Maximal operators of Fejér means of Walsh-Kaczmarz-Fourier series

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Abstract. The main aim of this paper is to prove that there exists a martingale \( f \in H_{1/2} \) such that the maximal Fejér operator with respect to Walsh-Kaczmarz system does not belong to the space \( L_{1/2} \). For the two-dimensional case, we prove that there exists a martingale \( f \in H_{1/2}^{□} (f \in H_{1/2}) \) such that the restricted (unrestricted) maximal operator of Fejér means of two-dimensional Walsh-Kaczmarz-Fourier series does not belong to the space weak-\( L_{1/2} \).

1. Introduction

The first result with respect to the a.e. convergence of the Walsh-Fejér means \( \sigma_n f \) is due to Fine [1]. Later, Schipp [9] showed that the maximal operator \( \sigma^* f := \sup_n |\sigma_n f| \) is of weak type \( (1,1) \), from which the a.e. convergence follows by standard argument. Schipp’s result implies by interpolation also the boundedness of \( \sigma^*: L_p \to L_p \) (\( 1 < p \leq \infty \)). This fails to hold for \( p = 1 \), but Fujii [2] proved that \( \sigma^* \) is bounded from the dyadic Hardy space \( H_1 \) to the space \( L_1 \). Fujii’s theorem was extended by Weisz [19]. Namely, he proved that the maximal operator of the Fejér means of the one-dimensional Walsh-Fourier series is bounded from the martingale Hardy space \( H_p (G) \) to the space \( L_p (G) \) for \( p > 1/2 \). Simon [13] gave a counterexample, which shows that this boundedness does not hold for \( 0 < p < 1/2 \). In the endpoint case \( p = 1/2 \) Weisz [21] proved that \( \sigma^* \) is
bounded from the Hardy space $H_{1/2}(G)$ to the space weak-$L_{1/2}(G)$ (see also [14]). In [5] the first author proved that the maximal operator $σ^*$ is not bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$.

In 1948 Šneider [16] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n \to \infty} \frac{D_n(x)}{\log n} \geq C > 0$$

holds a.e. In 1974 Schipp [10] and Young [17] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [15] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to $f$ for any continuous functions $f$. Gát [3] proved, for any integrable functions, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function and Gát proved that $\|σ^*\|_{1} \leq C\|f\|_{H_1}$. The result of Gát was extended to the Hardy space by Simon [11], who proved that $σ^*$ is of type $(H_p, L_p)$ for $p > 1/2$. Weisz [21] showed that in endpoint case $p = 1/2$ the maximal operator is of weak type $(H_{1/2}, L_{1/2})$.

In this paper we will prove a stronger result than the unboundedness of the maximal operator from the Hardy space $H_{1/2}$ to the space $L_{1/2}$, in particular, we prove that there exists a martingale $f \in H_{1/2}$ such that

$$\|σ^*\|_{1/2} = +\infty.$$

For the two-dimensional Walsh-Kaczmarz-Fourier series Simon proved [12] that the restricted maximal operator $σ^*_{\lambda}$ is bounded from the Hardy space $H_p$ to the space $L_p$ for all $p > 1/2$.

In the paper [7] it was proved that the assumption $p > 1/2$ is essential. Namely, the maximal operator $σ^* := \sup_n |σ^*_{n,n}|$ of the Fejér means of double Fourier series with respect to the Walsh-Kaczmarz system is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. In this paper we will prove a stronger result than in the paper [7], in particular, we prove that there exists a martingale $f \in H_{1/2}^{\square}(f \in H_{1/2})$ such that

$$\|σ^*_{\lambda}\|_{weak-L_{1/2}} = +\infty (\|σ^*\|_{weak-L_{1/2}} = +\infty).$$

Thus, as regards boundedness of $σ^*$, the case of two-dimensional Walsh-Kaczmarz series differs from the case of one-dimensional Walsh-Kaczmarz series.

Let denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on $\mathbb{Z}_2$ is given in the way that the measure of a singleton is $1/2$. 
Let $G := \bigoplus_{k=0}^{\infty} \mathbb{Z}_2$, $G$ be called the Walsh group. The elements of $G$ are sequences $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0, 1\}$ ($k \in \mathbb{N}$).

The group operation on $G$ is the coordinate-wise addition (denoted by $+$), the normalized Haar measure (denoted by $\mu$) and the topology are the product measure and topology. Dyadic intervals are defined by $I_0(x) := G, I_n(x) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}$ for $x \in G, n \in \mathbb{N}$. They form a base for the neighborhoods of $G$. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of $G$ and $I_n := I_n(0)$ for $n \in \mathbb{N}$.

Let $L_p$ denote the usual Lebesgue spaces on $G$ (with the corresponding norm or quasinorm $\| \cdot \|_p$). The space weak-$L_p$ consists of all measurable functions $f$ for which

$$\|f\|_{\text{weak-}L_p} := \sup_{\lambda > 0} \lambda \mu(|f| > \lambda)^{1/p} < +\infty.$$  

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} (x \in G, k \in \mathbb{N}).$$

Let the Walsh-Paley functions be the product functions of the Rademacher functions. Namely, each natural number $n$ can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0, 1\} (i \in \mathbb{N}),$$

where only a finite number of $n_i$’s different from zero. Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$. Walsh-Paley functions are $w_0 = 1$ and for $n \geq 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}.$$  

The Walsh-Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$
The set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions is the same in dyadic blocks. Namely,

\[ \{ \kappa_n : 2^k \leq n < 2^{k+1} \} = \{ w_n : 2^k \leq n < 2^{k+1} \} \]

for all \( k \in \mathbb{P} \) and \( \kappa_0 = w_0 \).

V. A. Skvortsov (see [15]) gave a relation between the Walsh-Kaczmarz functions and the Walsh-Paley functions by the help of the transformation \( \tau_A : G \rightarrow G \) defined by

\[ \tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, x_{A+1}, \ldots) \]

for \( A \in \mathbb{N} \). By the definition of \( \tau_A \), we have

\[ \kappa_n(x) = r_{n|n|}(x)w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, x \in G). \]

The Dirichlet kernels are defined by

\[ D_n^\alpha := \sum_{k=0}^{n-1} \alpha_k, \]

where \( \alpha_n = w_n \) or \( \kappa_n \) \((n \in \mathbb{P})\), \( D_0^\alpha := 0 \). The \( 2^n \)th Dirichlet kernels have a closed form (see e.g. [8])

\[ D_{2^n}^w(x) = D_{2^n}^\kappa(x) = D_{2^n}(x) = \begin{cases} 0, & \text{if } x \notin I_n \\ 2^n, & \text{if } x \in I_n. \end{cases} \]

The \( \sigma \)-algebra generated by the dyadic intervals of measure \( 2^{-k} \) will be denoted by \( F_k \) \((k \in \mathbb{N})\).

Denote by \( f = (f^{(n)}, n \in \mathbb{N}) \) a martingale with respect to \((F_n, n \in \mathbb{N})\) (for details see, e. g. [20]). The maximal function of a martingale \( f \) is defined by

\[ f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|. \]

In case \( f \in L_1(G) \), the maximal function can also be given by

\[ f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) \, d\mu(u) \right|, \quad x \in G. \]

For \( 0 < p < \infty \) the Hardy martingale space \( H_p(G) \) consists of all martingales for which
\[ \| f \|_{H_p} := \| f^* \|_p < \infty. \]

If \( f \in L_1(G) \), then it is easy to show that the sequence \( (S_{2^n} f : n \in \mathbb{N}) \) is a martingale. If \( f \) is a martingale, that is \( f = (f^{(0)}, f^{(1)}, ...) \) then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

\[ \hat{f}(i) = \lim_{k \to \infty} \int_G f^{(k)}(x) \alpha_i(x) \, d\mu(x) \quad (\alpha = w \text{ or } \kappa). \]

The Walsh-(Kaczmarz)-Fourier coefficients of \( f \in L_1(G) \) are the same as the ones of the martingale \( (S_{2^n} f : n \in \mathbb{N}) \) obtained from \( f \).

The two-dimensional dyadic cubes are of the form

\[ I_{n,n}(x,y) := I_n(x) \times I_n(y). \]

By \( F_{n,n} \), we denote the \( \sigma \)-algebra generated by the dyadic rectangles \( \{I_{n,n}(x,y) : (x,y) \in G \times G\} \).

Denote by \( f = (f^{(n,n)}, n \in \mathbb{N}) \) a martingale with respect to \( (F_{n,n}, n \in \mathbb{N}) \) (for details see, e. g. [20]). The maximal function of a martingale \( f \) is defined by

\[ f^\square = \sup_{n \in \mathbb{N}} \left| f^{(n,n)} \right|. \]

In case \( f \in L_1(G \times G) \), the maximal function can also be given by

\[ f^\square(x,y) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_{n,n}(x,y))} \left| \int_{I_{n,n}(x,y)} f(u,v) \, d\mu(u,v) \right|, \]

\[ (x,y) \in G \times G, \]

For \( 0 < p < \infty \) the Hardy martingale space \( H_p^\square(G \times G) \) consists of all martingales for which

\[ \| f \|_{H_p} := \| f^\square \|_p < \infty. \]

Let

\[ I_{n,m}(x,y) := I_n(x) \times I_m(y). \]

We denote by \( F_{n,m}(n,m \in \mathbb{N}) \), the \( \sigma \)-algebra generated by the dyadic rectangles \( \{I_{n,m}(x,y) : (x,y) \in G \times G\} \).

Denote by \( f = (f^{(n,m)}, n,m \in \mathbb{N}) \) a martingale with respect to \( (F_{n,m}, n,m \in \mathbb{N}) \) (for details see, e. g. [20]).
The maximal function of a martingale \( f \) is defined by
\[
f^* = \sup_{n,m \in \mathbb{N}} |f^{(n,m)}|.
\]

For \( 0 < p < \infty \) the Hardy martingale space \( H_p(G \times G) \) consists of all martingales for which
\[
\|f\|_{H_p} := \|f^*\|_p < \infty.
\]

In case \( f \in L_1(G \times G) \), maximal functions can also be given by
\[
f^* (x,y) = \sup_{n,m \in \mathbb{N}} \frac{1}{\mu(I_{n,m}(x,y))} \left| \int_{I_{n,m}(x,y)} f(u,v) \, d\mu(u,v) \right|.
\]

2. The one-dimensional maximal operator

For \( n = 1, 2, \ldots \) and a martingale \( f \) the Fejér means of the Walsh-(Kaczmarz)-Fourier series of the function \( f \) is given by
\[
\sigma_n^\alpha f (x) = \frac{1}{n} \sum_{j=0}^{n-1} S_j^\alpha (f; x) \quad (\alpha = w \text{ or } \kappa).
\]

For a martingale \( f \) we consider the maximal operator
\[
\sigma_n^{\alpha,*} f = \sup_{n \in \mathbb{P}} |\sigma_n^\alpha f (x)| \quad (\alpha = w \text{ or } \kappa).
\]

The \( n \)th Fejér kernel of the Walsh-(Kaczmarz)-Fourier series defined by
\[
K_n^\alpha (x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k^\alpha (x) \quad (\alpha = w \text{ or } \kappa).
\]

A bounded measurable function \( a \) is a \( p \)-atom, if there exists a dyadic interval \( I \), such that
\begin{enumerate}
\item \( \int_I ad\mu = 0; \)
\item \( \|a\|_\infty \leq \mu(I)^{-1/p}; \)
\item \( \text{supp } a \subset I. \)
\end{enumerate}

The basic result of atomic decomposition is the following one.

**Theorem A.** (Weisz [20].) A martingale \( f = (f^{(n)}; n \in \mathbb{N}) \) is in \( H_p(0 < p \leq 1) \) if and only if there exists a sequence \( (a_k, k \in \mathbb{N}) \) of \( p \)-atoms
and a sequence \((\mu_k, k \in \mathbb{N})\) of real numbers such that for every \(n \in \mathbb{N},\)

\[
\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = f^{(n)},
\]

\[
\sum_{k=0}^{\infty} |\mu_k|^p < \infty.
\]

Moreover,

\[
\|f\|_{H^p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},
\]

where the infimum is taken over all decompositions of \(f\) of the form (1).

We will use the following lemma of Goginava:

**Lemma 1.** (Goginava [6]) Let \(2 < A \in \mathbb{P}\) and \(q_A := 2^{2A} + 2^{A-2} + \ldots + 2^2 + 2^0\). Then

\[
q_{A-1} \left| K_{q_{A-1}}^w (x) \right| \geq 2^{2m+2s-3}
\]

for \(x \in I_{2A} (0, ..., 0, x_{2m} = 1, 0, ..., 0, x_{2s} = 1, x_{2s+1}, ..., x_{2A-1}), m = 0, 1, ..., A - 3, s = m + 2, m + 3, ..., A - 1.\)

We will prove the following theorem.

**Theorem 1.** There exists a martingale \(f \in H_{1/2} (G)\) such that

\[
\|\sigma^{\kappa,*} f\|_{1/2} = +\infty.
\]

**Proof.** Since \(\frac{2^{m_k}}{m_k} \uparrow \infty\) as \(k \to \infty\) it is easy to show that there exists an increasing sequence of positive integers \((m_k : k \in \mathbb{N})\) such that

\[
\sum_{k=0}^{\infty} \frac{1}{m_k^{1/2}} < \infty,
\]

\[
\sum_{l=0}^{k-1} \frac{2^{4m_l}}{m_l} < \frac{2^{4m_k}}{m_k},
\]

\[
\frac{k2^{4m_k-1}}{m_k} \leq \frac{2^{2m_k}}{m_k}.
\]

Let

\[
f^{(A)} (x) := \sum_{k, 2m_k < A} \lambda_k a_k, \text{ where } \lambda_k := \frac{1}{m_k}
\]
and

\[ a_k(x) := 2^{2m_k} \left( D_{2^{2m_k+1}}(x) - D_{2^{2m_k}}(x) \right). \]

The martingale \( f := (f^{(0)}, f^{(1)}, \ldots, f^{(A)}, \ldots) \) is in \( H_{1/2}(G) \). Indeed, since

\[ S_{2^A} a_k(x) = \begin{cases} 0, & \text{if } A \leq 2m_k, \\ a_k(x), & \text{if } A > 2m_k, \end{cases} \]

and

\[ f^{(A)}(x) = \sum_{k:2^m_k < A} \lambda_k a_k(x) = \sum_{k=0}^{\infty} \lambda_k S_{2^A} a_k(x) \]

by (2) and Theorem A we conclude that \( f \in H_{1/2}(G) \).

Now, we investigate the Fourier coefficients.

Let \( j \in \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\} \) for some \( k = 0, 1, 2, \ldots \). Then it is evident that

\[ \hat{f}^\kappa(j) := \lim_{A \to \infty} \hat{f}^{(A)}(j) = \frac{2^{2m_k}}{m_k} \]

and \( \hat{f}^\kappa(j) = 0 \), if \( j \notin \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\}, k = 0, 1, 2, \ldots \).

Now, we decompose the \( q_{m_k} \) th Walsh-Kaczmarz-Fejér means as follows. (For the definition of \( q_{m_k} \) see Lemma 1 of Goginava.)

\[ \sigma_{q_{m_k}} f(x) = \frac{1}{q_{m_k}} \sum_{j=0}^{2^{2m_k} - 1} S_j f(x) + \frac{1}{q_{m_k}} \sum_{j=2^{2m_k}}^{q_{m_k} - 1} S_j f(x) = I + II. \]

Let \( j < 2^{2m_k} \). Then (3) gives that

\[ |S_j f(x)| \leq \sum_{l=0}^{k-1} \sum_{v=2^m_l}^{2^{2m_l+1} - 1} |\hat{f}^\kappa(v)| \leq \sum_{l=0}^{k-1} \frac{2^{4m_l}}{m_l} < \frac{2^{4m_{k-1}}}{m_{k-1}} \]

and

\[ I \leq \frac{1}{q_{m_k}} \sum_{j=0}^{2^{2m_k} - 1} |S_j f(x)| \leq \frac{2^{4m_{k-1}}}{m_{k-1}}. \]

Now, we discuss \( II \).
For $2^{2m_k} \leq j < q_{m_k}$ we have the following:

$$S_j^\kappa f(x) = \sum_{v=0}^{2^{2m_{k-1}}+1} \hat{f}^\kappa(v) \kappa_v(x) + \sum_{v=2^{2m_k}}^{j-1} \hat{f}^\kappa(v) \kappa_v(x)$$

$$= \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \hat{f}^\kappa(v) \kappa_v(x) + \sum_{v=2^{2m_k}}^{j-1} \hat{f}^\kappa(v) \kappa_v(x)$$

$$= \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \frac{\hat{f}}{m_l} \kappa_v(x) + \frac{2^{2m_k}}{m_k} \sum_{v=2^{2m_k}}^{j-1} \kappa_v(x)$$

(7) $$= \sum_{l=0}^{k-1} \frac{2^{2m_l}}{m_l} \left( D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x) \right) + \frac{2^{2m_k}}{m_k} \left( D_j^\kappa(x) - D_{2^{2m_k}}(x) \right).$$

This gives that

$$II = \left( \frac{q_{m_k} - 2^{2m_k}}{q_{m_k}} \right) \sum_{l=0}^{k-1} \frac{2^{2m_l}}{m_l} \left( D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x) \right)$$

$$+ \frac{2^{2m_k}}{q_{m_k} m_k} \sum_{j=2^{2m_k}}^{q_{m_k}-1} \left( D_j^\kappa(x) - D_{2^{2m_k}}(x) \right)$$

$$=: II_1 + II_2.$$

To discuss $II_1$, we use (3) and $|D_{2^n}(x)| \leq 2^n$. Thus, we can write

(8) $$|II_1| \leq c \sum_{l=0}^{k-1} \frac{2^{4m_l}}{m_l} < \frac{2^{4m_k-1}}{m_{k-1}}.$$

From $\sigma_{q_{m_k}}^\kappa f(x) = I + II_1 + II_2$, and (6), (8) we have

(9) $$|\sigma_{q_{m_k}}^\kappa f(x)| \geq |II_2| - |I| - |II_1| \geq |II_2| - c \frac{2^{4m_k-1}}{m_{k-1}}.$$

Now, we discuss $II_2$.

We can write the $n$th Dirichlet kernel with respect to the Walsh-Kaczmarz system in the following form:

$$D_n^\kappa(x) = D_{2^n}(x) + \sum_{k=2^{2|n|}}^{n-1} r_{|k|}(x) w_{k-2^{2|n|}}(\tau_{|k|}(x))$$

$$= D_{2^n}(x) + r_{|n|}(x) D_{n-2^{2|n|}}^w(\tau_{|n|}(x)).$$

(10)
By the help of this, we immediately get

\[ |I_2| = \frac{2^{2m_k}}{q_{m_k}^2} \left| \sum_{j=0}^{q_{m_k-1}^2} \left( D_{j+2m_k}^\epsilon(x) - D_{2m_k}^\epsilon(x) \right) \right| \]

\[ \geq \frac{2^{2m_k}}{q_{m_k}^2} \left| \phi_{2m_k}^\epsilon(x) \sum_{j=0}^{q_{m_k-1}^2} D_j^\epsilon(\phi_{2m_k}^\epsilon(x)) \right| \]

Thus, from (9) we have

\[ |\sigma_{q_{m_k}}^\epsilon f(x)| \geq c q_{m_k-1}^{\epsilon} |K_{q_{m_k-1}}^\epsilon (\phi_{2m_k}^\epsilon(x))| - c \frac{2^{4m_k-1}}{m_k}. \]

Define the set \( J_{2A}^{l,s}(x) \) for \( l < s < A \) by

\[ J_{2A}^{l,s}(x) := I_{2A}(x_0, x_1, \ldots, x_{2A-2l-2}, x_{2A-2s-1} = 1, 0, \ldots, 0, x_{2A-2l-1} = 1, 0, \ldots, 0). \]

Let \( x \in J_{2m_k}^{l,s}(x) \), for some \( l = \lfloor m_k/2 \rfloor, \lfloor m_k/2 \rfloor + 1, \ldots, m_k - 3 \), and \( s = l + 2, l + 3, \ldots, m_k - 1 \), then from Lemma 1 and (4) we have

\[ \left| \sigma_{q_{m_k}}^\epsilon f(x) \right| \geq c \frac{2^{2l+2s}}{m_k} - c \frac{2^{4m_k-1}}{m_k}. \]

Hence,

\[ \int_G |\sigma_{q_{m_k}}^\epsilon f(x)|^{1/2} d\mu(x) \]

\[ \geq \int_G |\sigma_{q_{m_k}}^\epsilon f(x)|^{1/2} d\mu(x) \]

\[ \geq \frac{c}{m_k^{1/2}} \sum_{l=\lfloor m_k/2 \rfloor}^{m_k-3} \sum_{s=l+2}^{m_k-1} \sum_{x_i=0}^{1} \int_{J_{2m_k}^{l,s}(x)} |\sigma_{q_{m_k}}^\epsilon f(x)|^{1/2} d\mu(x) \]

\[ \geq \frac{c}{m_k^{1/2}} \sum_{l=\lfloor m_k/2 \rfloor}^{m_k-3} \sum_{s=l+2}^{m_k-1} \frac{1}{2^{2m_k-2l-2}} \]

\[ \geq \frac{c}{m_k^{1/2}} \sum_{l=\lfloor m_k/2 \rfloor}^{m_k-3} \sum_{s=l+2}^{m_k-1} \frac{2l^2}{2s} \geq cm_k^{1/2} \to \infty \text{ as } k \to \infty. \]
That is \( \| \sigma_{\kappa,*} f \|_{1/2} = +\infty \). The proof is complete. \( \square \)

3. The two-dimensional restricted maximal operator

For \( \alpha = w \) or \( \kappa \) the rectangular partial sums of the double Walsh-(Kaczmarz)-Fourier series are defined as follows:

\[
S_{M,N}^\alpha f (x,y) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \hat{f}(i,j) \alpha_i(x) \alpha_j(y),
\]

where the number

\[
\hat{f}(i,j) = \int_{G \times G} f(x,y) \alpha_i(x) \alpha_j(y) \, d\mu(x,y).
\]

is said to be the \((i,j)\)th Walsh-(Kaczmarz)-Fourier coefficient of the function \( f \). If \( f \in L_1(G \times G) \) then it is easy to show that the sequence \((S_{2^n,2^n}^\alpha f : n \in \mathbb{N})\) is a martingale. If \( f \) is a martingale, that is \( f = (f^{(n,n)} : n \in \mathbb{N}) \) then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

\[
\hat{f}^\alpha (i,j) = \lim_{k \to \infty} \int_{G \times G} f^{(k)}(x,y) \alpha_i(x) \alpha_j(y) \, d\mu(x,y).
\]

The Walsh-(Kaczmarz)-Fourier coefficients of \( f \in L_1(G \times G) \) are the same as the ones of the martingale \((S_{2^n,2^n}^\alpha f : n \in \mathbb{N})\) obtained from \( f \).

For \( n, m \in \mathbb{P} \) and a martingale \( f \) the \((n,m)\)th Fejér mean of the double Walsh-(Kaczmarz)-Fourier series is given by

\[
\sigma_{n,m}^\alpha f (x,y) = \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} S_{i,j}^\alpha f (x,y).
\]

For the martingale \( f \) the restricted maximal operator is defined by

\[
\sigma_\lambda^{\square} f (x,y) = \sup_{2^{-\lambda \leq n/m \leq 2^\lambda}} |\sigma_{n,m}^\alpha f (x,y)|.
\]

A bounded measurable function \( a \) is a \( p \)-atom, if there exists a dyadic 2-dimensional cube \( I \times I \), such that

a) \( \int_{I \times I} a \, d\mu = 0; \)

b) \( \|a\|_\infty \leq \mu(I \times I)^{-1/p}; \)

c) \( \text{supp } a \subset I \times I. \)
The basic result of atomic decomposition is the following one.

**Theorem B.** (Weisz [20]). A martingale \( f = (f^{(n,n)} : n \in \mathbb{N}) \) is in \( H_{p}^\square (0 < p \leq 1) \) if and only if there exists a sequence \((a_k, k \in \mathbb{N})\) of \( p \)-atoms and a sequence \((\mu_k, k \in \mathbb{N})\) of real numbers such that for every \( n \in \mathbb{N} \),

\[
\sum_{k=0}^{\infty} \mu_k S_{2^n,2^n} a_k = f^{(n,n)},
\]

\[
\sum_{k=0}^{\infty} |\mu_k|^p < \infty.
\]

Moreover,

\[
\|f\|_{H_p^\square} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},
\]

We will prove the following theorem.

**Theorem 2.** There exists a martingale \( f \in H_{1/2}^\square (G \times G) \) such that

\[
\left\| \sigma^{\kappa, \square} f \right\|_{\text{weak-}L_{1/2}} = +\infty.
\]

**Proof.** To prove Theorem 2 we modify the sequence \( \{m_k : k \in \mathbb{P}\} \) and atoms \( a_k \) given in the previous section in the following way.

Let \( \{m_k : k \in \mathbb{N}\} \) be an increasing sequence of positive integers such that

\[
\sum_{k=0}^{\infty} \frac{1}{m_k^{1/2}} < \infty,
\]

\[
\sum_{l=0}^{k-1} \frac{2^{8m_l}}{m_l} < \frac{2^{8m_k}}{m_k},
\]

\[
\frac{2^{8m_k-1}}{m_{k-1}} < \frac{2^{m_k}}{km_k},
\]

Let

\[
f^{(A,A)} (x,y) := \sum_{k,2^{m_k} < A} \lambda_k a_k (x,y), \text{ where } \lambda_k := \frac{1}{m_k}
\]

and

\[
a_k (x,y) := 2^{4m_k} (D_{2^{m_k+1}} (x) - D_{2^{m_k}} (x)) (D_{2^{2m_k+1}} (y) - D_{2^{2m_k}} (y)).
\]
The martingale $f := \{f^{(0,0)}, f^{(1,1)}, \ldots, f^{(A,A)}, \ldots\} \in \mathcal{H}_{1/2} (G \times G)$.

Indeed,

$$S_{2^A, 2^A} a_k (x, y) = \begin{cases} 0, & \text{if } A \leq 2m_k, \\ a_k (x, y), & \text{if } A > 2m_k, \end{cases}$$

$$f^{(A,A)} (x) = \sum_{k, 2m_k < A} \lambda_k a_k (x, y) = \sum_{k=0}^{\infty} \lambda_k S_{2^A, 2^A} a_k (x, y)$$

from (12) and Theorem B we conclude that $f \in \mathcal{H}_{1/2} (G \times G)$.

Now, we investigate the Fourier coefficients. Since

$$\int_{G \times G} f^{(A)} (x, y) \kappa_i (x) \kappa_j (y) d\mu (x, y)$$

$$= \begin{cases} 0, & (i, j) \notin \bigcup_{k=0}^{\infty} \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\}, \\
0, & (i, j) \in \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\}, \\
\frac{\lambda_k a_k (x, y)}{m_k}, & (i, j) \in \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\}, \\
& A > 2m_k, \end{cases}$$

we can write

$$\hat{f}^\kappa (i, j) = \begin{cases} \frac{\lambda_k a_k (x, y)}{m_k}, & (i, j) \in \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\}, \\
0, & (i, j) \notin \bigcup_{k=1}^{\infty} \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\}. \end{cases}$$

We decompose the $(q_{mk}, q_{mk})$th Fejér means as follows

$$\sigma_{q_{mk}, q_{mk}}^\kappa f (x, y)$$

$$= \frac{1}{q_{mk}^2} \sum_{i=0}^{q_{mk} - 1} \sum_{j=0}^{q_{mk} - 1} S_{i,j}^\kappa f (x, y)$$

$$= \frac{1}{q_{mk}^2} \sum_{i=0}^{2^m_{mk} - 1} \sum_{j=0}^{2^m_{mk} - 1} S_{i,j}^\kappa f (x, y) + \frac{1}{q_{mk}^2} \sum_{i=2^m_{mk}}^{q_{mk} - 1} \sum_{j=0}^{2^m_{mk} - 1} S_{i,j}^\kappa f (x, y)$$

$$+ \frac{1}{q_{mk}^2} \sum_{i=0}^{2^m_{mk} - 1} \sum_{j=2^m_{mk}}^{q_{mk} - 1} S_{i,j}^\kappa f (x, y) + \frac{1}{q_{mk}^2} \sum_{i=2^m_{mk}}^{q_{mk} - 1} \sum_{j=2^m_{mk}}^{q_{mk} - 1} S_{i,j}^\kappa f (x, y)$$

$$= I + II + III + IV.$$
Let
\[(i, j) \in \left(\{2^{2m_k}, \ldots, q_{m_k} - 1\} \times \{0, 1, \ldots, 2^{2m_k} - 1\}\right) \cup \left(\{0, 1, \ldots, 2^{2m_k} - 1\} \times \{2^{2m_k}, \ldots, q_{m_k} - 1\}\right) \cup \left(\{0, 1, \ldots, 2^{2m_k} - 1\} \times \{0, 1, \ldots, 2^{2m_k} - 1\}\right).
\]
for some \(k\). Then from (15) and (13) it is easy to show that
\[|S^n_{i,j} f(x,y)| \leq \sum_{l=0}^{k-1} \sum_{\nu=2^{2m_l}} \sum_{\mu=2^{2m_l}} |\tilde{f}^\kappa(\nu, \mu)| \leq \sum_{l=0}^{k-1} \frac{2^{2m_l}}{m_l} \leq C2^{m_k},
\]
where
\[|S^n_{i,j} f(x,y)| \leq \frac{1}{q_{2m_k}} \sum_{i=0}^{2^{2m_k} - 1} \sum_{j=0}^{2^{2m_k} - 1} |S^n_{i,j} f(x,y)| \leq C\frac{2^{4m_k} q_{2m_k}}{q_{2m_k}^2} \leq C\frac{2^{m_k}}{km_k},
\]
Consequently, we have
\[|I| \leq \frac{1}{q_{2m_k}} \sum_{i=0}^{2^{2m_k} - 1} \sum_{j=0}^{2^{2m_k} - 1} |S^n_{i,j} f(x,y)| \leq C\frac{2^{4m_k} q_{2m_k}}{q_{2m_k}^2} \leq C\frac{2^{m_k}}{km_k},
\]
and
\[|II| \leq \frac{2^{2m_k}(q_{2m_k} - 2^{2m_k}) \frac{2^{m_k}}{q_{2m_k}^2}}{km_k} \leq C\frac{2^{m_k}}{km_k}
\]
and
\[|III| \leq C\frac{2^{m_k}}{km_k}.
\]
Combining (16)-(19) we obtain that
\[|\sigma^q_{mv_k q_{m_k}} f(x,y)| \geq |IV| - C\frac{2^{m_k}}{km_k}.
\]
Now, we discuss IV.
Let \((i, j) \in \{2^{2m_k}, \ldots, q_{m_k} - 1\} \times \{2^{2m_k}, \ldots, q_{m_k} - 1\}\). Then from (15) we have
\[S^n_{i,j} f(x,y) = \sum_{\nu=0}^{i-1} \sum_{\mu=0}^{j-1} \tilde{f}^\kappa(\nu, \mu) \kappa_\nu(x) \kappa_\mu(y)
\]
\[= \sum_{l=0}^{k-1} \sum_{\nu=2^{2m_l}} \sum_{\mu=2^{2m_l}} \tilde{f}^\kappa(\nu, \mu) \kappa_\nu(x) \kappa_\mu(y)
\]
\[+ \sum_{\nu=2^{2m_k}} \sum_{\mu=2^{2m_k}} \tilde{f}^\kappa(\nu, \mu) \kappa_\nu(x) \kappa_\mu(y).
\]
By (13), (14) and IV and 

\[
\begin{align*}
\text{By the help of the equation (10) we immediately have for }\quad (21) \quad & \quad \sum_{l=0}^{k-1} \frac{q^{4m_l}}{m_l} (D_{2^{m_l+1}}(x) - D_{2^{m_l}}(x)) \times (D_{2^{m_l+1}}(y) - D_{2^{m_l}}(y)) \\
& \quad + \frac{q^{4m_k}}{m_k} (D_{r_{2m_k}}(x) - D_{2^{m_k}}(x)) (D_{r_{2m_k}}(y) - D_{2^{m_k}}(y))
\end{align*}
\]

and

\[
IV = \frac{1}{q_{m_k}^2} (q_{m_k} - 2^{2m_k})^2
\]

\[
\times \sum_{l=0}^{k-1} \frac{q^{4m_l}}{m_l} (D_{2^{m_l+1}}(x) - D_{2^{m_l}}(x)) (D_{2^{m_l+1}}(y) - D_{2^{m_l}}(y))
\]

\[
+ \frac{1}{q_{m_k}^2} \frac{q^{4m_k}}{m_k} \left( \sum_{i=2^{m_k}}^{2^{m_k-1}} \sum_{j=2^{m_k}}^{2^{m_k-1}} (D_{r_{2m_k}}(x) - D_{2^{m_k}}(x)) (D_{r_{2m_k}}(y) - D_{2^{m_k}}(y)) \right)
\]

\[
= IV_1 + IV_2.
\]

By (13), (14) and \(|D_{2^n}(x)| \leq 2^n\) we get that

\[
|IV_1| \leq C \sum_{l=0}^{k-1} \frac{q^{8m_l}}{m_l} \leq C \frac{q^{2m_k}}{km_k}
\]

and

\[
\left| \sigma_{q_{m_k}}^{r_{2m_k}} f(x, y) \right| \geq \left| IV_2 \right| - \frac{C 2^{m_k}}{km_k}.
\]

By the help of the equation (10) we immediately have for \(IV_2\)

\[
IV_2 = \frac{1}{q_{m_k}^2} \frac{q^{4m_k}}{m_k} r_{2m_k}(x) r_{2m_k}(y) \sum_{i=0}^{q_{m_k}-1} D_i^w (\tau_{2m_k}(x)) \sum_{j=0}^{q_{m_k}-1} D_j^w (\tau_{2m_k}(y))
\]

\[
= \frac{1}{q_{m_k}^2} \frac{q^{4m_k}}{m_k} r_{2m_k}(x) r_{2m_k}(y) q_{m_k}^2 K_{q_{m_k}}^{r_{2m_k}} (\tau_{2m_k}(x)) K_{q_{m_k}-1}^{r_{2m_k}} (\tau_{2m_k}(y)).
\]

Let \((x, y) \in J_{l_1, l_2+2}^{2m_k} \times J_{l_2, l_2+2}^{2m_k}(y)\), where \((l_1, l_2) \in \{0, 1, ..., m_k - 3\} \times \{0, 1, ..., m_k - 3\}\.

Then from Lemma 1 we can write

\[
q_{m_k-1} \left| K_{q_{m_k}-1}^{r_{2m_k}} (\tau_{2m_k}(x)) \right| \geq C 2^{4l_1} \quad \text{and} \quad q_{m_k-1} \left| K_{q_{m_k}-1}^{r_{2m_k}} (\tau_{2m_k}(y)) \right| \geq C 2^{4l_2}.
\]
Consequently, 

\( (22) \quad |\sigma_{q_{m_k},q_{m_k}} f(x,y)| \geq \frac{C}{m_k} 2^{4l_1+4l_2} - \frac{2^{m_k}}{km_k}. \)

Denote

\[ A(m_k) := \{ (l_1, l_2) : 0 \leq l_2 \leq m_k - 3, 0 \leq l_1 \leq \frac{m_k}{4}, l_1 + l_2 \geq \frac{m_k}{4} \} \]

and

\[ \lambda'_k := \frac{c^{2m_k}}{m_k}. \]

For \((l_1, l_2) \in A(m_k)\), we have

\[ |\sigma_{q_{m_k},q_{m_k}} f(x,y)| \geq \frac{c^{2m_k}}{m_k} \]

and

\[ \mu \left\{ (x,y) \in G \times G : \sigma^{\kappa,\square} f(x,y) \geq C \lambda'_k \right\} \geq \]

\[ \geq \sum_{(l_1, l_2) \in A(m_k)} \mu \left\{ (x,y) \in J_{2m_k}^{l_1,l_1+2}(x) \times J_{2m_k}^{l_2,l_2+2}(y) : |\sigma_{q_{m_k},q_{m_k}} f(x,y)| \geq \lambda'_k \right\} \]

\[ \geq C \sum_{l_1=0}^{[m_k/4]} \sum_{l_2=[m_k/4]-l_1}^{m_k-3} \sum_1^{m_k-3} \sum_0^{l_1} \prod \mu \left\{ J_{2m_k}^{l_1,l_1+2}(x) \times J_{2m_k}^{l_2,l_2+2}(y) \right\} \]

\[ \geq C \sum_{l_1=0}^{[m_k/4]} \sum_{l_2=[m_k/4]-l_1}^{m_k-3} \sum_0^{l_2} 2^{2l_1+2l_2} \geq \frac{C m_k}{2m_k/2}. \]

Consequently,

\[ \lambda'_k \left( \mu \left\{ (x,y) : \sigma^{\kappa,\square} f(x,y) \geq C \lambda'_k \right\} \right)^2 \geq \frac{c^{2m_k}}{m_k} \frac{m_k^2}{2m_k} = C m_k \rightarrow \infty \text{ as } k \rightarrow \infty. \]

This completes the proof of this theorem. \(\square\)

4. The two-dimensional unrestricted maximal operator

If \( f \in L_1(G \times G) \) then it is easy to show that the sequence \((S_{2^n,2^m} f) : n, m \in \mathbb{N})\) is a martingale. If \( f \) is a martingale, that is
If \( f = (f^{(n,m)} : n, m \in \mathbb{N}) \) then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

\[
\hat{f}(i,j) = \lim_{\min(k,l) \to \infty} \int_{G \times G} f^{(k,l)}(x,y) \alpha_i(x) \alpha_j(y) \, d\mu(x,y).
\]

The Walsh-(Kaczmarz)-Fourier coefficients of \( f \in L_1(G \times G) \) are the same as the ones of the martingale \((S_{2^n,2^m}(f) : n, m \in \mathbb{N})\) obtained from \( f \).

For the martingale \( f \) the unrestricted maximal operator of the Fejér mean is defined by

\[
\sigma^{\alpha,*} f(x,y) = \sup_{n,m \in \mathbb{N}} |\sigma_{n,m}^\alpha(f;x,y)|.
\]

A function \( a \in L_2 \) is called a rectangle \( p \)-atom if there exists a dyadic rectangle \( R \) such that

(a) \( \text{supp} \, a \subset R \),
(b) \( \|a\|_2 \leq |R|^{1/2 - 1/p} \),
(c) \( \int_G a(x,y) \, d\mu(x) = \int_G a(x,y) \, d\mu(y) = 0 \) for all \( x,y \in G \).

The basic result of atomic decomposition is the following one.

**Theorem C.** (Weisz [20]). A martingale \( f = (f^{(n,m)} : n, m \in \mathbb{N}) \) is in \( H_p(0 < p \leq 1) \) if there exists a sequence \( (a_k, k \in \mathbb{N}) \) of rectangle \( p \)-atoms and a sequence \( (\mu_k, k \in \mathbb{N}) \) of real numbers such that for every \( n, m \in \mathbb{N} \),

\[
\sum_{k=0}^{\infty} \mu_k S_{2^n,2^m} a_k = f^{(n,m)},
\]

\[
\sum_{k=0}^{\infty} |\mu_k|^p < \infty.
\]

Moreover,

\[
\|f\|_{H_p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}.
\]

We will prove the following theorem.

**Theorem 3.** There exists a martingale \( f \in H_{1/2}(G \times G) \) such that

\[
\|\sigma^{\alpha,*} f\|_{\text{weak-L}_{1/2}} = +\infty.
\]
Proof. Now, we use the sequence \( \{ \mu_k : k \in \mathbb{N} \} \) and the atoms \( a_k \) defined in the previous proof. Let

\[
 f^{(A,B)}(x,y) := \sum_{l:2m_l < \min(A,B)} \lambda_l a_l(x,y).
\]

First, we prove that the martingale \( f := (f^{(A,B)} : A, B \in \mathbb{N}) \) belongs to the Hardy space \( H_{1/2}(G \times G) \). Indeed, since

\[
 \|a_l\|_2 \leq c2^{6m_l},
\]

\[
 S_{2^A, 2^B} a_k(x,y) = \begin{cases} 0, & \text{if } \min(A,B) \leq 2m_k, \\ a_k(x,y), & \text{if } \min(A,B) > 2m_k, \end{cases}
\]

\[
 f^{(A,B)}(x,y) := \sum_{l:2m_l < \min(A,B)} \lambda_l a_l(x,y) = \sum_{k=0}^\infty \lambda_k S_{2^A, 2^B} a_k(x,y)
\]

from (12) and Theorem C we conclude that \( f \in H_{1/2}(G \times G) \).

Now, we investigate the Fourier coefficients. Since

\[
 \int_{G \times G} f^{(A,B)}(x,y) \kappa_i(x) \kappa_j(y) d\mu(x,y)
\]

\[
 = \begin{cases} 0, & (i,j) \notin \bigcup_{k=0}^\infty \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\}, \\ 0, & (i,j) \in \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\}, \\ \frac{2^{4m_k}}{mk}, & (i,j) \in \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\}, \\ & \min(A,B) > 2m_k, \end{cases}
\]

we can write

\[
 \hat{f}^\kappa(i,j) = \begin{cases} \frac{2^{4m_k}}{mk}, & (i,j) \in \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\}, \\ 0, & (i,j) \notin \bigcup_{k=1}^\infty \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\} \times \{2^{2m_k}, \ldots, 2^{2m_k+1} - 1\}. \end{cases}
\]

The estimation of

\[
 \mu \left\{ (x,y) : \sigma^{\kappa,a} f(x,y) \geq C\lambda_k \right\}
\]

is analogous to the estimation of

\[
 \mu \left\{ (x,y) : \sigma^{\kappa,\square} f(x,y) \geq C\lambda_k \right\}
\]
and we have that
\[ \sup_{\lambda > 0} \lambda \mu \left\{ (x, y) : \sigma^{x,y} f(x, y) \geq \lambda \right\}^2 = \infty. \]

Theorem 3 is proved. \[ \square \]

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