The Boundedness of Commutators of Singular Integral Operators with Besov Functions¹

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Abstract. In this paper, we prove the boundedness of commutator generated by singular integral operator and Besov function from some L^d to Triebel-Lizorkin spaces.

1. Introduction

Let K be a Calderón-Zygmund singular integral kernel being of the form

$$K(x) = \frac{\Omega(x)}{|x|^n},$$

where $\Omega \in C^\infty(\Sigma^{n-1})$ is homogeneous of degree 0 and

$$\int_{\Sigma^{n-1}} \Omega(x) dx = 0.$$

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The Calderón-Zygmund singular integral operator T is defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)f(y)dy.$$

The fractional integral operator I^α of order α , $0 < \alpha < n$, is the convolution operator with the kernel $\omega_\alpha |x|^{\alpha-n}$. In [3], a well known result of Coifman, Rochberg and Weiss states that the commutator of singular integral operator

$$(1.1) \quad [b, T]f(x) = b(x)Tf(x) - T(bf)(x)$$

is bounded on some L^p , $1 < p < \infty$, if and only if $b \in \text{BMO}$. In 1982, Chanillo [1] introduced the commutator of fractional integral operator $[b, I^\alpha]$:

$$(1.2) \quad [b, I^\alpha]f(x) = b(x)I^\alpha f(x) - I^\alpha(bf)(x),$$

and proved a similar result when T is replaced by the fractional integral operator. In [5], Janson investigated the commutator of singular integral operator and fractional integral operator with Lipschitz functions. In 1995, Paluszynski [6] proved that $[b, T]$ is bounded from L^p to $F_p^{\beta, \infty}$, the Triebel-Lizorkin space, if and only if $b \in \dot{\Lambda}_\beta$, where $\dot{\Lambda}_\beta$ is the homogeneous Lipschitz space which is the set of functions satisfying

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{\substack{x, h \in \mathbb{R}^n \\ h \neq 0}} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty,$$

where Δ_h^k denotes the k -th difference operator (see [6]).

A generalization of Lipschitz space is Besov space $\dot{\Lambda}_\beta^{p, q}(\mathbb{R}^n)$ defined by the set of all functions satisfying

$$\|b\|_{\dot{\Lambda}_\beta^{p, q}} = \left(\int_{\mathbb{R}^n} \frac{\|b(x+t) - b(x)\|_p^q}{|t|^{n+\beta q}} dt \right)^{1/q} < \infty.$$

For the properties of Besov functions we refer [9]. In 2006, Y. Chen and B. Ma in [2] introduced the commutator of singular integral operators and Besov functions and proved that $[b, T]^k$ is bounded from some L^d to certain L^r .

In this paper, we investigate the boundedness of commutators generated by singular integral operators and Besov functions from Lebesgue integrable spaces to Triebel-Lizorkin spaces, as well as commutators of fractional

integral operators. We simply denote T_b by $[b, T]$ and I_b^α by $[b, I^\alpha]$. Our results are stated below.

Theorem 1.1. *Let $1 < q < p < \infty$, $0 < \beta - n/p < 1$ and $b \in \dot{\Lambda}_\beta^{p,q}$. If $d > q/(q-1)$, then the commutator T_b is a bounded operator from $L^d(\mathbb{R}^n)$ to $\dot{F}_d^{(\beta-n/p),\infty}(\mathbb{R}^n)$.*

Theorem 1.2. *Let $1 < p < q < \infty$, $0 < \beta - n/p < 1$ and $b \in \dot{\Lambda}_\beta^{p,q}$. If $d > p/(p-1)$, then the commutator T_b is a bounded operator from $L^d(\mathbb{R}^n)$ to $\dot{F}_d^{(\beta-n/p),\infty}(\mathbb{R}^n)$.*

Theorem 1.3. *Let $1 < q < p < \infty$, $0 < \beta - n/p < 1$, $b \in \dot{\Lambda}_\beta^{p,q}$ and $0 < \alpha < n$. If $d > q/(q-1)$, $1/d - 1/r = \alpha/n$, then the commutator $I_b^\alpha f$ is a bounded operator from $L^d(\mathbb{R}^n)$ to $\dot{F}_r^{(\beta-n/p),\infty}(\mathbb{R}^n)$.*

Theorem 1.4. *Let $1 < p < q < \infty$, $0 < \beta - n/p < 1$, $b \in \dot{\Lambda}_\beta^{p,q}$ and $0 < \alpha < n$. If $d > p/(p-1)$, $1/d - 1/r = \alpha/n$, then the commutator I_b^α is a bounded operator from $L^d(\mathbb{R}^n)$ to $\dot{F}_r^{(\beta-n/p),\infty}(\mathbb{R}^n)$.*

From the results stated above it can be seen that if $p, q \rightarrow \infty$ then all the theorems are the ones obtained by Paluszynski in [6]. The proofs of our theorems are based on an estimate similar to the mean oscillation of a Besov function over cubes. We shall provide it as a lemma in the next section and then prove our theorems.

2. Some Lemmas and Proofs of Results

In this section, we get an estimate of the oscillation of Besov function b on cubes. Using the property of Besov function, we discuss the boundedness of the commutator. In the following, let Q be a cube and denote by f_Q the mean of f over Q , i.e.,

$$f_Q = \frac{1}{|Q|} \int_Q f(y) dy.$$

Lemma 2.1. ([6]) *For $0 < \beta < 1$, $1 < p < \infty$, we have*

$$\|f\|_{\dot{F}_p^{\beta,\infty}} \approx \left\| \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f - f_Q| \right\|_p.$$

Lemma 2.2. (a) For $0 < \beta < 1$, $1 < q \leq p < \infty$, we have

$$(2.1) \quad \sup_Q \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q |b(y) - b_Q| dy \leq \sup_Q \frac{1}{|Q|^{\beta/n+1/q-1/p}} \left(\int_Q |b(y) - b_Q|^q dy \right)^{1/q} \leq C \|b\|_{\dot{\Lambda}_\beta^{p,q}}.$$

(b) For $0 < \beta < 1$, $1 < p \leq q < \infty$, we have:

$$(2.2) \quad \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\int_Q |b(y) - b_Q|^p dy \right)^{1/p} \leq C \|b\|_{\dot{\Lambda}_\beta^{p,q}}.$$

Proof. Fix a cube Q , and set $\wedge = \{t = y - z : y, z \in Q\}$. So for any $t \in \wedge$, we have $|t| \leq C|Q|^{1/n}$. Using Hölder's inequality, we first estimate $\int_Q |b(y) - b_Q|^q dy$ as follows

$$\begin{aligned} \int_Q |b(y) - b_Q|^q dy &= \int_Q \left| b(y) - \frac{1}{|Q|} \int_Q |b(z)| dz \right|^q dy \\ &\leq \int_Q \left(\frac{1}{|Q|} \int_Q |b(y) - b(z)| dz \right)^q dy \\ &\leq \int_Q \left(\frac{1}{|Q|} \int_Q |b(y) - b(z)|^q dz \right) \left(\frac{1}{|Q|} \int_Q dz \right)^{q-1} dy \\ &\leq |Q|^{-1} \int_Q \int_Q |b(y) - b(z)|^q dz dy \\ &\leq C|Q|^{-1} \int_{t \in \wedge} \int_Q |b(z+t) - b(z)|^q dz dt \\ &=: S_1. \end{aligned}$$

Since $p/q > 1$, using Hölder's inequality on z , we obtain

$$\begin{aligned} S_1 &\leq C|Q|^{-1} \int_{t \in \wedge} \left(\int_Q |b(z+t) - b(z)|^p dz \right)^{q/p} dt |Q|^{1-q/p} \\ &\leq C|Q|^{-1} \int_{t \in \wedge} \frac{(\int_Q |b(z+t) - b(z)|^p dz)^{q/p}}{|t|^{n+\beta q}} |t|^{n+\beta q} dt |Q|^{1-q/p} \\ &\leq C|Q|^{(-1+1-q/p)} |Q|^{1+\beta q/n} \|b\|_{\dot{\Lambda}_\beta^{p,q}}^q \\ &\leq C|Q|^{(1+q\beta/n-q/p)} \|b\|_{\dot{\Lambda}_\beta^{p,q}}^q. \end{aligned}$$

Therefore we obtain the second inequality in (2.1).

With regard to the first inequality in (2.1), using the second one, together with Hölder inequality, we have

$$\begin{aligned} \int_Q |b(y) - b_Q| dy &\leq \left(\int_Q |b(y) - b_Q|^q dy \right)^{1/q} |Q|^{1-1/q} \\ &\leq C|Q|^{(1-1/q+1/q+\beta/n-1/p)} \|b\|_{\dot{\lambda}_\beta^{p,q}} \\ &\leq C|Q|^{(1+\beta/n-1/p)} \|b\|_{\dot{\lambda}_\beta^{p,q}}. \end{aligned}$$

Thus, the part (a) of Lemma 2.2 is proved.

Now we consider the part (b). Clearly we have

$$\int_Q |b(y) - b_Q|^p dy = \int_Q \left| b(y) - \frac{1}{|Q|} \int_Q b(z) dz \right|^p dy.$$

Also by Hölder's inequality, we obtain

$$\begin{aligned} \int_Q |b(y) - b_Q|^p dy &\leq \int_Q \left(\frac{1}{|Q|} \int_Q |b(y) - b(z)| dz \right)^p dy \\ &\leq \int_Q \left(\frac{1}{|Q|} \int_Q |b(y) - b(z)|^p dz \right) \left(\frac{1}{|Q|} \int_Q dz \right)^{p-1} dy \\ &\leq |Q|^{-1} \int_Q \int_Q |b(y) - b(z)|^p dz dy. \end{aligned}$$

Set $y = z + t$, $z = z$, by a change of variable, we get

$$\int_Q |b(y) - b_Q|^p dy \leq C|Q|^{-1} \int_{t \in \wedge} \int_Q |b(z+t) - b(z)|^p dz dt =: S_2.$$

Since $q/p > 1$, using Hölder's inequality, we have

$$\begin{aligned} S_2 &\leq C|Q|^{-1} \left(\int_{t \in \wedge} \left(\int_Q |b(z+t) - b(z)|^p dz \right)^{q/p} dt \right)^{p/q} \left(\int_{t \in \wedge} dt \right)^{1-p/q} \\ &\leq C|Q|^{-p/q} \left(\int_{t \in \wedge} \frac{(\int_Q |b(z+t) - b(z)|^p dz)^{q/p}}{|t|^{n+\beta q}} |t|^{n+\beta q} dt \right)^{p/q} \\ &\leq C|Q|^{(-q/p)} |Q|^{p/q+p\beta/n} \left(\int_{\mathbb{R}^n} \frac{(\int_Q |b(z+t) - b(z)|^p dz)^{q/p}}{|t|^{n+\beta q}} dt \right)^{p/q} \\ &\leq C|Q|^{p\beta/n} \|b\|_{\dot{\lambda}_\beta^{p,q}}^p. \end{aligned}$$

Thus the Lemma 2.2 is proved completely. $\square$

We now proceed with proofs of our theorems.

Proof of the Theorem 1.1: Fix a cube $Q = Q(x_Q, s)$, centered at x_Q , with the side length s , and $x \in Q$. For $f \in L^p$, decompose f as $f_1 = f\chi_{2Q}$, $f_2 = f - f_1$. Observing that $T_b f = T_{(b-b_Q)} f$, we have

$$\begin{aligned} \int_Q |T_b f(y) - (T_b f)_Q| dy &\leq 2 \int_Q |T_{(b-b_Q)} f(y) - T((b-b_Q)f_2)(x_Q)| dy \\ &\leq 2 \int_Q |(b(y) - b_Q) T f(y)| dy + 2 \int_Q |T((b-b_Q)f_1)(y)| dy \\ &\quad + 2|Q| \sup_{y \in Q} |T((b-b_Q)f_2)(y) - T((b-b_Q)f_2)(x_Q)| \\ &= I + II + III. \end{aligned}$$

We estimate these terms respectively.

To the term I , for $x \in Q$, by Hölder's inequality and the part (a) in Lemma 2.2, we have

$$\begin{aligned} I &\leq 2 \left(\int_Q |b(y) - b_Q|^q dy \right)^{1/q} \left(\int_Q |T f(y)|^{q'} dy \right)^{1/q'} \\ &\leq C |Q|^{\beta/n-1/p+1/q} |Q|^{1-1/q} \|b\|_{\dot{\lambda}_\beta^{p,q}} \left(|Q|^{-1} \int_Q |T f(y)|^{q'} dy \right)^{1/q'} \\ (2.3) \quad &\leq C |Q|^{1+\beta/n-1/p} \|b\|_{\dot{\lambda}_\beta^{p,q}} M_{q'}(|T f|)(x). \end{aligned}$$

To get II , choosing $1 < r < q$ such that $r q / (q - r) > 1$, we firstly estimate $\|(b - b_Q)f_1\|_r$ as

$$\begin{aligned} \|(b - b_Q)f_1\|_r^r &= \int_{2Q} |(b(y) - b_Q)f| dy \\ &\leq \left(\int_{2Q} |b(y) - b_Q|^q dy \right)^{r/q} \left(\int_{2Q} |f(y)|^{r q / (q-r)} dy \right)^{(q-r)/q} \\ &\leq C |2Q|^{r(\beta/n+1/q-1/p)} \|b\|_{\dot{\lambda}_\beta^{p,q}}^r \\ &\quad \times \left(|Q|^{-1} \int_{2Q} |f(y)|^{r q / (q-r)} dy \right)^{(q-r)/q} |Q|^{(q-r)/q} \\ (2.4) \quad &\leq C |Q|^{r(1/r+\beta/n-1/p)} \|b\|_{\dot{\lambda}_\beta^{p,q}}^r M_{r q / (q-r)}^r(f)(x). \end{aligned}$$

Take r' such that $1/r+1/r' = 1$. By the boundedness of Calderon-Zygmund Singular integral operator and the estimate of $\|(b - b_Q)f_1\|_r$ in (2.4), we

obtain

$$\begin{aligned}
II &= 2 \int_Q |T((b - b_Q)f_1)(y)| dy \\
&\leq C \|T((b - b_Q)f_1)\|_r |Q|^{1/r'} \\
&\leq C \|((b - b_Q)f_1)\|_r |Q|^{1/r'} \\
&\leq C |Q|^{1/r+\beta/n-1/p} |Q|^{1/r'} \|b\|_{\dot{\lambda}_\beta^{p,q}} M_{rq/(q-r)}(f)(x) \\
(2.5) \quad &\leq C |Q|^{1+\beta/n-1/p} \|b\|_{\dot{\lambda}_\beta^{p,q}} M_{rq/(q-r)}(f)(x).
\end{aligned}$$

Now we estimate *III*. Before we start the proof, we recall the smoothness condition of Calderon-Zygmund kernel, that is,

$$|k(x - y) - k(-y)| \leq \frac{|x|}{|y|^{n+1}}.$$

Then we have

$$\begin{aligned}
&|T((b - b_Q)f_2)(y) - T((b - b_Q)f_2)(x_Q)| \\
&= \left| \int_{\mathbb{R}^n} (K(y - z) - K(x_Q - z))(b(z) - b_Q)f_2(z) dz \right| \\
&\leq C \int_{(2Q)^c} \frac{|y - x_Q|}{|x_Q - z|^{n+1}} |b(z) - b_Q| |f(z)| dz \\
(2.6) \quad &\leq C \sum_{k=2}^{\infty} \int_{2^k Q \setminus 2^{k-1} Q} 2^{-k} |2^k Q|^{-1} |b(z) - b_Q| |f(z)| dz.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
|b_Q - b_{2Q}| &\leq \frac{1}{|Q|} \int_Q |b(y) - b_{2Q}| dy \\
&\leq \frac{2^n}{|2Q|} \int_{2Q} |b(y) - b_{2Q}| dy \\
&\leq 2^n |2Q|^{\beta/n-1/p} \|b\|_{\dot{\lambda}_\beta^{p,q}}
\end{aligned}$$

and also

$$(2.7) \quad |b_{2^k Q} - b_Q| \leq 2^n k |2^k Q|^{\beta/n-1/p} \|b\|_{\dot{\lambda}_\beta^{p,q}}.$$

Thus by Hölder's inequality, (2.1) and (2.7), the last inequality in (2.6) is dominated by

$$\begin{aligned}
& \sum_{k=2}^{\infty} 2^{-k} |2^k Q|^{-1} \int_{2^k Q \setminus 2^{k-1} Q} |b(z) - b_Q| |f(z)| dz \\
& \leq \sum_{k=2}^{\infty} 2^{-k} |2^k Q|^{-1} \left(\int_{2^k Q} |b(z) - b_{2^k Q}| |f(z)| dz + \int_{2^k Q} |b_{2^k Q} - b_Q| |f(z)| dz \right) \\
& \leq C \sum_{k=2}^{\infty} 2^{-k} |2^k Q|^{-1} \left(\int_{2^k Q} |b(z) - b_Q|^q dz \right)^{1/q} \left(\int_{2^k Q} |f(z)|^{q'} dz \right)^{1/q'} + \\
& \quad + C \sum_{k=2}^{\infty} 2^{-k} |2^k Q|^{\beta/n-1/p} \|b\|_{\dot{\lambda}_{\beta}^{p,q}} |2^k Q|^{-1} \int_{2^k Q} |f(z)| dz \\
& \leq C |Q|^{\beta/n-1/p} \|b\|_{\dot{\lambda}_{\beta}^{p,q}} \left(\sum_{k=2}^{\infty} 2^{k(-1+\beta-n/p)} M_{q'}(f)(x) + \sum_{k=2}^{\infty} 2^{k(-1+\beta-n/p)} k M(f)(x) \right) \\
& \leq C |Q|^{\beta/n-1/p} \|b\|_{\dot{\lambda}_{\beta}^{p,q}} (M_{q'}(f)(x) + M(f)(x)),
\end{aligned}$$

where we use the assumption $0 < \beta - n/p < 1$. Thus we have

$$(2.8) \quad III \leq C |Q|^{1+\beta/n-1/p} \|b\|_{\dot{\lambda}_{\beta}^{p,q}} (M_{q'}(f)(x) + M(f)(x)).$$

Putting these estimates (2.3), (2.5) and (2.8) together, we obtain

$$\begin{aligned}
& \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q |T_b f(y) - (T_b f)_Q| dy \\
& \leq C \|b\|_{\dot{\lambda}_{\beta}^{p,q}} (M_{q'}(f)(x) + M_{q'}(|Tf|)(x) + M_{rq/(q-r)}(f)(x) + M(f)(x)).
\end{aligned}$$

Since $0 < \beta - n/p < 1$, $d > q/(q-1) = q'$, we can choose $1 < r$ and closely to 1 such that $d > rq/(q-r) > q/(q-1)$. Taking the supremum over all Q such that $x \in Q$, and L^d norm on both sides, and using Lemma 2.1, we conclude that

$$\|T_b f\|_{\dot{F}_d^{(\beta-n/p),\infty}} \leq C \|b\|_{\dot{\lambda}_{\beta}^{p,q}} \|f\|_d.$$

Thus, the proof of Theorem 1.1 is completed. $\square$

The proof of Theorem 1.2 is similar to that of Theorem 1.1.

Proof of Theorem 1.3 Fix $f \in L^d(\mathbb{R}^n)$ and recall that x_Q denotes the center of a cube Q , and $f_1 = f \chi_{2Q}$, $f_2 = f - f_1$. Using Lemma 2.2(a) and

the fact that $I_b^\alpha f - (I_b^\alpha f)_Q = I_{(b-b_Q)}^\alpha f - (I_{(b-b_Q)}^\alpha f)_Q$, we have

$$\begin{aligned}
\|I_b^\alpha f\|_{\dot{F}_r^{(\beta-n/p),\infty}} &\approx C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{(1+\beta/n-1/p)}} \int_Q |I_{b-b_Q}^\alpha f - (I_{b-b_Q}^\alpha f)_Q| \right\|_r \\
&\leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{(1+\beta/n-1/p)}} \int_Q |I_{b-b_Q}^\alpha f - I^\alpha((b-b_Q)f_2)(x_Q)| \right\|_r \\
&\leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{(1+\beta/n-1/p)}} \int_Q |(b-b_Q)I^\alpha f| \right\|_r + \\
&\quad + C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{(1+\beta/n-1/p)}} \int_Q |I^\alpha((b-b_Q)f) - I^\alpha((b-b_Q)f_2)(x_Q)| \right\|_r \\
&\leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{(1+\beta/n-1/p)}} \int_Q |(b-b_Q)I^\alpha f| \right\|_r + \\
&\quad + \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+(\beta+\alpha)/n-1/p}} \int_Q |I^\alpha((b-b_Q)f) - I^\alpha((b-b_Q)f_2)(x_Q)| \right\|_d.
\end{aligned}$$

Since $1 < d < r < \infty$, $0 < \beta - n/p < 1$, $1/d - 1/r = \alpha/n$. The last inequality follows from the following lemma, with $\beta - n/p$ in the place of γ . $\square$

Lemma 2.3. ([8] pp.71-72) *Let $1 < d < r < \infty$, $1/d - 1/r = \alpha/n$. Suppose that for each cube Q we have a function h^Q , defined on this cube. Then, for $\gamma \geq 0$,*

$$\left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\gamma/n}} \int_Q |h^Q| \right\|_r \leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\gamma/n+\alpha/n}} \int_Q |h^Q| \right\|_d,$$

where the constant C depends only on d, r, α and n .

Lemma 2.3, as in [8], specifies that the function h^Q have the form $h - h^Q$, but the proof carries over immediately to the more general case above. Thus,

$$\begin{aligned}
\|I_b^\alpha f\|_{\dot{F}_r^{(\beta-n/p),\infty}} &\leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{(1+\beta/n-1/p)}} \int_Q |(b-b_Q)I^\alpha f| \right\|_r \\
&\quad + C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{(1+(\alpha+\beta)/n-1/p)}} \int_Q |I^\alpha((b-b_Q)f_1)| \right\|_d \\
&\quad + C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{((\alpha+\beta)/n-1/p)}} \times \right. \\
&\quad \times \left. \sup_{y \in Q} |I^\alpha((b-b_Q)f_2)(y) - I^\alpha((b-b_Q)f_2)(x_Q)| \right\|_d \\
&\leq I + II + III.
\end{aligned}$$

Using Lemma 2.2 (a), we can estimate I as follows

$$\begin{aligned} \int_Q |b(y) - b_Q| |I^\alpha f(y)| dy &\leq C \left(\int_Q |b(y) - b_Q|^q dy \right)^{1/q} \left(\int_Q |I^\alpha f(y)|^{q'} dy \right)^{1/q'} \\ &\leq C |Q|^{1+\beta/n-1/p} \|b\|_{\dot{\lambda}_\beta^{p,q}} M_{q'}(I^\alpha f)(x). \end{aligned}$$

Since $r > q'$, we have

$$\begin{aligned} I &\leq C \|b\|_{\dot{\lambda}_\beta^{p,q}} \|M_{q'}(I^\alpha f)\|_r \\ &\leq C \|b\|_{\dot{\lambda}_\beta^{p,q}} \|I^\alpha f\|_r \\ (2.9) \quad &\leq C \|b\|_{\dot{\lambda}_\beta^{p,q}} \|f\|_d. \end{aligned}$$

We now estimate II . Choose $1 < h < q$, and $\bar{h}$ such that then $(1/h) - (1/\bar{h}) = \alpha/n$. Consider $\|(b - b_Q)f_1\|_h$ as

$$\begin{aligned} &\left(\int_Q |b(y) - b_Q|^h |f_1(y)|^h dy \right)^{1/h} \\ &\leq \left(\int_Q |b(y) - b_Q|^q dy \right)^{1/q} \left(\int_Q |f_1(y)|^{hq/(q-h)} dy \right)^{(1/h-1/q)} \\ &\leq C |Q|^{(\beta/n+1/q-1/p)} \|b\|_{\dot{\lambda}_\beta^{p,q}} M_{qh/(q-h)}(f)(x) |Q|^{1/h-1/q} \\ &\leq C |Q|^{(\beta/n+1/h-1/p)} \|b\|_{\dot{\lambda}_\beta^{p,q}} M_{qh/(q-h)}(f)(x). \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{|Q|^{(1+(\alpha+\beta)/n-1/p)}} \int_Q |I^\alpha((b - b_Q)f_1)(y)| dy \\ &\leq \frac{1}{|Q|^{(1+(\alpha+\beta)/n-1/p)}} \|I^\alpha((b - b_Q)f_1)\|_{\bar{h}} |Q|^{1-1/\bar{h}} \\ &\leq C |Q|^{-1-\alpha/n-\beta/n+1-1/\bar{h}+1/p} \|(b - b_Q)f_1\|_h \\ &= C |Q|^{-1/h-\beta/n+1/p} \|(b - b_Q)f_1\|_h \\ &\leq C \|b\|_{\dot{\lambda}_\beta^{p,q}} M_{qh/(q-h)}(f)(x), \end{aligned}$$

Since $d > q/(q-1)$, we can choose $1 < h$ such that $d > qh/(q-h) > q/(q-1)$, and obtain

$$\begin{aligned} II &\leq C \|b\|_{\dot{\lambda}_\beta^{p,q}} \|M_{qh/(q-h)}(f)\|_d \\ (2.10) \quad &\leq C \|b\|_{\dot{\lambda}_\beta^{p,q}} \|f\|_d. \end{aligned}$$

Now we estimate *III*. Set $y \in Q$. By (2.7) and Lemma 2.2, we have

$$\begin{aligned}
& \frac{1}{|Q|^{(\alpha+\beta)/n-1/p}} |I^\alpha((b-b_Q)f_2)(y) - I^\alpha((b-b_Q)f_2)(x_Q)| \\
& \leq \frac{C}{|Q|^{(\alpha+\beta)/n-1/p}} \int_{(2Q)^c} \frac{|y-x_Q|}{|x_Q-z|^{n+1-\alpha}} |b(z) - b_Q| |f(z)| dz \\
& \leq \frac{C}{|Q|^{(\alpha+\beta)/n-1/p}} \sum_{k=2}^{\infty} \int_{2^k Q \setminus 2^{k-1} Q} 2^{-k} |2^k Q|^{-1+\alpha/n} |b(z) - b_{2^k Q}| |f(z)| dz \\
& \leq \frac{C}{|Q|^{(\alpha+\beta)/n-1/p}} \left(\sum_{k=2}^{\infty} 2^{-k} |2^k Q|^{-1+\alpha/n} \int_{2^k Q} |b(z) - b_{2^k Q}| |f(z)| dz + \right. \\
& \quad \left. + \sum_{k=2}^{\infty} 2^{-k} k |2^k Q|^{(\alpha+\beta)/n-1/p} \|b\|_{\dot{\lambda}_\beta^{p,q}} |2^k Q|^{-1} \int_{2^k Q} |f(z)| dz \right) \\
& \leq \frac{C \|b\|_{\dot{\lambda}_\beta^{p,q}}}{|Q|^{(\alpha+\beta)/n-1/p}} \left(\sum_{k=2}^{\infty} 2^{-k} |2^k Q|^{(\alpha+\beta)/n-1/p} M_{q'}(f)(x) + \right. \\
& \quad \left. + \sum_{k=2}^{\infty} 2^{-k} k |2^k Q|^{(\alpha+\beta)/n-1/p} M(f)(x) \right) \\
& \leq C \|b\|_{\dot{\lambda}_\beta^{p,q}} \left(\sum_{k=2}^{\infty} 2^{k(-1+\alpha+\beta-n/p)} M_{q'}(f)(x) + \right. \\
& \quad \left. + \sum_{k=2}^{\infty} 2^{k(-1+\alpha+\beta-n/p)} k M(f)(x) \right) \\
& \leq C \|b\|_{\dot{\lambda}_\beta^{p,q}} (M_{q'}(f)(x) + M(f)(x)),
\end{aligned}$$

where we first assume that $\alpha + \beta - n/p < 1$. Thus,

$$\begin{aligned}
(2.11) \quad III & \leq C \|b\|_{\dot{\lambda}_\beta^{p,q}} (\|M_{q'}(f)\|_d + \|M(f)\|_d) \\
& \leq C \|b\|_{\dot{\lambda}_\beta^{p,q}} \|f\|_d.
\end{aligned}$$

Putting (2.9), (2.10) and (2.11) together, Theorem 1.3 is proved under the assumption $\alpha + \beta - n/p < 1$. Following the facts 1.9 and 1.10 in [6] on pages 10 and 11, we can also get the result when $0 < \alpha < n$. $\square$

The proof of the Theorem 1.4 is similar to that of Theorem 1.3, and we omit the details.

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