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# Maximal operators of Fejér means of Walsh-Kaczmarz-Fourier series

**Ushangi Goginava and Károly Nagy**

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**Abstract.** The main aim of this paper is to prove that there exists a martingale $f \in H_{1/2}$ such that the maximal Fejér operator with respect to Walsh-Kaczmarz system does not belong to the space $L_{1/2}$. For the two-dimensional case, we prove that there exists a martingale $f \in H_{1/2}^{\square}(f \in H_{1/2})$ such that the restricted (unrestricted) maximal operator of Fejér means of two-dimensional Walsh-Kaczmarz-Fourier series does not belong to the space weak-$L_{1/2}$.

## 1. Introduction

The first result with respect to the a.e. convergence of the Walsh-Fejér means $\sigma_n f$ is due to Fine [1]. Later, Schipp [9] showed that the maximal operator $\sigma^* f := \sup_n |\sigma_n f|$ is of weak type $(1,1)$, from which the a. e. convergence follows by standard argument. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L_p \to L_p$ $(1 < p \leq \infty)$. This fails to hold for $p = 1$, but Fujii [2] proved that $\sigma^*$ is bounded from the dyadic Hardy space $H_1$ to the space $L_1$. Fujii's theorem was extended by Weisz [19]. Namely, he proved that the maximal operator of the Fejér means of the one-dimensional Walsh-Fourier series is bounded from the martingale Hardy space $H_p(G)$ to the space $L_p(G)$ for $p > 1/2$. Simon [13] gave a counterexample, which shows that this boundedness does not hold for $0 < p < 1/2$. In the endpoint case $p = 1/2$ Weisz [21] proved that $\sigma^*$ is

bounded from the Hardy space $H_{1/2}(G)$ to the space weak-$L_{1/2}(G)$ (see also [14]). In [5] the first author proved that the maximal operator $\sigma^*$ is not bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$.

In 1948 Šneider [16] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty} \frac{D_n^{\kappa}(x)}{\log n} \geq C > 0$$

holds a.e. In 1974 Schipp [10] and Young [17] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [15] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to $f$ for any continuous functions $f$. Gát [3] proved, for any integrable functions, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function and Gát proved that $\|\sigma^{\kappa *}\|_1 \leq C\||f|\|_{H_1}$. The result of Gát was extended to the Hardy space by Simon [11], who proved that $\sigma^{\kappa *}$ is of type $(H_p, L_p)$ for $p > 1/2$. Weisz [21] showed that in endpoint case $p = 1/2$ the maximal operator is of weak type $\left(H_{1/2}, L_{1/2}\right)$.

In this paper we will prove a stronger result than the unboundedness of the maximal operator from the Hardy space $H_{1/2}$ to the space $L_{1/2}$, in particular, we prove that there exists a martingale $f \in H_{1/2}$ such that

$$\|\sigma^{\kappa,*}\|_{1/2} = +\infty.$$

For the two-dimensional Walsh-Kaczmarz-Fourier series Simon proved [12] that the restricted maximal operator $\sigma_{\lambda}^{\kappa *}$ is bounded from the Hardy space $H_p$ to the space $L_p$ for all $p > 1/2$.

In the paper [7] it was proved that the assumption $p > 1/2$ is essential. Namely, the maximal operator $\sigma^{\kappa *} := \sup_n |\sigma_{n,n}^{\kappa}|$ of the Fejér means of double Fourier series with respect to the Walsh-Kaczmarz system is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. In this paper we will prove a stronger result than in the paper [7], in particular, we prove that there exists a martingale $f \in H_{1/2}^{\square}(f \in H_{1/2})$ such that

$$\|\sigma^{\kappa,\square}\|_{weak-L_{1/2}} = +\infty \quad (\|\sigma^{\kappa,*}\|_{weak-L_{1/2}} = +\infty).$$

Thus, as regards boundedness of $\sigma^{\kappa,*}$ the case of two-dimensional Walsh-Kaczmarz series differs from the case of one-dimensional Walsh-Kaczmarz series.

Let denote by $\mathbf{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on $\mathbf{Z}_2$ is given in the way that the measure of a singleton is 1/2.

Let $G := \underset{k=0}{\overset{\infty}{\times}} \mathbf{Z}_2$, $G$ be called the Walsh group. The elements of $G$ are sequences $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0,1\}$ $(k \in \mathbf{N})$.

The group operation on $G$ is the coordinate-wise addition (denoted by $+$), the normalized Haar measure (denoted by $\mu$) and the topology are the product measure and topology. Dyadic intervalls are defined by

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1} ...)\}$$

for $x \in G, n \in \mathbf{P}$. They form a base for the neighborhoods of $G$. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of $G$ and $I_n := I_n(0)$ for $n \in \mathbf{N}$.

Let $L_p$ denote the usual Lebesgue spaces on $G$ (with the corresponding norm or quasinorm $\|.\|_p$). The space weak-$L_p$ consists of all measurable functions $f$ for which

$$\|f\|_{\text{weak}-L_p} := \sup_{\lambda>0} \lambda \mu \left(|f| > \lambda\right)^{1/p} < +\infty.$$

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Let the Walsh-Paley functions be the product functions of the Rademacher functions. Namely, each natural number $n$ can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0,1\} \ (i \in \mathbf{N}),$$

where only a finite number of $n_i$'s different from zero. Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. Walsh-Paley functions are $w_0 = 1$ and for $n \geq 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}.$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions is the same in dyadic blocks. Namely,

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{w_n : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbf{P}$ and $\kappa_0 = w_0$.

V. A. Skvortsov (see [15]) gave a relation between the Walsh-Kaczmarz functions and the Walsh-Paley functions by the help of the transformation $\tau_A : G \to G$ defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_1, x_0, x_A, x_{A+1}, ...)$$

for $A \in \mathbf{N}$. By the definition of $\tau_A$, we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

The Dirichlet kernels are defined by

$$D_n^\alpha := \sum_{k=0}^{n-1} \alpha_k,$$

where $\alpha_n = w_n$ or $\kappa_n$ $(n \in \mathbf{P})$, $D_0^\alpha := 0$. The $2^n$th Dirichlet kernels have a closed form (see e.g. [8])

$$D_{2^n}^w(x) = D_{2^n}^\kappa(x) = D_{2^n}(x) = \begin{cases} 0, & \text{if } x \notin I_n \\ 2^n, & \text{if } x \in I_n. \end{cases}$$

The $\sigma$-algebra generated by the dyadic intervals of measure $2^{-k}$ will be denoted by $F_k\,(k \in \mathbf{N})$.

Denote by $f = \left(f^{(n)}, n \in \mathbf{N}\right)$ a martingale with respect to $(F_n, n \in \mathbf{N})$ (for details see, e. g. [20]). The maximal function of a martingale $f$ is defined by

$$f^* = \sup_{n \in \mathbf{N}} \left| f^{(n)} \right|.$$

In case $f \in L_1(G)$, the maximal function can also be given by

$$f^*(x) = \sup_{n \in \mathbf{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|, \quad x \in G.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G)$, then it is easy to show that the sequence $(S_{2^n} f : n \in \mathbf{N})$ is a martingale. If $f$ is a martingale, that is $f = (f^{(0)}, f^{(1)}, ...)$ then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i) = \lim_{k\to\infty} \int_G f^{(k)}(x)\, \alpha_i(x)\, d\mu(x) \quad (\alpha = w \text{ or } \kappa).$$

The Walsh-(Kaczmarz)-Fourier coefficients of $f \in L_1(G)$ are the same as the ones of the martingale $(S_{2^n} f : n \in \mathbf{N})$ obtained from $f$.

The two-dimensional dyadic cubes are of the form

$$I_{n,n}(x,y) := I_n(x) \times I_n(y).$$

By $F_{n,n}$, we denote the $\sigma$-algebra generated by the dyadic rectangles $\{I_{n,n}(x,y) : (x,y) \in G \times G\}$.

Denote by $f = \left(f^{(n,n)}, n \in N\right)$ a martingale with respect to $(F_{n,n}, n \in \mathbf{N})$ (for details see, e. g. [20]). The maximal function of a martingale $f$ is defined by

$$f^{\square} = \sup_{n\in N} \left| f^{(n,n)} \right|.$$

In case $f \in L_1(G \times G)$ , the maximal function can also be given by

$$f^{\square}(x,y) = \sup_{n\in\mathbf{N}} \frac{1}{\mu(I_{n,n}(x,y))} \left| \int_{I_{n,n}(x,y)} f(u,v)\, d\mu(u,v) \right|,$$

$$(x,y) \in G \times G,$$

For $0 < p < \infty$ the Hardy martingale space $H_p^{\square}(G \times G)$ consists of all martingales for which

$$\|f\|_{H_p} := \left\| f^{\square} \right\|_p < \infty.$$

Let

$$I_{n,m}(x,y) := I_n(x) \times I_m(y).$$

We denote by $F_{n,m}\,(n,m \in \mathbf{N})$, the $\sigma$-algebra generated by the dyadic rectangles $\{I_{n,m}(x,y) : (x,y) \in G \times G\}$.

Denote by $f = \left(f^{(n,m)}, n, m \in \mathbf{N}\right)$ a martingale with respect to $(F_{n,m}, n, m \in \mathbf{N})$ (for details see, e. g. [20]).

The maximal function of a martingale $f$ is defined by

$$f^* = \sup_{n,m\in\mathbf{N}} \left| f^{(n,m)} \right|.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G \times G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty$$

.

In case $f \in L_1(G \times G)$, maximal functions can also be given by

$$f^*(x,y) = \sup_{n,m\in\mathbf{N}} \frac{1}{\mu(I_{n,m}(x,y))} \left| \int\limits_{I_{n,m}(x,y)} f(u,v)\, d\mu(u,v) \right|$$

## 2. The one-dimensional maximal operator

For $n = 1, 2, ...$ and a martingale $f$ the Fejér means of the Walsh-(Kaczmarz)-Fourier series of the function $f$ is given by

$$\sigma_n^\alpha f(x) = \frac{1}{n} \sum_{j=0}^{n-1} S_j^\alpha(f;x) \quad (\alpha = w \text{ or } \kappa).$$

For a martingale $f$ we consider the maximal operator

$$\sigma^{\alpha,*} f = \sup_{n\in\mathbf{P}} |\sigma_n^\alpha f(x)| \quad (\alpha = w \text{ or } \kappa).$$

The $n$th Fejér kernel of the Walsh-(Kaczmarz)-Fourier series defined by

$$K_n^\alpha(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k^\alpha(x) \quad (\alpha = w \text{ or } \kappa).$$

A bounded measurable function $a$ is a $p$-atom, if there exists a dyadic interval $I$, such that

a) $\int\limits_I a d\mu = 0$;
b) $\|a\|_\infty \le \mu(I)^{-1/p}$;
c) $\operatorname{supp} a \subset I$.

The basic result of atomic decomposition is the following one.

**Theorem A.** *(Weisz* [20]*). A martingale* $f = \left(f^{(n)} : n \in \mathbf{N}\right)$ *is in* $H_p\,(0 < p \le 1)$ *if and only if there exists a sequence* $(a_k, k \in \mathbf{N})$ *of* $p$*-atoms*

*and a sequence* $(\mu_k, k \in \mathbf{N})$ *of real numbers such that for every* $n \in \mathbf{N}$,

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = f^{(n)}, \tag{1}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

*Moreover,*

$$\|f\|_{H_p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

*where the infimum is taken over all decompositions of* $f$ *of the form (1).*

We will use the following lemma of Goginava:

**Lemma 1.** *(Goginava* [6]*) Let* $2 < A \in \mathbf{P}$ *and* $q_A := 2^{2A} + 2^{2A-2} + \cdots + 2^2 + 2^0$. *Then*

$$q_{A-1} \left| K^w_{q_{A-1}}(x) \right| \geq 2^{2m+2s-3}$$

*for* $x \in I_{2A}(0, ..., 0, x_{2m} = 1, 0, ..., 0, x_{2s} = 1, x_{2s+1}, ..., x_{2A-1})$, $m = 0, 1, ..., A-3$, $s = m+2, m+3, ..., A-1$.

We will prove the following theorem.

**Theorem 1.** *There exists a martingale* $f \in H_{1/2}(G)$ *such that*

$$\|\sigma^{\kappa,*} f\|_{1/2} = +\infty.$$

*Proof.* Since $\frac{2^{m_k}}{m_k} \uparrow \infty$ as $k \to \infty$ it is easy to show that there exists an increasing sequence of positive integers $(m_k : k \in \mathbf{N})$ such that

$$\sum_{k=0}^{\infty} \frac{1}{m_k^{1/2}} < \infty, \tag{2}$$

$$\sum_{l=0}^{k-1} \frac{2^{4m_l}}{m_l} < \frac{2^{4m_k}}{m_k}, \tag{3}$$

$$\frac{k 2^{4m_{k-1}}}{m_{k-1}} \leq \frac{2^{2m_k}}{m_k}. \tag{4}$$

Let

$$f^{(A)}(x) := \sum_{k, 2m_k < A} \lambda_k a_k, \text{ where } \lambda_k := \frac{1}{m_k}$$

and

$$a_k(x) := 2^{2m_k}\left(D_{2^{2m_k+1}}(x) - D_{2^{2m_k}}(x)\right).$$

The martingale $f := \left(f^{(0)}, f^{(1)}, ..., f^{(A)}, ...\right)$ is in $H_{1/2}(G)$. Indeed, since

$$S_{2^A} a_k(x) = \begin{cases} 0, & \text{if } A \le 2m_k, \\ a_k(x), & \text{if } A > 2m_k, \end{cases}$$

and

$$f^{(A)}(x) = \sum_{k:2m_k<A} \lambda_k a_k(x) = \sum_{k=0}^{\infty} \lambda_k S_{2^A} a_k(x)$$

by (2) and Theorem A we conclude that $f \in H_{1/2}(G)$.

Now, we investigate the Fourier coefficients.

Let $j \in \{2^{2m_k}, ..., 2^{2m_k+1}-1\}$ for some $k = 0, 1, 2, ....$ Then it is evident that

$$\widehat{f}^{\kappa}(j) := \lim_{A\to\infty} \widehat{f^{(A)}}^{\kappa}(j) = \frac{2^{2m_k}}{m_k}$$

and $\widehat{f}^{\kappa}(j) = 0$, if $j \notin \{2^{2m_k}, ..., 2^{2m_k+1}-1\}$, $k = 0, 1, 2, ....$

Now, we decompose the $q_{m_k}$th Walsh-Kaczmarz-Fejér means as follows. (For the definition of $q_{m_k}$ see Lemma 1 of Goginava.)

$$\sigma^{\kappa}_{q_{m_k}} f(x) = \frac{1}{q_{m_k}} \sum_{j=0}^{2^{2m_k}-1} S^{\kappa}_j f(x) + \frac{1}{q_{m_k}} \sum_{j=2^{2m_k}}^{q_{m_k}-1} S^{\kappa}_j f(x) = I + II. \tag{5}$$

Let $j < 2^{2m_k}$. Then (3) gives that

$$\left|S^{\kappa}_j f(x)\right| \le \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} |\widehat{f}^{\kappa}(v)| \le \sum_{l=0}^{k-1} \frac{2^{4m_l}}{m_l} < 2\frac{2^{4m_{k-1}}}{m_{k-1}}$$

and

$$I \le c\frac{1}{q_{m_k}} \sum_{j=0}^{2^{2m_k}-1} \left|S^{\kappa}_j f(x)\right| \le c\frac{2^{4m_{k-1}}}{m_{k-1}}. \tag{6}$$

Now, we discuss $II$.

For $2^{2m_k} \leq j < q_{m_k}$ we have the following:

$$\begin{aligned}
S_j^{\kappa} f(x) &= \sum_{v=0}^{2^{2m_{k-1}+1}-1} \widehat{f}^{\kappa}(v)\,\kappa_v(x) + \sum_{v=2^{2m_k}}^{j-1} \widehat{f}^{\kappa}(v)\,\kappa_v(x) \\
&= \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \widehat{f}^{\kappa}(v)\,\kappa_v(x) + \sum_{v=2^{2m_k}}^{j-1} \widehat{f}^{\kappa}(v)\,\kappa_v(x) \\
&= \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \frac{2^{2m_l}}{m_l}\kappa_v(x) + \frac{2^{2m_k}}{m_k} \sum_{v=2^{2m_k}}^{j-1} \kappa_v(x) \\
&= \sum_{l=0}^{k-1} \frac{2^{2m_l}}{m_l}\left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right) + \frac{2^{2m_k}}{m_k}\left(D_j^{\kappa}(x) - D_{2^{2m_k}}(x)\right). \qquad (7)
\end{aligned}$$

This gives that

$$\begin{aligned}
II &= \frac{\left(q_{m_k} - 2^{2m_k}\right)}{q_{m_k}} \sum_{l=0}^{k-1} \frac{2^{2m_l}}{m_l}\left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right) \\
&\quad + \frac{2^{2m_k}}{q_{m_k} m_k} \sum_{j=2^{2m_k}}^{q_{m_k}-1} \left(D_j^{\kappa}(x) - D_{2^{2m_k}}(x)\right) \\
&=: II_1 + II_2.
\end{aligned}$$

To discuss $II_1$, we use (3) and $|D_{2^n}(x)| \leq 2^n$. Thus, we can write

$$|II_1| \leq c \sum_{l=0}^{k-1} \frac{2^{4m_l}}{m_l} < c\frac{2^{4m_{k-1}}}{m_{k-1}}. \qquad (8)$$

From $\sigma_{q_{m_k}}^{\kappa} f(x) = I + II_1 + II_2$. and (6), (8) we have

$$|\sigma_{q_{m_k}}^{\kappa} f(x)| \geq |II_2| - |I| - |II_1| \geq |II_2| - c\frac{2^{4m_{k-1}}}{m_{k-1}}. \qquad (9)$$

Now, we discuss $II_2$.

We can write the $n$th Dirichlet kernel with respect to the Walsh-Kaczmarz system in the following form:

$$\begin{aligned}
D_n^{\kappa}(x) &= D_{2^{|n|}}(x) + \sum_{k=2^{|n|}}^{n-1} r_{|k|}(x) w_{k-2^{|n|}}(\tau_{|k|}(x)) \\
&= D_{2^{|n|}}(x) + r_{|n|}(x) D_{n-2^{|n|}}^{w}(\tau_{|n|}(x)). \qquad (10)
\end{aligned}$$

By the help of this, we immediately get

$$\begin{aligned}
|II_2| &= \frac{2^{2m_k}}{q_{m_k}m_k}\left|\sum_{j=0}^{q_{m_k-1}-1}\left(D^{\kappa}_{j+2^{2m_k}}(x)-D_{2^{2m_k}}(x)\right)\right| \\
&= \frac{2^{2m_k}}{q_{m_k}m_k}\left|r_{2m_k}(x)\sum_{j=0}^{q_{m_k-1}-1}D^{w}_{j}(\tau_{2m_k}(x))\right| \\
&= \frac{2^{2m_k}}{m_k}\frac{q_{m_k-1}}{q_{m_k}}\left|K^{w}_{q_{m_k-1}}(\tau_{2m_k}(x))\right| \\
&\geq c\frac{q_{m_k-1}}{m_k}\left|K^{w}_{q_{m_k-1}}(\tau_{2m_k}(x))\right|.
\end{aligned}$$

Thus, from (9) we have

$$|\sigma^{\kappa}_{q_{m_k}}f(x)| \geq c\frac{q_{m_k-1}}{m_k}\left|K^{w}_{q_{m_k-1}}(\tau_{2m_k}(x))\right| - c\frac{2^{4m_{k-1}}}{m_{k-1}}.$$

Define the set $J^{l,s}_{2A}(x)$ for $l<s<A$ by

$$J^{l,s}_{2A}(x) := I_{2A}(x_0,x_1,...,x_{2A-2s-2},x_{2A-2s-1}=1,0,...,0,x_{2A-2l-1}=1,0,...,0).$$

Let $x\in J^{l,s}_{2m_k}(x)$, for some $l=[m_k/2],[m_k/2]+1,...,m_k-3$, and $s=l+2,l+3,...,m_k-1$, then from Lemma 1 and (4) we have

$$\left|\sigma^{\kappa}_{q_{m_k}}f(x)\right| \geq \frac{c2^{2l+2s}}{m_k} - \frac{c2^{4m_{k-1}}}{m_{k-1}} \geq \frac{c2^{2l+2s-1}}{m_k}.$$

Hence,

$$\begin{aligned}
&\int_G |\sigma^{\kappa,*}f(x)|^{1/2}d\mu(x) \\
&\quad\geq \int_G |\sigma^{\kappa}_{q_{m_k}}f(x)|^{1/2}d\mu(x) \\
&\quad\geq \sum_{l=[m_k/2]}^{m_k-3}\sum_{s=l+2}^{m_k-1}\sum_{\substack{x_i=0\\ i\in\{0,1,...,2m_k-2s-2\}}}^{1}\int_{J^{l,s}_{2m_k}(x)}|\sigma^{\kappa}_{q_{m_k}}f(x)|^{1/2}d\mu(x) \\
&\quad\geq \frac{c}{m_k^{1/2}}\sum_{l=[m_k/2]}^{m_k-3}\sum_{s=l+2}^{m_k-1}2^{2m_k-2s}\frac{1}{2^{2m_k}}2^{l+s} \\
&\quad\geq \frac{c}{m_k^{1/2}}\sum_{l=[m_k/2]}^{m_k-3}\sum_{s=l+2}^{m_k-1}\frac{2^l}{2^s} \geq cm_k^{1/2}\to\infty \text{ as } k\to\infty.
\end{aligned}$$

That is $\|\sigma^{\kappa,*}f\|_{1/2}=+\infty$. The proof is complete. $\square$

## 3. The two-dimensional restricted maximal operator

For $\alpha=w$ or $\kappa$ the rectangular partial sums of the double Walsh-(Kaczmarz)-Fourier series are defined as follows:

$$S_{M,N}^{\alpha}f(x,y):=\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\widehat{f}(i,j)\,\alpha_i(x)\,\alpha_j(y),$$

where the number

$$\widehat{f}(i,j)=\int_{G\times G}f(x,y)\,\alpha_i(x)\,\alpha_j(y)\,d\mu(x,y).$$

is said to be the $(i,j)$th Walsh-(Kaczmarz)-Fourier coefficient of the function $f$. If $f\in L_1(G\times G)$ then it is easy to show that the sequence $(S_{2^n,2^n}(f):n\in\mathbf{N})$ is a martingale. If $f$ is a martingale, that is $f=(f^{(n,n)}:n\in\mathbf{N})$ then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}^{\alpha}(i,j)=\lim_{k\to\infty}\int_{G\times G}f^{(k)}(x,y)\,\alpha_i(x)\,\alpha_j(y)\,d\mu(x,y).$$

The Walsh-(Kaczmarz)-Fourier coefficients of $f\in L_1(G\times G)$ are the same as the ones of the martingale $(S_{2^n,2^n}(f):n\in\mathbf{N})$ obtained from $f$.

For $n,m\in\mathbf{P}$ and a martingale $f$ the $(n,m)$th Fejér mean of the double Walsh-(Kaczmarz)-Fourier series is given by

$$\sigma_{n,m}^{\alpha}f(x,y)=\frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}S_{i,j}^{\alpha}f(x,y).$$

For the martingale $f$ the restricted maximal operator is defined by

$$\sigma_{\lambda}^{\alpha,\square}f(x,y)=\sup_{2^{-\lambda}\le n/m\le 2^{\lambda}}|\sigma_{n,m}^{\alpha}f(x,y)|.$$

A bounded measurable function $a$ is a $p$-atom, if there exists a dyadic 2-dimensional cube $I\times I$, such that

a) $\int_{I\times I}a\,d\mu=0$;
b) $\|a\|_{\infty}\le\mu(I\times I)^{-1/p}$;
c) $\operatorname{supp}a\subset I\times I$.

The basic result of atomic decomposition is the following one.

**Theorem B.** *(Weisz [20]). A martingale* $f = \left(f^{(n,n)} : n \in \mathbf{N}\right)$ *is in* $H_p^{\square}\,(0 < p \leq 1)$ *if and only if there exists a sequence* $(a_k, k \in \mathbf{N})$ *of* $p$*-atoms and a sequence* $(\mu_k, k \in \mathbf{N})$ *of real numbers such that for every* $n \in \mathbf{N}$ *,*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n,2^n} a_k = f^{(n,n)}, \tag{11}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

*Moreover,*

$$\|f\|_{H_p^{\square}} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

We will prove the following theorem.

**Theorem 2.** *There exists a martingale* $f \in H_{1/2}^{\square}\,(G \times G)$ *such that*

$$\left\| \sigma^{\kappa,\square} f \right\|_{weak-L_{1/2}} = +\infty.$$

*Proof.* To prove Theorem 2 we modify the sequence $\{m_k : k \in \mathbf{P}\}$ and atoms $a_k$ given in the previous section in the following way.

Let $\{m_k : k \in \mathbf{N}\}$ be an increasing sequence of positive integers such that

$$\sum_{k=0}^{\infty} \frac{1}{m_k^{1/2}} < \infty, \tag{12}$$

$$\sum_{l=0}^{k-1} \frac{2^{8m_l}}{m_l} < \frac{2^{8m_k}}{m_k}, \tag{13}$$

$$\frac{2^{8m_{k-1}}}{m_{k-1}} < \frac{2^{m_k}}{km_k}. \tag{14}$$

Let

$$f^{(A,A)}\,(x,y) := \sum_{k,2m_k<A} \lambda_k a_k\,(x,y)\,, \text{ where } \lambda_k := \frac{1}{m_k}$$

and

$$a_k\,(x,y) := 2^{4m_k}\left(D_{2^{2m_k+1}}(x) - D_{2^{2m_k}}(x)\right)\left(D_{2^{2m_k+1}}(y) - D_{2^{2m_k}}(y)\right).$$

The martingale $f := \left(f^{(0,0)}, f^{(1,1)}, ..., f^{(A,A)}, ...\right) \in H^{\square}_{1/2}(G \times G)$. Indeed,

$$S_{2^A,2^A} a_k(x,y) = \begin{cases} 0, & \text{if } A \leq 2m_k, \\ a_k(x,y), & \text{if } A > 2m_k, \end{cases}$$

$$f^{(A,A)}(x) = \sum_{k, 2m_k < A} \lambda_k a_k(x,y) = \sum_{k=0}^{\infty} \lambda_k S_{2^A,2^A} a_k(x,y)$$

from (12) and Theorem B we conclude that $f \in H^{\square}_{1/2}(G \times G)$.

Now, we investigate the Fourier coefficients. Since

$$\int\limits_{G \times G} f^{(A)}(x,y) \kappa_i(x) \kappa_j(y) \, d\mu(x,y)$$

$$= \begin{cases} 0, & (i,j) \notin \bigcup\limits_{k=0}^{\infty} \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, \\ 0, & (i,j) \in \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, \\ & \qquad A = 0, 1, ..., 2m_k, \\ \frac{2^{4m_k}}{m_k}, & (i,j) \in \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, \\ & \qquad A > 2m_k, \end{cases}$$

we can write

(15)

$$\widehat{f}^{\kappa}(i,j) = \begin{cases} \frac{2^{4m_k}}{m_k}, & (i,j) \in \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, \\ 0, & (i,j) \notin \bigcup\limits_{k=1}^{\infty} \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}. \end{cases}$$

We decompose the $(q_{m_k}, q_{m_k})$th Fejér means as follows

$$\begin{aligned} & \sigma^{\kappa}_{q_{m_k} q_{m_k}} f(x,y) \\ (16) \quad = \quad & \frac{1}{q^2_{m_k}} \sum_{i=0}^{q_{m_k}-1} \sum_{j=0}^{q_{m_k}-1} S^{\kappa}_{i,j} f(x,y) \\ = \quad & \frac{1}{q^2_{m_k}} \sum_{i=0}^{2^{2m_k}-1} \sum_{j=0}^{2^{2m_k}-1} S^{\kappa}_{i,j} f(x,y) + \frac{1}{q^2_{m_k}} \sum_{i=2^{2m_k}}^{q_{m_k}-1} \sum_{j=0}^{2^{2m_k}-1} S^{\kappa}_{i,j} f(x,y) \\ & + \frac{1}{q^2_{m_k}} \sum_{i=0}^{2^{2m_k}-1} \sum_{j=2^{2m_k}}^{q_{m_k}-1} S^{\kappa}_{i,j} f(x,y) + \frac{1}{q^2_{m_k}} \sum_{i=2^{2m_k}}^{q_{m_k}-1} \sum_{j=2^{2m_k}}^{q_{m_k}-1} S^{\kappa}_{i,j} f(x,y) \\ = \quad & I + II + III + IV. \end{aligned}$$

Let

$$\begin{aligned}(i,j) \quad \in \quad & \left(\left\{2^{2m_k},...,q_{m_k}-1\right\}\times\left\{0,1,...,2^{2m_k}-1\right\}\right)\\ & \cup\left(\left\{0,1,...,2^{2m_k}-1\right\}\times\left\{2^{2m_k},...,q_{m_k}-1\right\}\right)\\ & \cup\left(\left\{0,1,...,2^{2m_k}-1\right\}\times\left\{0,1,...,2^{2m_k}-1\right\}\right).\end{aligned}$$

for some $k$. Then from (15) and (13) it is easy to show that

$$\left|S_{i,j}^{\kappa}f(x,y)\right| \leq \sum_{l=0}^{k-1}\sum_{\nu=2^{2m_l}}^{2^{2m_l+1}-1}\sum_{\mu=2^{2m_l}}^{2^{2m_l+1}-1}\left|\widehat{f}^{\kappa}(\nu,\mu)\right| \leq \sum_{l=0}^{k-1}\frac{2^{8m_l}}{m_l} \leq \frac{C2^{m_k}}{km_k}.$$

Consequently, we have

$$|I| \leq \frac{1}{q_{m_k}^2}\sum_{i=0}^{2^{2m_k}-1}\sum_{j=0}^{2^{2m_k}-1}\left|S_{i,j}^{\kappa}f(x,y)\right| \leq C\frac{2^{4m_k}}{q_{m_k}^2}\frac{2^{m_k}}{km_k} \leq \frac{C2^{m_k}}{km_k}, \tag{17}$$

$$|II| \leq \frac{2^{2m_k}(q_{m_k}-2^{2m_k})}{q_{m_k}^2}\frac{2^{m_k}}{km_k} \leq C\frac{2^{m_k}}{km_k} \tag{18}$$

and

$$|III| \leq \frac{C2^{m_k}}{km_k}. \tag{19}$$

Combining (16)-(19) we obtain that

$$\left|\sigma_{q_{m_k},q_{m_k}}^{\kappa}f(x,y)\right| \geq |IV| - \frac{C2^{m_k}}{km_k}. \tag{20}$$

Now, we discuss $IV$.

Let $(i,j) \in \left\{2^{2m_k},...,q_{m_k}-1\right\}\times\left\{2^{2m_k},...,q_{m_k}-1\right\}$. Then from (15) we have

$$\begin{aligned}S_{i,j}^{\kappa}f\,(x,y) \quad &= \quad \sum_{\nu=0}^{i-1}\sum_{\mu=0}^{j-1}\widehat{f}^{\kappa}\,(\nu,\mu)\,\kappa_{\nu}(x)\kappa_{\mu}(y)\\ &= \quad \sum_{l=0}^{k-1}\sum_{\nu=2^{2m_l}}^{2^{2m_l+1}-1}\sum_{\mu=2^{2m_l}}^{2^{2m_l+1}-1}\widehat{f}^{\kappa}\,(\nu,\mu)\,\kappa_{\nu}(x)\kappa_{\mu}(y)\\ &\qquad + \sum_{\nu=2^{2m_k}}^{i-1}\sum_{\mu=2^{2m_k}}^{j-1}\widehat{f}^{\kappa}\,(\nu,\mu)\,\kappa_{\nu}(x)\kappa_{\mu}(y)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{k-1} \frac{2^{4m_l}}{m_l} \left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right) \times \left(D_{2^{2m_l+1}}(y) - D_{2^{2m_l}}(y)\right) \\
&\quad + \frac{2^{4m_k}}{m_k} \left(D_i^{\kappa}(x) - D_{2^{2m_k}}(x)\right) \left(D_j^{\kappa}(y) - D_{2^{2m_k}}(y)\right) \qquad (21)
\end{aligned}
$$

and

$$
\begin{aligned}
IV &= \frac{1}{q_{m_k}^2} \left(q_{m_k} - 2^{2m_k}\right)^2 \\
&\quad \times \sum_{l=0}^{k-1} \frac{2^{4m_l}}{m_l} \left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right) \left(D_{2^{2m_l+1}}(y) - D_{2^{2m_l}}(y)\right) \\
&\quad + \frac{1}{q_{m_k}^2} \frac{2^{4m_k}}{m_k} \left( \sum_{i=2^{2m_k}}^{q_{m_k}-1} \sum_{j=2^{2m_k}}^{q_{m_k}-1} \left(D_i^{\kappa}(x) - D_{2^{2m_k}}(x)\right) \left(D_j^{\kappa}(y) - D_{2^{2m_k}}(y)\right) \right) \\
&= IV_1 + IV_2.
\end{aligned}
$$

By (13), (14) and $|D_{2^n}(x)| \le 2^n$ we get that

$$
|IV_1| \le C \sum_{l=0}^{k-1} \frac{2^{8m_l}}{m_l} \le C \frac{2^{m_k}}{km_k}
$$

and

$$
\left| \sigma_{q_{m_k}, q_{m_k}}^{\kappa} f(x,y) \right| \ge |IV_2| - \frac{C 2^{m_k}}{km_k}.
$$

By the help of the equation (10) we immediatelly have for $IV_2$

$$
\begin{aligned}
IV_2 &= \frac{1}{q_{m_k}^2} \frac{2^{4m_k}}{m_k} r_{2m_k}(x) r_{2m_k}(y) \sum_{i=0}^{q_{m_k-1}-1} D_i^w(\tau_{2m_k}(x)) \sum_{j=0}^{q_{m_k-1}-1} D_j^w(\tau_{2m_k}(y)) \\
&= \frac{1}{q_{m_k}^2} \frac{2^{4m_k}}{m_k} r_{2m_k}(x) r_{2m_k}(y) q_{m_k-1}^2 K_{q_{m_k-1}}^w(\tau_{2m_k}(x)) K_{q_{m_k-1}}^w(\tau_{2m_k}(y)).
\end{aligned}
$$

Let $(x,y) \in J_{2m_k}^{l_1,l_1+2}(x) \times J_{2m_k}^{l_2,l_2+2}(y)$, where $(l_1,l_2) \in \{0,1,...,m_k-3\} \times \{0,1,...,m_k-3\}$.

Then from Lemma 1 we can write

$$
q_{m_k-1} \left| K_{q_{m_k-1}}^w(\tau_{2m_k}(x)) \right| \ge C 2^{4l_1} \text{ and } q_{m_k-1} \left| K_{q_{m_k-1}}^w(\tau_{2m_k}(y)) \right| \ge C 2^{4l_2}.
$$

Consequently,

$$\left|\sigma^{\kappa}_{q_{m_k},q_{m_k}} f(x,y)\right| \geq \frac{C}{m_k} 2^{4l_1+4l_2} - C\frac{2^{m_k}}{km_k}. \tag{22}$$

Denote

$$A(m_k) := \left\{(l_1,l_2) : 0 \leq l_2 \leq m_k - 3, 0 \leq l_1 \leq \frac{m_k}{4}, l_1 + l_2 \geq \frac{m_k}{4}\right\}$$

and

$$\lambda'_k := \frac{c2^{m_k}}{m_k}.$$

For $(l_1,l_2) \in A(m_k)$, we have

$$\left|\sigma^{\kappa}_{q_{m_k},q_{m_k}} f(x,y)\right| \geq \frac{c2^{m_k}}{m_k}$$

and

$$\begin{aligned}
& \mu\left\{(x,y) \in G \times G : \sigma^{\kappa,\square} f(x,y) \geq C\lambda'_k\right\} \geq \\
\geq & \sum_{(l_1,l_2)\in A(m_k)} \mu\left\{(x,y) \in J^{l_1,l_1+2}_{2m_k}(x) \times J^{l_2,l_2+2}_{2m_k}(y) : \left|\sigma^{\kappa}_{q_{m_k},q_{m_k}} f(x,y)\right| \geq \lambda'_k\right\} \\
\geq & \; C \sum_{l_1=0}^{[m_k/4]} \sum_{l_2=[m_k/4]-l_1}^{m_k-3} \sum_{x_0=0}^{1} \cdots \\
& \times \sum_{x_{2m_k-2l_1-2}=0}^{1} \sum_{y_0=0}^{1} \cdots \sum_{y_{2m_k-2l_2-2}=0}^{1} \mu\left(J^{l_1,l_1+2}_{2m_k}(x) \times J^{l_2,l_2+2}_{2m_k}(y)\right) \\
\geq & \; C \sum_{l_1=0}^{[m_k/4]} \sum_{l_2=[m_k/4]-l_1}^{m_k-3} \frac{1}{2^{2l_1+2l_2}} \geq \frac{Cm_k}{2^{m_k/2}}.
\end{aligned}$$

Consequently,

$$\lambda'_k \left(\mu\left\{(x,y) : \sigma^{\kappa,\square} f(x,y) \geq C\lambda'_k\right\}\right)^2 \geq C\frac{2^{m_k}}{m_k}\frac{m_k^2}{2^{m_k}} = Cm_k \to \infty \text{ as } k \to \infty.$$

This completes the proof of this theorem. $\square$

## 4. The two-dimensional unrestricted maximal operator

If $f \in L_1(G \times G)$ then it is easy to show that the sequence $(S_{2^n,2^m}(f) : n,m \in \mathbf{N})$ is a martingale. If $f$ is a martingale, that is

$f = (f^{(n,m)} : n, m \in \mathbf{N})$ then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i,j) = \lim_{\min(k,l)\to\infty} \int_{G\times G} f^{(k,l)}(x,y)\,\alpha_i(x)\,\alpha_j(y)\,d\mu(x,y).$$

The Walsh-(Kaczmarz)-Fourier coefficients of $f \in L_1(G\times G)$ are the same as the ones of the martingale $(S_{2^n,2^m}(f) : n, m \in \mathbf{N})$ obtained from $f$.

For the martingale $f$ the unrestricted maximal operator of the Fejér mean is defined by

$$\sigma^{\alpha,*} f(x,y) = \sup_{n,m\in\mathbf{N}} |\sigma^{\alpha}_{n,m}(f;x,y)|.$$

A function $a \in L_2$ is caled a rectangle $p$-atom if there exists a dyadic rectangle $R$ such that

(a) $\operatorname{supp} a \subset R$,
(b) $\|a\|_2 \le |R|^{1/2-1/p}$,
(c) $\int_G a(x,y)\,d\mu(x) = \int_G a(x,y)\,d\mu(y) = 0$ for all $x, y \in G$.

The basic result of atomic decomposition is the following one.

**Theorem C.** *(Weisz [20]). A martingale $f = \left(f^{(n,m)} : n, m \in \mathbf{N}\right)$ is in $H_p\,(0 < p \le 1)$ if there exists a sequence $(a_k, k \in \mathbf{N})$ of rectangle $p$-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that for every $n, m \in \mathbf{N}$ ,*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n,2^m} a_k = f^{(n,m)}, \tag{23}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

*Moreover,*

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p},$$

We will prove the following theorem.

**Theorem 3.** *There exists a martingale $f \in H_{1/2}(G\times G)$ such that*

$$\|\sigma^{\kappa,*} f\|_{weak-L_{1/2}} = +\infty.$$

*Proof.* Now, we use the sequence $\{\mu_k : k \in \mathbf{N}\}$ and the atoms $a_k$ defined in the previous proof. Let

$$f^{(A,B)}(x,y) := \sum_{l:2m_l<\min(A,B)} \lambda_l a_l(x,y).$$

First, we prove that the martingale $f := \left(f^{(A,B)} : A, B \in \mathbf{N}\right)$ belongs to the Hardy space $H_{1/2}(G \times G)$. Indeed, since

$$\|a_l\|_2 \leq c2^{6m_l},$$

$$S_{2^A,2^B} a_k(x,y) = \begin{cases} 0, & \text{if } \min(A,B) \leq 2m_k, \\ a_k(x,y), & \text{if } \min(A,B) > 2m_k, \end{cases}$$

$$f^{(A,B)}(x,y) := \sum_{l:2m_l<\min(A,B)} \lambda_l a_l(x,y) = \sum_{k=0}^{\infty} \lambda_k S_{2^A,2^B} a_k(x,y)$$

from (12) and Theorem C we conclude that $f \in H_{1/2}(G \times G)$.

Now, we investigate the Fourier coefficients. Since

$$\int_{G\times G} f^{(A,B)}(x,y)\,\kappa_i(x)\,\kappa_j(y)\,d\mu(x,y)$$

$$= \begin{cases} 0, & (i,j) \notin \bigcup_{k=0}^{\infty} \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, \\ 0, & (i,j) \in \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, \\ & \qquad \min(A,B) \leq 2m_k, \\ \frac{2^{4m_k}}{m_k}, & (i,j) \in \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, \\ & \qquad \min(A,B) > 2m_k, \end{cases}$$

we can write

$$\widehat{f}^{\kappa}(i,j) = \begin{cases} \frac{2^{4m_k}}{m_k}, & (i,j) \in \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, \\ 0, & (i,j) \notin \bigcup_{k=1}^{\infty} \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}. \end{cases}$$

The estimation of

$$\mu\{(x,y) : \sigma^{\kappa,*} f(x,y) \geq C\lambda_k\}$$

is analogous to the estimation of

$$\mu\left\{(x,y) : \sigma^{\kappa,\square} f(x,y) \geq C\lambda_k\right\}$$

and we have that

$$\sup_{\lambda>0} \lambda \mu \left\{(x,y) : \sigma^{\kappa,*} f(x,y) \geq \lambda\right\}^2 = \infty.$$

Theorem 3 is proved. □

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Institute of Mathematics
Faculty of Exact and Natural Sciences
Tbilisi State University
Chavchavadze str. 1, Tbilisi 0128
Georgia
(*E-mail : z_goginava@hotmail.com*)

Institute of Mathematics and Computer Sciences
College of Nyíregyháza
P.O. Box 166, Nyíregyháza
H-4400 Hungary
(*E-mail : nkaroly@nyf.hu*)

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