} + |a|^{\frac{6}{\alpha}-3} + |a|^{\frac{8}{\alpha}-4} \right. \right. \\ &\quad \left. \left. + |a|^{\frac{10}{\alpha}-5} + |a|^{\frac{12}{\alpha}-6} \right) |u| \right] \frac{|u|^2}{1 + |u| |a|^{\frac{2}{\alpha}-1}}. \end{aligned}$$

So there exist positive numbers C_4 , C'_4 and $\delta(|a|)$ such that for $|u| < \delta(|a|)$, we have

$$C'_4 Z_0^{1-\alpha} |u|^2 \leq \frac{N(a, u)}{D(a, u)} \leq C_4 Z_0^{1-\alpha} |u|^2,$$

which completes the proof. $\square$

We now prove Corollary 3.

Proof of Corollary 3. Assume $n = 1$ and $\frac{1}{\alpha} = r$ is a positive integer; the function $z \mapsto z^r$ is analytic in the whole complex plane. The Taylor formula completes the proof. $\square$

In the special case when $n = 1$ and $m \leq 1$, we shall prove Corollary 4.

Proof of Corollary 4. Assume $\alpha \geq 2$. Then, in the proof of Theorem 2, the terms $Z_0^{1-\alpha}$, $\left[|a|^{\frac{4}{\alpha}-3} + |a|^{\frac{6}{\alpha}-3} + |a|^{\frac{8}{\alpha}-4} + |a|^{\frac{10}{\alpha}-5} + |a|^{\frac{12}{\alpha}-6} \right]$ and $|a|^{\frac{2}{\alpha}-1}$ are uniformly bounded when $|a| > A_4$. $\square$

6. The range of Ber

The Berezin transform $Ber : X \mapsto \tilde{X}$, defined for an operator on $\mathcal{A}^2(\mu_m)$ such that both domains of X and $\tilde{X}$ contain the set $\{k_a, \in \mathbb{C}^n\}$ is linear and one-to-one (see). If Op denotes the algebra of all linear bounded operators on $\mathcal{A}^2(\mu_m)$, we showed that $Ber(Op)$ is a linear subspace of locally Lipschitz functions on $\mathbb{C}^n$. As in the classical case (see [5]), we can show that smooth bounded Lipschitz functions can fail to be in $Ber(Op)$.

Theorem 5. Assume $m > 0$. There is no X in Op such that $\tilde{X}(a) = \frac{K_m(ta, a)}{K_m(a, a)}$, where K_m is the reproducing kernel of $\mathcal{A}^2(\mu_m)$ and $t \in]-\infty, -1[$, $|t|^{\frac{1}{\alpha}} \cos \frac{\pi}{\alpha} < 1$.

Proof. We use the notation K instead of K_m and $d\mu$ instead of $d\mu_m$. For a real number t , let R_t be the linear operator on $\mathcal{A}^2(\mu_m)$ defined by $R_t(f)(z) := f(tz)$. First, we consider the boundedness of R_t . Using the

properties of the reproducing kernel, we compute

$$\| R_t k_a \|^2 = \frac{1}{K(a, a)} K(at, at)$$

Observe that $\| R_t k_a \|^2 \leq 1$ for all $|t| \leq 1$. On the other hand, if $|t| > 1$, we have

$$\| R_t k_a \|^2 \sim e^{|a|^{\frac{2}{\alpha}} (|t|^{\frac{2}{\alpha}} - 1)} |t|^{\frac{2n}{\alpha} (1 - \alpha)}, \text{ when } |a| \rightarrow +\infty.$$

Therefore $\lim_{|a| \rightarrow +\infty} \| R_t k_a \|^2 = +\infty$ and hence R_t is unbounded. The Berezin transform of R_t is given by

$$\tilde{R}_t(a) = \langle R_t k_a, k_a \rangle = \frac{1}{K(a, a)} K(at, a).$$

We are going to show that there is no bounded operator X on $\mathcal{A}^2(\mu_m)$ such that

$$\tilde{X}(a) = \frac{K(ta, a)}{K(a, a)}.$$

We observe that the function $g_t(a) = \frac{K(ta, a)}{K(a, a)}$ is real-differentiable at every order. Assume that $t < 0$. We will show that g and its first order derivatives are bounded on $\mathbb{C}^n$. We have

$$g_t(a) = \frac{F(t|a|^2)}{F(|a|^2)}$$

and

$$\frac{\partial g}{\partial a_j}(a) = \bar{a}_j \left[\frac{tF'(t|a|^2)F(|a|^2) - F'(t|a|^2)}{F^2(|a|^2)} \right].$$

With the notations of Section 2, for $z \in \mathbb{C} \setminus \{0\}$, we have the asymptotic expansions

$$F(z) = \sum_{s \in \mathcal{S}} \frac{e^{Z_s}}{z^n} P_n(Z_s) + \tilde{\mathcal{R}}^{(n-1)}(z),$$

$$F'(z) = \sum_{s \in \mathcal{S}} \frac{e^{Z_s}}{z^{n+1}} P_{n+1}(Z_s) + \tilde{\mathcal{R}}^{(n)}(z),$$

where $Z_s = |z|^{\frac{1}{\alpha}} \exp \left[\frac{i\pi}{\alpha} (1 + 2s) \right]$, for s integer, and for $p \in \{n-1, n\}$,

$$\tilde{\mathcal{R}}^p(z) = \sum_{k=0}^{N-p} (-1)^{p+1} \frac{(k+p)!}{p!} \frac{1}{\Gamma(-k\alpha)} \frac{1}{z^{p+1+k}} + \mathcal{R}_N^{(p)}(z).$$

First consider $F(t|a|^2)$. Setting $z = t|a|^2$ and $Z_+ = e^{\frac{i\pi}{\alpha}}|t|^{\frac{1}{\alpha}}|a|^{\frac{2}{\alpha}}$ and $Z_- = e^{-\frac{i\pi}{\alpha}}|t|^{\frac{1}{\alpha}}|a|^{\frac{2}{\alpha}}$, we have $\arg z = \pi$, and hence

$$F(z) = \frac{1}{z^n} [e^{Z_+} P_n(Z_+) + e^{Z_-} P_n(Z_-)] + \rho_{0,\pi}(z)$$

and

$$F'(z) = \frac{1}{z^{n+1}} [e^{Z_+} P_{n+1}(Z_+) + e^{Z_-} P_{n+1}(Z_-)] + \rho_{1,\pi}(z)$$

with $\rho_{0,\pi}(z) = o\left(e^{Z_+} Z_+^{n(1-\alpha)}\right)$ and $\rho_{1,\pi}(z) = o\left(e^{Z_+} Z_+^{(n+1)(1-\alpha)}\right)$ when $|z| \rightarrow +\infty$.

Also, when $|a| \rightarrow +\infty$, setting $Z_0 = |a|^{\frac{2}{\alpha}}$ we obtain

$$F(|a|^2) = \frac{1}{|a|^{2n}} e^{Z_0} P_n(Z_0) + \rho_{0,0}(|a|^2),$$

and

$$F'(|a|^2) = \frac{1}{|a|^{2n+2}} e^{Z_0} P_{n+1}(Z_0) + \rho_{1,0}(|a|^2),$$

where $\rho_{j,0}(|a|^2) = o\left(e^{|a|^{\frac{2}{\alpha}}} |a|^{\frac{2}{\alpha}(n+j)(1-\alpha)}\right)$, $j = 0, 1$. Now we compute $N_t(a) = tF'(t|a|^2)F(|a|^2) - F'(|a|^2)F(t|a|^2)$. The main term in $N_t(a)$ is

$$\begin{aligned} \widetilde{N_t(a)} &= \frac{1}{|t|^n |a|^{2(2n+1)}} \{e^{Z_0+Z_+} [P_n(Z_0)P_{n+1}(Z_+) - P_{n+1}(Z_0)P_n(Z_+)] \\ &\quad + e^{Z_0+Z_-} [P_n(Z_0)P_{n+1}(Z_-) - P_{n+1}(Z_0)P_n(Z_-)]\} \end{aligned}$$

Thus

$$\widetilde{N_t(a)} \sim \frac{Z_0^n e^{Z_0}}{\alpha^{2n+1} |t|^n |a|^{2(2n+1)}} 2\Re\{e^{Z_+} Z_+^n (Z_+ - Z_0)\}.$$

Observing that

$$|e^{Z_+} Z_+^n (Z_+ - Z_0)| = \exp\left[\cos\frac{\pi}{\alpha} |t|^{\frac{1}{\alpha}} |a|^{\frac{2}{\alpha}}\right] |t|^{\frac{n}{\alpha}} |a|^{\frac{2n+2}{\alpha}} |e^{i\frac{\pi}{\alpha}} t|^{\frac{1}{\alpha}} - 1|,$$

we obtain an estimation of $N_t(a)$; there exist positive constants A, C such that for $a \in \mathbb{C}^n$ with $|a| > A$,

$$|N_t(a)| \leq C |t|^{n(\frac{1}{\alpha}-1)} |e^{i\frac{\pi}{\alpha}} t|^{\frac{1}{\alpha}} - 1| |a|^{(4n+2)(\frac{1}{\alpha}-1)} \exp\left[\left(1 + |t|^{\frac{1}{\alpha}} \cos\frac{\pi}{\alpha}\right) |a|^{\frac{2}{\alpha}}\right].$$

On the other hand, we have ,

$$F(|a|^2) \sim \frac{1}{\alpha^n} |a|^{2n(\frac{1}{\alpha}-1)} \exp |a|^{\frac{2}{\alpha}}, \quad |a| \rightarrow +\infty.$$

Therefore there exist positive constant A' , C' such that for any $j \in [1, n]$, if $|a| > A'$,

$$|g_t(a)| \leq C' \exp \left[\left(-1 + |t|^{\frac{1}{\alpha}} \cos \frac{\pi}{\alpha} \right) |a|^{\frac{2}{\alpha}} \right],$$

$$\left| \frac{\partial g_t}{\partial a_j}(a) \right| \leq C' |a| |a|^{2(\frac{2}{\alpha}-1)} \exp \left[\left(-1 + |t|^{\frac{1}{\alpha}} \cos \frac{\pi}{\alpha} \right) |a|^{\frac{2}{\alpha}} \right].$$

Choosing $t \in]-\infty, -1[$ and $|t|^{\frac{1}{\alpha}} \cos \frac{\pi}{\alpha} < 1$, we see that g_t and its derivatives of the first order are bounded; in particular g_t is Lipschitz.

Suppose there exist a bounded operator X in Op with $\tilde{X}(a) = g_t(a)$. The operator $A := X - R_t$ is in Op and satisfies $\tilde{A}(a) = 0$. The injectivity of the mapping $A \mapsto \tilde{A}$ [7] leads to $A = 0$. Then $Xk_a = R_t k_a$, which contradicts the boundedness of X . $\square$

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