Function spaces on the Koch curve

Maryia Kabanava

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Abstract. We consider two types of Besov spaces on the Koch curve, defined by traces and with the help of the snowflaked transform. We compare these spaces and give their characterization in terms of Daubechies wavelets.

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1. Introduction

Let Γ be the Koch curve in $\mathbb{R}^2$. It is an example of a d -set with $d = \frac{\log 4}{\log 3}$. There are two possibilities to introduce Besov spaces on Γ . The first one is to define Besov spaces $B_{pq}^s(\Gamma, \mu)$ by traces

$$B_{pq}^s(\Gamma, \mu) = \text{tr}_\mu B_{pq}^{s+\frac{2-d}{p}}(\mathbb{T}^2), \quad 1 < p < \infty, \quad 0 < q < \infty, \quad 0 < s < \infty.$$

We prefer the periodic setting since we are interested to extend the theory to a closed snowflake. The second way is to use the snowflaked transform $H : \mathbb{T} \rightarrow \Gamma$, $\mathbb{T}$ – 1-torus and define $\mathbb{B}_{pq}^s(\Gamma)$ by

$$\mathbb{B}_{pq}^s(\Gamma) = \{f \circ H^{-1} : f \in B_{pq}^s(\mathbb{T})\} = B_{pq}^s(\mathbb{T}) \circ H^{-1},$$

where $B_{pq}^s(\mathbb{T})$ are periodic Besov spaces.

The question arises how the function spaces $B_{pq}^s(\Gamma, \mu)$ and $\mathbb{B}_{pq}^s(\Gamma)$ are interrelated. We concentrate mainly on the case $1 < p = q < \infty$, $0 < s < 1$. In particular, we shift the characterization in terms of Daubechies wavelets from $(\mathbb{T}, \rho = |x - y|^{1/d}, \mu_L)$, [7, p. 360], to Γ .

This paper is organized as follows. In Section 2, we describe the trace method of defining Besov spaces. In Section 3, we present the wavelet characterization of the periodic Besov spaces $B_{pq}^s(\mathbb{T})$ and then shift it to Γ . In Section 4, we compare $B_{pp}^s(\Gamma, \mu)$ and $\mathbb{B}_{pp}^s(\Gamma)$. The main result is contained in Theorem 3.

2. Trace spaces

2.1. Periodic Besov spaces on $\mathbb{T}^n$. Let

$$\mathbb{T}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1, i = 1, \dots, n\}$$

$x \in \mathbb{T}^n$ and $y \in \mathbb{T}^n$ are identified if and only if $x - y = k$, $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$.

By $D(\mathbb{T}^n)$, we denote the collection of all complex-valued infinitely differentiable functions on $\mathbb{T}^n$. The topology in $D(\mathbb{T}^n)$ is generated by the family of semi-norms $\sup_{x \in \mathbb{T}^n} |D^\alpha f(x)|$, where α is an arbitrary multi-index. $D'(\mathbb{T}^n)$ is defined to be the topological dual of $D(\mathbb{T}^n)$. Any $f \in D'(\mathbb{T}^n)$ can be represented as

$$f = \sum_{k \in \mathbb{Z}^n} a_k e^{2\pi i k x}, \quad x \in \mathbb{T}^n, \quad (\text{convergence in } D'(\mathbb{T}^n))$$

where the Fourier coefficients $\{a_k\} \subset \mathbb{C}$ are of at most polynomial growth,

$$|a_k| \leq c(1 + |k|)^\kappa, \quad \text{for some } c > 0, \quad \kappa > 0 \text{ and all } k \in \mathbb{Z}^n.$$

Definition 1. Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be a dyadic resolution of unity, $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ and

$$\|f|B_{pq}^s(\mathbb{T}^n)\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \sum_{k \in \mathbb{Z}^n} \varphi_j(k) a_k e^{2\pi i k x} \right\|_{L_p(\mathbb{T}^n)}^q \right)^{\frac{1}{q}}$$

(with the usual modification if $q = \infty$). Then the Besov space $B_{pq}^s(\mathbb{T}^n)$ consists of all $f \in D'(\mathbb{T}^n)$ such that $\|f|B_{pq}^s(\mathbb{T}^n)\| < \infty$, [6, Chapter 3].

2.2. Trace spaces.

Definition 2. A compact set Γ in $\mathbb{R}^n$ is called a d -set with $0 < d \leq n$ if there is a Radon measure μ in $\mathbb{R}^n$ with support Γ such that for some positive constants c_1 and c_2

$$(1) \quad c_1 r^d \leq \mu(B(\gamma, r)) \leq c_2 r^d, \quad \gamma \in \Gamma, \quad 0 < r < 1, \quad 0 < d \leq n.$$

where $B(x, r)$ is a ball in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ and of radius $r > 0$.

If Γ is a d -set, then the restriction to Γ of the d -dimensional Hausdorff measure satisfies (1) and any measure μ satisfying (1) is equivalent to $H^d|_\Gamma$.

When Γ is the Koch curve, it is a d -set in $\mathbb{R}^2$ with $d = \frac{\log 4}{\log 3}$. Moreover Γ is a subset of $\mathbb{T}^2$. In order to avoid problems in the endpoints $(0, 0)$ and $(1, 0)$ of Γ , we define the Besov spaces as a trace of the periodic Besov spaces $B_{pq}^s(\mathbb{T}^2)$.

Suppose that for some

$$(2) \quad s > 0, \quad 1 < p < \infty, \quad 0 < q < \infty$$

there is a constant $c > 0$ such that

$$(3) \quad \int_{\Gamma} |\varphi(\gamma)| \mu(d\gamma) \leq c \|\varphi|_{B_{pq}^s(\mathbb{T}^2)}\|, \quad \text{for all } \varphi \in D(\mathbb{T}^2),$$

where $\varphi(\gamma)$ denotes the pointwise trace of $\varphi \in D(\mathbb{T}^2)$ on Γ (sometimes we write $\varphi|_\Gamma$ to denote the pointwise trace of φ). $D(\mathbb{T}^2)$ is dense in $B_{pq}^s(\mathbb{T}^2)$. Then (3) can be extended by completion to any $f \in B_{pq}^s(\mathbb{T}^2)$ and the resulting function on Γ is denoted by $\text{tr}_\mu f$. By standard arguments, it is independent of the approximation of f in $B_{pq}^s(\mathbb{T}^2)$ by $D(\mathbb{T}^2)$ -functions. Any function $g \in \text{tr}_\mu B_{pq}^s(\mathbb{T}^2) \subset L_1(\Gamma, \mu)$ is quasi-normed by

$$\|g\|_{\text{tr}_\mu B_{pq}^s(\mathbb{T}^2)} = \inf \{ \|f\|_{B_{pq}^s(\mathbb{T}^2)} : \text{tr}_\mu f = g \}.$$

If one has (3) for some s, p, q satisfying (2), then one has also (3) for all spaces $B_{pv}^{s+\varepsilon}(\mathbb{T}^n)$ with $\varepsilon > 0$ and $0 < v \leq \infty$.

From the Corollary 1.175 in [7], it follows that the trace operator

$$\text{tr}_\mu : B_{pq}^s(\mathbb{T}^2) \hookrightarrow L_1(\Gamma, \mu)$$

exists if $s > \frac{2-d}{p}$. This justifies the following definition

Definition 3. Let Γ be the Koch curve. Let $1 < p < \infty$, $0 < q < \infty$ and $s > 0$. Then

$$(4) \quad B_{pq}^s(\Gamma, \mu) = \text{tr}_\mu B_{pq}^{s + \frac{2-d}{p}}(\mathbb{T}^2).$$

Here $B_{pq}^s(\Gamma, \mu)$ are considered as subsets of $L_1(\Gamma, \mu)$.
Let

$$B_p^s(\Gamma) = B_{pp}^s(\Gamma, \mu), \quad 1 < p < \infty, \quad 0 < s < 1.$$

It was shown in [3] that $B_p^s(\Gamma)$ with $1 < p < \infty$ and $0 < s < 1$ can be equivalently normed by $\|f|B_p^s(\Gamma)\|_*$ with

$$(5) \quad \|f|B_p^s(\Gamma)\|_*^p = \int_\Gamma |f(\gamma)|^p \mu(d\gamma) + \int_{\Gamma \times \Gamma} \frac{|f(\gamma) - f(\delta)|^p}{|\gamma - \delta|^{d+sp}} \mu(d\gamma) \mu(d\delta)$$

with $\mu = H^d|_\Gamma$.

3. Wavelets on $\mathbb{T}$ and Γ

3.1. Self-similar sets and the snowflaked transform. Let K be a self-similar set in $\mathbb{R}^n$ with respect to the contractions $\{F_i\}_{i=1}^N$, $K = \bigcup_{i=1}^N F_i(K)$. We can use iterations of the maps F_i to give the "address" of a point in K . We introduce the following spaces: let Σ be a set of all infinite sequences

$$\Sigma = \{(\omega_1, \omega_2, \dots) : \omega_i \in \{1, 2, \dots, N\}\}.$$

We use W_m to denote the collection of words of length m :

$$W_m = \{(w_1, w_2, \dots, w_m) : w_i \in \{1, 2, \dots, N\}\}.$$

Set $W_* = \bigcup_{m=0}^\infty W_m$. We write $K_w = F_w(K)$ for $F_{w_1} \circ F_{w_2} \circ \dots \circ F_{w_m}(K)$, where $w = (w_1, w_2, \dots, w_m) \in W_*$. For any $\omega = (\omega_1, \omega_2, \dots) \in \Sigma$ define $\pi : \Sigma \rightarrow K$ by

$$\pi(\omega) = \bigcap_{m=1}^\infty K_{\omega_1 \omega_2 \dots \omega_m},$$

see [4, Ch. 1.2].

The unit interval $I = [0, 1]$ can be considered as a self-similar set with respect to the similarities $T_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$,

$$T_1(x) = \frac{1}{2}x, \quad T_2(x) = \frac{1}{2}x + \frac{1}{2}.$$

The Koch curve Γ is a self-similar set with respect to the similarities $F_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $i = 1, 2$,

$$\begin{aligned} F_1 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \\ F_2 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix}. \end{aligned}$$

We denote a mapping π corresponding to I by π_I and to Γ by π_Γ ,

$$\pi_I(\omega) = \bigcap_{m=1}^{\infty} I_{\omega_1 \omega_2 \dots \omega_m}, \quad \pi_\Gamma(\omega) = \bigcap_{m=1}^{\infty} \Gamma_{\omega_1 \omega_2 \dots \omega_m}.$$

The mapping

$$(6) \quad H = \pi_\Gamma \circ \pi_I^{-1}$$

is a homeomorphism between I and Γ . It is called the snowflaked transform. Note that

$$|H(x) - H(y)| \sim |x - y|^{\frac{1}{d}},$$

[7, Proposition 8.16].

Since the 1-torus $\mathbb{T}$ can be identified in the usual way with the unit interval, it can be regarded as a self-similar set with respect to T_1 and T_2 .

3.2. Self-similar measures. Let $p_1, p_2, \dots, p_N$ be numbers such that

$$\sum_{i=1}^N p_i = 1, \quad 0 < p_i < 1.$$

Then we can define the probability measure μ with the weight $(p_1, p_2, \dots, p_N)$ on the hierarchy of sets by repeated subdivision of the measure in the ratio $p_1 : p_2 : \dots : p_N$, so that

$$\mu(K_{w_1 w_2 \dots w_m}) = p_{w_1} p_{w_2} \dots p_{w_m}$$

and this extends to a Borel measure supported by K , [2, Ch. 2.2].

FIGURE 1. The snowflaked transform

If $\{F_i\}_{i=1}^N$ are similarities with factors r_i , $i = 1, 2, \dots, N$ and s is the unique number with $\sum_{i=1}^N r_i^s = 1$, then μ with weight $(r_1^s, r_2^s, \dots, r_N^s)$ is the measure equivalent to the restriction $H^s|_K$ of the Hausdorff measure H^s in $\mathbb{R}^n$ to K , [4, Theorem 1.5.7]. When K is the Koch curve Γ , then the measure μ with the weight $(\frac{1}{2}, \frac{1}{2})$ is equivalent to $H^{\frac{\ln 4}{\ln 3}}|_\Gamma$. When K is the unit interval I , measure ν with the weight $(\frac{1}{2}, \frac{1}{2})$ is the Lebesgue measure.

Since the image of the measure ν under a mapping H is the measure μ , one has for a function $\tilde{f}$ defined on Γ

$$(7) \quad \int_{\Gamma} \tilde{f}(\gamma) \mu(d\gamma) = \int_0^1 (\tilde{f} \circ H)(x) \nu(dx) = \int_0^1 (\tilde{f} \circ H)(x) dx,$$

[5, Theorem 1.19].

3.3 Wavelet characterization of $B_p^s(\mathbb{T})$ and $\mathbb{B}_p^s(\Gamma)$. Let

$$\mathbb{B}_{pq}^s(\Gamma) = \{f \circ H^{-1} : f \in B_{pq}^s(\mathbb{T})\} = B_{pq}^s(\mathbb{T}) \circ H^{-1}.$$

By $B_p^s(\mathbb{T})$ and $\mathbb{B}_p^s(\Gamma)$, $1 < p < \infty$, $0 < s < 1$ we denote the spaces $B_{pp}^s(\mathbb{T})$ and $\mathbb{B}_{pp}^s(\Gamma)$ respectively. We are interested in wavelet expansions for the spaces $\mathbb{B}_p^s(\Gamma)$. We start with the wavelet characterization of $B_p^s(\mathbb{T})$ and then transfer it with the help of mapping H to Γ .

Let $C^u(\mathbb{R})$, $u \in \mathbb{N}$ denote the collection of all complex-valued continuous functions on $\mathbb{R}$ having continuous bounded derivatives up to order u inclusively. Let $\psi_F \in C^u(\mathbb{R})$ and $\psi_M \in C^u(\mathbb{R})$ be a father and a mother Daubechies wavelet on $\mathbb{R}$ respectively. Define ψ_F^k and ψ_j^k by

$$\psi_F^k = \psi_F(x - k), \quad \psi_j^k(x) = 2^{\frac{j}{2}} \psi_M(2^j x - k), \quad j \in \mathbb{N}_0, \quad k \in \mathbb{Z}.$$

Then $\{\psi_F^k, \psi_j^k\}_{j \in \mathbb{N}_0, k \in \mathbb{Z}}$ is an orthonormal system in $L_2(\mathbb{R})$. Let $L \in \mathbb{N}$. One can replace ψ_F and ψ_M by

$$\psi_F^L(\cdot) = \psi_F(2^L \cdot), \quad \psi_M^L(\cdot) = \psi_M(2^L \cdot),$$

ψ_F^k and ψ_j^k by

$$\psi_F^{L,k}(x) = 2^{\frac{L}{2}} \psi_F(2^L x - k), \quad \psi_j^{L,k}(x) = 2^{\frac{j+L}{2}} \psi_M(2^{j+L} x - k), \quad k \in \mathbb{Z}.$$

We choose and fix L such that

$$\text{supp } \psi_F^L \subset \left\{x : |x| < \frac{1}{2}\right\}, \quad \text{supp } \psi_M^L \subset \left\{x : |x| < \frac{1}{2}\right\}.$$

Then one has

$$\text{supp } \psi_j^{L,0} = \text{supp } \psi_M^L(2^j \cdot) \subset \left\{x : |x| < \frac{1}{2^{j+1}}\right\}.$$

Let

$$\mathbb{P}_j = \{m \in \mathbb{Z} : 0 \leq m < 2^{j+L}\}, \quad j \in \mathbb{N}_0.$$

Given the functions $\psi_F^{L,k}$, $\psi_j^{L,k}$ on the real line we can construct their 1-periodic counterparts by the procedure

$$\psi_{F,per}^{L,k}(x) = \sum_{l=-\infty}^{\infty} \psi_F^{L,k}(x + l), \quad \psi_{j,per}^{L,k}(x) = \sum_{l=-\infty}^{\infty} \psi_j^{L,k}(x + l).$$

Define $\psi_F^{L,k,per}$ and $\psi_j^{L,k,per}$ on the 1-torus $\mathbb{T}$ by

$$\psi_F^{L,k,per}(x) = \psi_{F,per}^{L,k}(x), \quad \psi_j^{L,k,per}(x) = \psi_{j,per}^{L,k}(x), \quad x \in \mathbb{T}.$$

Then according to the Proposition 1.34 in [8]

$$\left\{ \psi_F^{L,k',per}, \psi_j^{k,L,per}, k' \in \mathbb{P}_0, j \in \mathbb{N}_0, k \in \mathbb{P}_j \right\}$$

is an orthornormal basis in $L_2(\mathbb{T})$.

It is easy to see that

$$\psi_j^{L,k,per}(x) = \psi_j^{L,0,per}\left(x - \frac{k}{2^{j+L}}\right), \quad k \in \mathbb{P}_j,$$

on $\mathbb{T}$ with the usual interpretation.

The shift operation is well-defined on the real line, but it can not be defined on the Koch curve. Therefore we would like to replace this operation in order to be able to construct its counterpart on the Koch curve. First of all when j is fixed, the 1-torus $\mathbb{T}$ treated as a self-similar set can be represented as follows

$$\mathbb{T} = \bigcup_{w \in W_{j+L}} \mathbb{T}_w.$$

Let us introduce the order relation on the set W_{j+L} of words of length $j+L$. We say that $v = (v_1, \dots, v_{j+L})$ is less than $w = (w_1, \dots, w_{j+L})$ if and only if the first v_i which is different from w_i is less than w_i :

$$v < w \Leftrightarrow v_{\min\{i:w_i \neq v_i\}} < w_{\min\{i:w_i \neq v_i\}}.$$

The words are ordered in such a way that whenever u follows w in W_{j+L} , the interval $\mathbb{T}_u$ is the right neighbour of $\mathbb{T}_w$. We agree that $\mathbb{T}_{\underbrace{11\dots 1}_{j+L}}$ is the

right neighbour of $\mathbb{T}_{\underbrace{22\dots 2}_{j+L}}$.

Then we notice that

$$x - \frac{k}{2^{j+L}} = 2^{-(j+L)} (2^{j+L}x - k).$$

When k is fixed, there is a unique sequence of contractions

$$T_{w_1}, T_{w_2}, \dots, T_{w_{j+L}}, \quad w = (w_1, w_2, \dots, w_{j+L}) \in W_{j+L}$$

such that

$$(8) \quad 2^{-(j+L)} (2^{j+L}x - k) = T_1^{j+L} \circ T_{w_{j+L}}^{-1} \circ \dots \circ T_{w_2}^{-1} \circ T_{w_1}^{-1}(x).$$

The mapping (8) transfers $\mathbb{T}_{w_1 w_2 \dots w_{j+L}}$ to $\mathbb{T}_{\underbrace{11 \dots 1}_{j+L}}$. The values of $\psi_j^{L,k,per}$ on $\mathbb{T}_u$, where $u = (u_1, u_2, \dots, u_{j+L})$ follows w , coincide with the values of $\psi_j^{L,0,per}$ on $\mathbb{T}_{\underbrace{11 \dots 12}_{j+L-1}}$. Thus there is the following connection between functions $\psi_j^{L,k,per}$ and $\psi_j^{L,0,per}$:

$$\begin{aligned} \psi_j^{L,k,per}(x) &= \\ &= \begin{cases} \psi_j^{L,0,per} \circ T_1^{j+L} \circ T_{w_{j+L}}^{-1} \circ \dots \circ T_{w_2}^{-1} \circ T_{w_1}^{-1}(x), & \text{if } x \in \mathbb{T}_{w_1 w_2 \dots w_{j+L}}, \\ \psi_j^{L,0,per} \circ T_1^{j+L-1} \circ T_2 \circ T_{u_{j+L}}^{-1} \circ \dots \circ T_{u_2}^{-1} \circ T_{u_1}^{-1}(x), & \text{if } x \in \mathbb{T}_{u_1 u_2 \dots u_{j+L}}, \\ \dots \end{cases} \end{aligned}$$

where $\dots$ indicates the procedure of assigning to $\psi_j^{L,k,per}$ on each next right neighbour of $\mathbb{T}_u$ the values of $\psi_j^{L,0,per}$ on each next right neighbour of $\mathbb{T}_{\underbrace{11 \dots 12}_{j+L-1}}$.

Let us simplify the notation and denote the functions $\psi_F^{L,k',per}$ and $\psi_j^{L,k,per}$ by

$$\begin{aligned} \psi_{F,w} &= \psi_{F,w_1 w_2 \dots w_L}, \quad w \in W_L, \\ \psi_w &= \psi_{w_1 w_2 \dots w_{j+L}}, \quad j \in \mathbb{N}_0, w \in W_{j+L} \end{aligned}$$

respectively, where w is chosen according to (8). Now we transfer the functions $\psi_{F,w}$, $w \in W_L$, ψ_w , $w \in \bigcup_{j \in \mathbb{N}_0} W_{j+L}$ from $\mathbb{T}$ to the Koch curve

Γ . Define $\tilde{\psi}_{F,w}$ and $\tilde{\psi}_w$ by

$$\begin{aligned} \tilde{\psi}_{F,w}(\gamma) &= \psi_{F,w} \circ H^{-1}(\gamma), \\ \tilde{\psi}_w(\gamma) &= \psi_w \circ H^{-1}(\gamma). \end{aligned}$$

From (7) follows that the system $\left\{ \tilde{\psi}_{F,v}, \tilde{\psi}_w \right\}_{v \in W_L, w \in \bigcup_{j \in \mathbb{N}_0} W_{j+L}}$ is orthonormal in $L_2(\Gamma, \mu)$.

Let $\gamma \in \Gamma_{w_1 w_2 \dots w_{j+L}}$. Then there is an $\omega \in \Sigma$ such that

$$\gamma = \bigcap_{m=1}^{\infty} \Gamma_{w_1 w_2 \dots w_{j+L} \omega_1 \omega_2 \dots \omega_m}.$$

Recall that $\gamma \in \Gamma$ corresponds to $x = H^{-1}(\gamma) \in \mathbb{T}_{w_1 w_2 \dots w_{j+L}}$ given by

$$x = \bigcap_{m=1}^{\infty} \mathbb{T}_{w_1 w_2 \dots w_{j+L} \omega_1 \omega_2 \dots \omega_m}.$$

The connection between the values of $\tilde{\psi}_{w_1 w_2 \dots w_{j+L}}$ and $\underbrace{\tilde{\psi}_{11 \dots 1}}_{j+L}$ on $\Gamma_{w_1 w_2 \dots w_{j+L}}$ is the following:

$$\begin{aligned} \tilde{\psi}_{w_1 w_2 \dots w_{j+L}}(\gamma) &= \psi_{w_1 w_2 \dots w_{j+L}} \circ H^{-1}(\gamma) = \\ &= \underbrace{\psi_{11 \dots 1}}_{j+L} \circ T_1^{j+L} \circ T_{w_{j+L}}^{-1} \circ \dots \circ T_{w_2}^{-1} \circ T_{w_1}^{-1} \left(\bigcap_{m=1}^{\infty} \mathbb{T}_{w_1 w_2 \dots w_{j+L} \omega_1 \omega_2 \dots \omega_m} \right) = \\ &= \underbrace{\psi_{11 \dots 1}}_{j+L} \left(\bigcap_{m=1}^{\infty} \mathbb{T}_{\underbrace{11 \dots 1}_{j+L} \omega_1 \omega_2 \dots \omega_m} \right) = \underbrace{\psi_{11 \dots 1}}_{j+L} \circ H^{-1} \left(\bigcap_{m=1}^{\infty} \Gamma_{\underbrace{11 \dots 1}_{j+L} \omega_1 \omega_2 \dots \omega_m} \right) = \\ &= \underbrace{\tilde{\psi}_{11 \dots 1}}_{j+L} \left(\bigcap_{m=1}^{\infty} \Gamma_{\underbrace{11 \dots 1}_{j+L} \omega_1 \omega_2 \dots \omega_m} \right) = \\ &= \underbrace{\tilde{\psi}_{11 \dots 1}}_{j+L} \circ F_1^{j+L} \circ F_{w_{j+L}}^{-1} \circ \dots \circ F_{w_2}^{-1} \circ F_{w_1}^{-1} \left(\bigcap_{m=1}^{\infty} \Gamma_{w_1 w_2 \dots w_{j+L} \omega_1 \omega_2 \dots \omega_m} \right) = \\ &= \underbrace{\tilde{\psi}_{11 \dots 1}}_{j+L} \circ F_1^{j+L} \circ F_{w_{j+L}}^{-1} \circ \dots \circ F_{w_2}^{-1} \circ F_{w_1}^{-1}(\gamma). \end{aligned}$$

In the same way we can follow the connection between the values of $\tilde{\psi}_{w_1 w_2 \dots w_{j+L}}$ and $\underbrace{\tilde{\psi}_{11 \dots 1}}_{j+L}$ on other Γ_v , $v \in W_{j+L}$. So we get

$$\begin{aligned} \tilde{\psi}_w(\gamma) &= \\ &= \begin{cases} \underbrace{\tilde{\psi}_{11 \dots 1}}_{j+L} \circ F_1^{j+L} \circ F_{w_{j+L}}^{-1} \circ \dots \circ F_{w_2}^{-1} \circ F_{w_1}^{-1}(\gamma), & \text{if } \gamma \in \Gamma_{w_1 w_2 \dots w_{j+L}}, \\ \underbrace{\tilde{\psi}_{11 \dots 1}}_{j+L} \circ F_1^{j+L-1} \circ F_2 \circ F_{u_{j+L}}^{-1} \circ \dots \circ F_{u_2}^{-1} \circ F_{u_1}^{-1}(\gamma), & \text{if } \gamma \in \Gamma_{u_1 u_2 \dots u_{j+L}}, \\ \dots & \end{cases} \end{aligned}$$

Similarly for $\tilde{\psi}_{F,w}$.

According to [8, Ch. 1.3.3] the following theorem holds.

Theorem 1. *Let $1 < p < \infty$, $0 < s < 1$ and $u > s$. Let $f \in L_p(\mathbb{T})$. Then $f \in B_p^s(\mathbb{T})$ if, and only if, it can be represented as*

$$f = \sum_{w \in W_L} a_w 2^{-\frac{L}{2}} \psi_{F,w} + \sum_{j=0}^{\infty} \sum_{w \in W_{j+L}} b_w 2^{-\frac{j+L}{2}} \psi_w,$$

unconditional convergence being in $L_p(\mathbb{T})$. Furthermore this representation is unique,

$$\begin{aligned} a_w &= 2^{\frac{L}{2}} (f, \psi_{F,w})_{\mathbb{T}}, & w \in W_L, \\ b_w &= 2^{\frac{j+L}{2}} (f, \psi_w)_{\mathbb{T}}, & w \in W_{j+L} \end{aligned}$$

and

$$I : f \rightarrow \{a_w, w \in W_L, b_w, w \in W_{j+L}, j \in \mathbb{N}_0\}$$

is an isomorphic map of $B_p^s(\mathbb{T})$ onto the sequence space such that

$$\sum_{w \in W_L} |a_w|^p + \sum_{j=0}^{\infty} 2^{j(s-\frac{1}{p})p} \sum_{w \in W_{j+L}} |b_w|^p < \infty.$$

Similar theorem holds for the spaces $\mathbb{B}_p^s(\Gamma)$.

Theorem 2. *Let $1 < p < \infty$, $0 < s < 1$ and $u > s$. Let $\tilde{f} \in L_p(\Gamma)$. Then $\tilde{f} \in \mathbb{B}_p^s(\Gamma)$ if, and only if, it can be represented as*

$$\tilde{f} = \sum_{w \in W_L} a_w 2^{-\frac{L}{2}} \tilde{\psi}_{F,w} + \sum_{j=0}^{\infty} \sum_{w \in W_{j+L}} b_w 2^{-\frac{j+L}{2}} \tilde{\psi}_w,$$

unconditional convergence being in $L_p(\Gamma)$. Furthermore this representation is unique,

$$\begin{aligned} a_w &= 2^{\frac{L}{2}} (\tilde{f}, \tilde{\psi}_{F,w})_{\Gamma}, & w \in W_L, \\ b_w &= 2^{\frac{j+L}{2}} (\tilde{f}, \tilde{\psi}_w)_{\Gamma}, & w \in W_{j+L} \end{aligned}$$

and

$$I : \tilde{f} \rightarrow \{a_w, w \in W_L, b_w, w \in W_{j+L}, j \in \mathbb{N}_0\}$$

is an isomorphic map of $\mathbb{B}_p^s(\Gamma)$ onto the sequence space such that

$$\sum_{w \in W_L} |a_w|^p + \sum_{j=0}^{\infty} 2^{j(s-\frac{1}{p})p} \sum_{w \in W_{j+L}} |b_w|^p < \infty.$$

4. Comparison of $B_p^s(\Gamma)$ and $\mathbb{B}_p^s(\Gamma)$

The spaces $B_p^s(\mathbb{T})$ can be normed by

$$(9) \quad \|f|B_p^s(\mathbb{T})\|_*^p = \int_0^1 |f(x)|^p dx + \int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x - y|^{1+sp}} dx dy.$$

Since (9) is equivalent to

$$\int_{\Gamma} |\tilde{f}(\gamma)|^p \mu(d\gamma) + \int_{\Gamma} \int_{\Gamma} \frac{|\tilde{f}(\gamma) - \tilde{f}(\delta)|^p}{|\gamma - \delta|^{d+sp}} \mu(d\gamma) \mu(d\delta),$$

where $\tilde{f} = f \circ H^{-1}$, we endow the spaces $\mathbb{B}_p^s(\Gamma)$ with the norm

$$(10) \quad \|\tilde{f}|\mathbb{B}_p^s(\Gamma)\|_*^p = \int_{\Gamma} |\tilde{f}(\gamma)|^p \mu(d\gamma) + \int_{\Gamma} \int_{\Gamma} \frac{|\tilde{f}(\gamma) - \tilde{f}(\delta)|^p}{|\gamma - \delta|^{d+sp}} \mu(d\gamma) \mu(d\delta).$$

Together with (5) this leads to

$$(11) \quad B_p^s(\Gamma) = \mathbb{B}_p^{\frac{s}{d}}(\Gamma).$$

The analogue of Theorem 2 for the spaces $B_p^s(\Gamma)$ reads as follows

Theorem 3. *Let $1 < p < \infty$, $0 < s < 1$ and $u > s$. Let $\tilde{f} \in L_p(\Gamma)$. Then $\tilde{f} \in B_p^s(\Gamma)$ if, and only if, it can be represented as*

$$\tilde{f} = \sum_{w \in W_L} a_w 2^{-\frac{L}{2}} \tilde{\psi}_{F,w} + \sum_{j=0}^{\infty} \sum_{w \in W_{j+L}} b_w 2^{-\frac{j+L}{2}} \tilde{\psi}_w,$$

unconditional convergence being in $L_p(\Gamma)$. Furthermore this representation is unique,

$$a_w = 2^{\frac{L}{2}} (\tilde{f}, \tilde{\psi}_{F,w})_{\Gamma}, \quad w \in W_L,$$

$$b_w = 2^{\frac{j+L}{2}} (\tilde{f}, \tilde{\psi}_w)_{\Gamma}, \quad w \in W_{j+L}$$

and

$$I : \tilde{f} \rightarrow \{a_w, w \in W_L, b_w, w \in W_{j+L}, j \in \mathbb{N}_0\}$$

is an isomorphic map of $B_p^s(\Gamma)$ onto the sequence space such that

$$\sum_{w \in W_L} |a_w|^p + \sum_{j=0}^{\infty} \left(\sqrt{3} \right)^{j(s-\frac{d}{p})p} \sum_{w \in W_{j+L}} |b_w|^p < \infty.$$

Proof. This follows from the observation (11) and

$$2^{j(\frac{s}{d}-\frac{1}{p})} = 2^{\frac{1}{d}j(s-\frac{d}{p})} = \left(\sqrt{3}\right)^{j(s-\frac{d}{p})}.$$

□

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Mathematical Institute
 Friedrich Schiller University Jena
 D-07737 Jena, Germany
 (E-mail : maryia.kabanava@uni-jena.de)

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