Boundedness of Lusin-area and g_λ^* functions on localized Morrey-Campanato spaces over doubling metric measure spaces

Haibo Lin, Eiichi Nakai and Dachun Yang¹

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Abstract. Let $\mathcal{X}$ be a doubling metric measure space and ρ an admissible function on $\mathcal{X}$. In this paper, the authors establish some equivalent characterizations for the localized Morrey-Campanato spaces $\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$ and Morrey-Campanato-BLO spaces $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$ when $\alpha \in (-\infty, 0)$ and $p \in [1, \infty)$. If $\mathcal{X}$ has the volume regularity Property (P) , the authors then establish the boundedness of the Lusin-area function, which is defined via kernels modeled on the semigroup generated by the Schrödinger operator, from $\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$ to $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$ without invoking any regularity of considered kernels. The same is true for the g_λ^* function and, unlike the Lusin-area function, in this case, $\mathcal{X}$ is even not necessary to have Property (P) . These results are also new even for $\mathbb{R}^d$ with the d -dimensional Lebesgue measure and have a wide applications.

1. Introduction

The theory of Morrey-Campanato spaces plays an important role in harmonic analysis and partial differential equations; see, for example, [1, 5, 16, 22, 23, 25, 27, 29, 31] and their references. It is well-known that

¹Corresponding author

the dual space of the Hardy space $H^p(\mathbb{R}^d)$ with $p \in (0, 1)$ is the Morrey-Campanato space $\mathcal{E}^{1/p-1, 1}(\mathbb{R}^d)$. Notice that the Morrey-Campanato spaces on $\mathbb{R}^d$ are essentially related to the Laplacian $\Delta \equiv \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$.

On the other hand, there exists an increasing interest on the study of Schrödinger operators on $\mathbb{R}^d$ and the sub-Laplace Schrödinger operators on connected and simply connected nilpotent Lie groups with nonnegative potentials satisfying the reverse Hölder inequality; see, for example, [6, 7, 8, 9, 17, 18, 26, 35, 37]. Let $\mathcal{L} \equiv -\Delta + V$ be the Schrödinger operator on $\mathbb{R}^d$, where the potential V is a nonnegative locally integrable function. Denote by $\mathcal{B}_q(\mathbb{R}^d)$ the class of nonnegative functions satisfying the reverse Hölder inequality of order q . For $V \in \mathcal{B}_{d/2}(\mathbb{R}^d)$ with $d \geq 3$, Dziubański et al. [6, 7, 8] studied the BMO-type space $\text{BMO}_{\mathcal{L}}(\mathbb{R}^d)$ and the Hardy space $H_{\mathcal{L}}^p(\mathbb{R}^d)$ with $p \in (d/(d+1), 1]$ and, especially, proved that the dual space of $H_{\mathcal{L}}^1(\mathbb{R}^d)$ is $\text{BMO}_{\mathcal{L}}(\mathbb{R}^d)$; moreover, they obtained the boundedness on these spaces of the Littlewood-Paley g -function associated to $\mathcal{L}$.

Let $\mathcal{X}$ be a doubling metric measure space, which means that $\mathcal{X}$ is a space of homogeneous type in the sense of Coifman and Weiss [2, 3], but $\mathcal{X}$ is endowed with a metric instead of a quasi-metric. Let ρ be a given admissible function modeled on the known auxiliary function determined by $V \in \mathcal{B}_{d/2}(\mathbb{R}^d)$ (see [35] or (2.4) below). The localized atomic Hardy space $H_{\rho}^{p,q}(\mathcal{X})$ with $p \in (0, 1]$ and $q \in [1, \infty] \cap (p, \infty]$, the localized Morrey-Campanato space $\mathcal{E}_{\rho}^{\alpha,p}(\mathcal{X})$ and localized Morrey-Campanato-BLO space $\tilde{\mathcal{E}}_{\rho}^{\alpha,p}(\mathcal{X})$ with $\alpha \in \mathbb{R}$ and $p \in (0, \infty)$ were introduced in [34]. Moreover, the boundedness from $\mathcal{E}_{\rho}^{\alpha,p}(\mathcal{X})$ to $\tilde{\mathcal{E}}_{\rho}^{\alpha,p}(\mathcal{X})$ of several maximal operators and the Littlewood-Paley g -function, which are defined via kernels modeled on the semigroup generated by the Schrödinger operator, was obtained in [34]. Meanwhile, the boundedness from localized BMO-type space $\text{BMO}_{\rho}(\mathcal{X})$ to BLO-type space $\text{BLO}_{\rho}(\mathcal{X})$ of the Lusin-area and g_λ^* functions was established in [19].

The purpose of this paper is to investigate behaviors of the Lusin-area and g_λ^* functions on Morrey-Campanato spaces over doubling metric measure spaces. Precisely, let $\mathcal{X}$ be a doubling metric measure space and ρ an admissible function on $\mathcal{X}$. In this paper, we first establish some equivalent characterizations for $\mathcal{E}_{\rho}^{\alpha,p}(\mathcal{X})$ and $\tilde{\mathcal{E}}_{\rho}^{\alpha,p}(\mathcal{X})$ when $\alpha \in (-\infty, 0)$ and $p \in [1, \infty)$. To obtain the boundedness of the Lusin-area function on the Morrey-Campanato spaces, we need to assume that $\mathcal{X}$ has the volume regularity Property (P), which was introduced in [19], motivated by Colding-Minicozzi II [4] and Tessera [30]. We remark that the volume regularity property is related to the Følner sequence of a compact generating set of a compactly generated locally compact group with polynomial growth

in [30] and used to establish the generalized Liouville theorems for harmonic sections of Hermitian vector bundles over a complete metric space in [4].

In this paper, if $\mathcal{X}$ has Property (P), we then establish the boundedness of the Lusin-area function from $\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$ to $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$ without invoking any regularity of considered kernels. The corresponding boundedness of g_λ^* function from $\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$ to $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$ is also established in this paper. Both the Lusin-area function and the g_λ^* function are defined via kernels modeled on the semigroup generated by the Schrödinger operator. Moreover, an interesting phenomena is that unlike the Lusin-area function, the boundedness of the g_λ^* function needs neither the regularity of the kernels nor Property (P) of $\mathcal{X}$, which reflects the speciality of the structure of the g_λ^* function. These results are new even on $\mathbb{R}^d$ with the d -dimensional Lebesgue measure and the Heisenberg group, and apply in a wide range of settings, for instance, to the Schrödinger operator or the degenerate Schrödinger operator on $\mathbb{R}^d$, or the sub-Laplace Schrödinger operator on Heisenberg groups or connected and simply connected nilpotent Lie groups.

This paper is organized as follows. Let $\mathcal{X}$ be a doubling metric measure space and ρ an admissible function on $\mathcal{X}$. In Section 2, we establish some equivalent characterizations for $\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$ and $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$ when $p \in [1, \infty)$ and $\alpha \in (-\infty, -1/p)$ or $\alpha \in [-1/p, 0)$; see Theorems 2.1 and 2.2 below. Moreover, under the assumption that $\sup_{x \in \mathcal{X}} \mu(B(x, \rho(x))) = \infty$, we prove that the Morrey-Campanato-BLO space $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$ is a proper subspace of the Morrey-Campanato space $\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$ when $p \in [1, \infty)$ and $\alpha \in [-1/p, 0)$; see Theorem 2.2(iii) below.

In Section 3, assuming that $\mathcal{X}$ has Property (P) and the Lusin-area function $S(f)$ is bounded on $L^p(\mathcal{X})$ with $p \in (1, \infty)$, we prove that if $f \in \mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$, then $[S(f)]^2 \in \tilde{\mathcal{E}}_\rho^{2\alpha,p/2}(\mathcal{X})$ with norm no more than $C\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}^2$, where C is a positive constant independent of f ; see Theorem 3.1 below. As a corollary, we obtain the boundedness of the Lusin-area function from $\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$ to $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$; see Corollary 3.1 below. If the g_λ^* function $g_\lambda^*(f)$ is bounded on $L^p(\mathcal{X})$ with $p \in (1, \infty)$, the corresponding results for $g_\lambda^*(f)$ are also established, and moreover, in this case, $\mathcal{X}$ is not necessary to have Property (P); see Theorem 3.2 and Corollary 3.2 below. We point out that Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.2 are true to the Schrödinger operator or the degenerate Schrödinger operator on $\mathbb{R}^d$, or the sub-Laplace Schrödinger operator on Heisenberg groups or connected and simply connected nilpotent Lie groups; see [34] for the detailed explanations. Notice that $\mathcal{E}_\rho^{0,p}(\mathcal{X}) = \text{BMO}_\rho(\mathcal{X})$ and $\tilde{\mathcal{E}}_\rho^{0,p}(\mathcal{X}) = \text{BLO}_\rho(\mathcal{X})$ when $p \in [1, \infty)$. Thus, the results in this section when $\alpha = 0$ were already obtained in [19].

We remark that the results obtained in Section 3 are also new even on $\mathbb{R}^d$ with the d -dimensional Lebesgue measure and the Heisenberg group, since we do not need any regularity of involved kernels. However, to establish the boundedness of Lusin-area function on a doubling metric measure space $\mathcal{X}$, we need certain regularity of $\mathcal{X}$, namely, the volume regularity Property (P), which reflects the speciality of the Lusin-area function, comparing with the corresponding results of the g_λ^* function. Moreover, $\mathbb{R}^d$ with the Lebesgue measure and the Heisenberg group have the volume regularity Property (P); see [19].

Finally, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_1 and K_1 , do not change in different occurrences. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$. We also use B to denote a ball of $\mathcal{X}$, and for $\lambda > 0$, λB denotes the ball with the same center as B , but radius λ times the radius of B . Moreover, set $B^c \equiv \mathcal{X} \setminus B$. Also, for any set $E \subset \mathcal{X}$, χ_E denotes its characteristic function. For all $f \in L^1_{\text{loc}}(\mathcal{X})$ and balls B , we always set $f_B \equiv \frac{1}{\mu(B)} \int_B f(y) d\mu(y)$.

2. Some characterizations of localized Morrey-Campanato spaces

Let $\mathcal{X}$ be a doubling metric measure space and ρ an admissible function on $\mathcal{X}$. In this section, we establish some equivalent characterizations for $\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$ and $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$ when $\alpha \in (-\infty, 0)$ and $p \in [1, \infty)$. Moreover, under the assumption that $\sup_{x \in \mathcal{X}} \mu(B(x, \rho(x))) = \infty$, we prove that the Morrey-Campanato-BLO space $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$ is a proper subspace of the Morrey-Campanato space $\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$ when $p \in [1, \infty)$ and $\alpha \in [-1/p, 0)$.

We begin with recalling the notion of doubling metric measure spaces [2, 3].

Definition 2.1. Let $(\mathcal{X}, d)$ be a metric space endowed with a Borel regular measure μ such that all balls defined by d have finite and positive measures. For any $x \in \mathcal{X}$ and $r \in (0, \infty)$, set the ball $B(x, r) \equiv \{y \in \mathcal{X} : d(x, y) < r\}$. The triple $(\mathcal{X}, d, \mu)$ is called a doubling metric measure space if there exists a constant $C_1 \in [1, \infty)$ such that for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$(2.1) \quad \mu(B(x, 2r)) \leq C_1 \mu(B(x, r)) \quad (\text{doubling property}).$$

From Definition 2.1, it is easy to see that there exist positive constants C_2 and n such that for all $x \in \mathcal{X}$, $r \in (0, \infty)$ and $\lambda \in [1, \infty)$,

$$(2.2) \quad \mu(B(x, \lambda r)) \leq C_2 \lambda^n \mu(B(x, r)).$$

Now we recall the notion of admissible functions introduced in [35].

Definition 2.2 ([35]). A positive function ρ on $\mathcal{X}$ is called admissible if there exist positive constants C_0 and k_0 such that for all $x, y \in \mathcal{X}$,

$$(2.3) \quad \frac{1}{\rho(x)} \leq C_0 \frac{1}{\rho(y)} \left(1 + \frac{d(x, y)}{\rho(y)} \right)^{k_0}.$$

Obviously, if ρ is a constant function, then ρ is admissible. Moreover, let $x_0 \in \mathcal{X}$ be fixed. The function $\rho(y) \equiv (1 + d(x_0, y))^s$ for all $y \in \mathcal{X}$ with $s \in (-\infty, 1)$ also satisfies Definition 2.2 with $k_0 = s/(1-s)$ when $s \in [0, 1)$ and $k_0 = -s$ when $s \in (-\infty, 0)$. Another non-trivial class of admissible functions is given by the well-known reverse Hölder class $\mathcal{B}_q(\mathcal{X}, d, \mu)$, which is written as $\mathcal{B}_q(\mathcal{X})$ for simplicity. Recall that a nonnegative potential V is said to be in $\mathcal{B}_q(\mathcal{X})$ with $q \in (1, \infty]$ if there exists a positive constant C such that for all balls B of $\mathcal{X}$,

$$\left(\frac{1}{|B|} \int_B [V(y)]^q dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) dy$$

with the usual modification made when $q = \infty$. It is known that if $V \in \mathcal{B}_q(\mathcal{X})$ for certain $q \in (1, \infty]$, then V is an $A_\infty(\mathcal{X})$ weight in the sense of Muckenhoupt, and also $V \in \mathcal{B}_{q+\epsilon}(\mathcal{X})$ for certain $\epsilon \in (0, \infty)$; see, for example, [27] and [28]. Thus $\mathcal{B}_q(\mathcal{X}) = \cup_{q_1 \in (q, \infty]} \mathcal{B}_{q_1}(\mathcal{X})$. For all $V \in \mathcal{B}_q(\mathcal{X})$ with $q \in (1, \infty]$ and all $x \in \mathcal{X}$, set

$$(2.4) \quad \rho(x) \equiv \sup \left\{ r > 0 : \frac{r^2}{\mu(B(x, r))} \int_{B(x, r)} V(y) dy \leq 1 \right\};$$

see, for example, [26] and also [35]. It was also proved in [35] that ρ in (2.4) is an admissible function if $n \in [1, \infty)$, $q \in (\max\{1, n/2\}, \infty]$ and $V \in \mathcal{B}_q(\mathcal{X})$.

The following localized Morrey-Campanato space and localized Morrey-Campanato-BLO space associated to the admissible function ρ were first introduced in [34].

Definition 2.3. Let ρ be an admissible function on $\mathcal{X}$, $\mathcal{D} \equiv \{B(x, r) : x \in \mathcal{X}, r \geq \rho(x)\}$, $p \in (0, \infty)$ and $\alpha \in \mathbb{R}$. Denote by B any ball of $\mathcal{X}$.

(i) A function $f \in L_{\text{loc}}^p(\mathcal{X})$ is said to be in the localized Morrey-Campanato space $\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})$ if

$$\|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})} \equiv \sup_{B \notin \mathcal{D}} \left\{ \frac{1}{[\mu(B)]^{1+p\alpha}} \int_B |f(y) - f_B|^p d\mu(y) \right\}^{1/p} + \sup_{B \in \mathcal{D}} \left\{ \frac{1}{[\mu(B)]^{1+p\alpha}} \int_B |f(y)|^p d\mu(y) \right\}^{1/p} < \infty.$$

(ii) A function $f \in L_{\text{loc}}^p(\mathcal{X})$ is said to be in the localized Morrey-Campanato-BLO space $\tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X})$ if

$$\|f\|_{\tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X})} \equiv \sup_{B \notin \mathcal{D}} \left\{ \frac{1}{[\mu(B)]^{1+p\alpha}} \int_B [f(y) - \text{essinf}_B f]^p d\mu(y) \right\}^{1/p} + \sup_{B \in \mathcal{D}} \left\{ \frac{1}{[\mu(B)]^{1+p\alpha}} \int_B |f(y)|^p d\mu(y) \right\}^{1/p} < \infty.$$

(iii) Let $\alpha \in (0, \infty)$. A function f on $\mathcal{X}$ is said to be in the localized Lipschitz space $\text{Lip}_\rho(\alpha; \mathcal{X})$ if there exists a nonnegative constant C such that for all $x, y \in \mathcal{X}$ and balls B containing x and y with $B \notin \mathcal{D}$, $|f(x) - f(y)| \leq C[\mu(B)]^\alpha$, and that for all balls $B \in \mathcal{D}$, $\|f\|_{L^\infty(B)} \leq C[\mu(B)]^\alpha$. The minimal nonnegative constant C as above is called the norm of f in $\text{Lip}_\rho(\alpha; \mathcal{X})$ and denoted by $\|f\|_{\text{Lip}_\rho(\alpha; \mathcal{X})}$.

(iv) Let $\alpha \in (-\infty, 0)$. A function $f \in L_{\text{loc}}^p(\mathcal{X})$ is said to be in the Morrey space $\mathcal{L}^{\alpha, p}(\mathcal{X})$ if

$$\|f\|_{\mathcal{L}^{\alpha, p}(\mathcal{X})} \equiv \sup_{B \subset \mathcal{X}} \left\{ \frac{1}{[\mu(B)]^{1+\alpha p}} \int_B |f(x)|^p d\mu(x) \right\}^{1/p} < \infty.$$

Remark 2.1. (i) For all $\alpha \in \mathbb{R}$ and $p \in (0, \infty)$, $\tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X}) \subset \mathcal{E}_\rho^{\alpha, p}(\mathcal{X})$.

(ii) When $\alpha = 0$ and $p \in [1, \infty)$, we denote $\mathcal{E}_\rho^{0, p}(\mathcal{X})$ by $\text{BMO}_\rho^p(\mathcal{X})$ and $\text{BMO}_\rho^1(\mathcal{X})$ by $\text{BMO}_\rho(\mathcal{X})$. And we also denote $\tilde{\mathcal{E}}_\rho^{0, p}(\mathcal{X})$ by $\text{BLO}_\rho^p(\mathcal{X})$ and $\text{BLO}_\rho^1(\mathcal{X})$ by $\text{BLO}_\rho(\mathcal{X})$. The localized BLO space was first introduced in [13] in the setting of $\mathbb{R}^d$ endowed with a nondoubling measure.

(iii) If $\mathcal{X}$ is the Euclidean space $\mathbb{R}^d$ and $\rho \equiv 1$, then $\text{BMO}_\rho(\mathcal{X})$ is just the localized BMO space of Goldberg [10], and $\text{Lip}_\rho(\alpha; \mathcal{X})$ with $\alpha \in (0, 1)$ is just the inhomogeneous Lipschitz space (see also [10]).

(iv) When $\alpha \in (0, \infty)$, $\tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X}) = \mathcal{E}_\rho^{\alpha, p}(\mathcal{X}) = \text{Lip}_\rho(\alpha; \mathcal{X})$ with equivalent norms; see [34] for details.

Theorem 2.1. *Let $\mathcal{X}$ be a doubling metric measure space and ρ an admissible function on $\mathcal{X}$. Let $p \in [1, \infty)$ and $\alpha \in (-\infty, -1/p)$.*

- (i) If $\mu(\mathcal{X}) = \infty$, then $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X}) = \mathcal{E}_\rho^{\alpha,p}(\mathcal{X}) = \mathcal{L}^{\alpha,p}(\mathcal{X}) = \{0\}$.
(ii) If $\mu(\mathcal{X}) < \infty$, then $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X}) = \tilde{\mathcal{E}}_\rho^{-1/p,p}(\mathcal{X})$, $\mathcal{E}_\rho^{\alpha,p}(\mathcal{X}) = \mathcal{E}_\rho^{-1/p,p}(\mathcal{X})$ and $\mathcal{L}^{\alpha,p}(\mathcal{X}) = \mathcal{L}^{-1/p,p}(\mathcal{X}) = L^p(\mathcal{X})$ with equivalent norms, respectively.

Proof. (i) In this case, since $\mu(\mathcal{X}) = \infty$, then, for all $x \in \mathcal{X}$, $\mu(B(x, r)) \rightarrow \infty$ as $r \rightarrow \infty$, which together with $\alpha \in (-\infty, -1/p)$ implies that if $f \in \mathcal{L}^{\alpha,p}(\mathcal{X})$, then for all $B \equiv B(x, r) \subset \mathcal{X}$,

$$\int_{B(x, r)} |f(y)|^p d\mu(y) \leq \|f\|_{\mathcal{L}^{\alpha,p}(\mathcal{X})}^p [\mu(B(x, r))]^{1+\alpha p} \rightarrow 0,$$

when $r \rightarrow \infty$, and if $f \in \mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$, then for all $B \equiv B(x, r) \subset \mathcal{X}$ with $\rho(x) \leq r$,

$$\int_{B(x, r)} |f(y)|^p d\mu(y) \leq \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}^p [\mu(B(x, r))]^{1+\alpha p} \rightarrow 0,$$

as $r \rightarrow \infty$. Thus, in both cases, we have that $\int_{\mathcal{X}} |f(y)|^p d\mu(y) \leq 0$, which implies that $f(y) = 0$ for almost all $y \in \mathcal{X}$. Therefore, $\mathcal{L}^{\alpha,p}(\mathcal{X}) = \mathcal{E}_\rho^{\alpha,p}(\mathcal{X}) = \{0\}$, which together with the fact that $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X}) \subset \mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$ yields (i).

(ii) Since $\mu(\mathcal{X}) < \infty$, by [24, Lemmma 5.1], there exists $r_0 > 0$ such that $\mathcal{X} \subset B(x, r_0)$ for all $x \in \mathcal{X}$, which together with that ρ is admissible and (2.2) implies that

$$\begin{aligned} 0 < \inf_{x \in \mathcal{X}} \mu(B(x, \rho(x)/2)) &= \inf_{x \in \mathcal{X}, \rho(x)/2 \leq r < \rho(x)} \mu(B(x, r)) \\ &\leq \sup_{x \in \mathcal{X}, \rho(x)/2 \leq r < \rho(x)} \mu(B(x, r)) \leq \mu(\mathcal{X}) < \infty. \end{aligned}$$

If $0 < r < \rho(x)/2$, by $1 + \alpha p < 0$, we then have

$$\begin{aligned} &\frac{1}{[\mu(B(x, r))]^{1+\alpha p}} \int_{B(x, r)} \left[f(y) - \operatorname{essinf}_{B(x, r)} f \right]^p d\mu(y) \\ &\leq \frac{1}{[\mu(B(x, \rho(x)/2))]^{1+\alpha p}} \int_{B(x, \rho(x)/2)} \left[f(y) - \operatorname{essinf}_{B(x, \rho(x)/2)} f \right]^p d\mu(y), \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{[\mu(B(x, r))]^{1+\alpha p}} \int_{B(x, r)} |f(y) - f_{B(x, r)}|^p d\mu(y) \\ &\leq \frac{1}{[\mu(B(x, \rho(x)/2))]^{1+\alpha p}} \int_{B(x, \rho(x)/2)} |f(y) - f_{B(x, r)}|^p d\mu(y) \\ &\leq \frac{2^p}{[\mu(B(x, \rho(x)/2))]^{1+\alpha p}} \int_{B(x, \rho(x)/2)} |f(y) - f_{B(x, \rho(x)/2)}|^p d\mu(y). \end{aligned}$$

Hence,

$$\begin{aligned}
& \sup_{x \in \mathcal{X}, 0 < r < \rho(x)} \frac{1}{[\mu(B(x, r))]^{1+\alpha p}} \int_{B(x, r)} \left[f(y) - \operatorname{essinf}_{B(x, r)} f \right]^p d\mu(y) \\
&= \sup_{x \in \mathcal{X}, \rho(x)/2 \leq r < \rho(x)} \frac{1}{[\mu(B(x, r))]^{1+\alpha p}} \int_{B(x, r)} \left[f(y) - \operatorname{essinf}_{B(x, r)} f \right]^p d\mu(y) \\
&\sim \sup_{x \in \mathcal{X}, \rho(x)/2 \leq r < \rho(x)} \int_{B(x, r)} \left[f(y) - \operatorname{essinf}_{B(x, r)} f \right]^p d\mu(y) \\
&\sim \sup_{x \in \mathcal{X}, 0 < r < \rho(x)} \int_{B(x, r)} \left[f(y) - \operatorname{essinf}_{B(x, r)} f \right]^p d\mu(y),
\end{aligned}$$

and

$$\begin{aligned}
& \sup_{x \in \mathcal{X}, 0 < r < \rho(x)} \frac{1}{[\mu(B(x, r))]^{1+\alpha p}} \int_{B(x, r)} |f(y) - f_{B(x, r)}|^p d\mu(y) \\
&\sim \sup_{x \in \mathcal{X}, \rho(x)/2 \leq r < \rho(x)} \frac{1}{[\mu(B(x, r))]^{1+\alpha p}} \int_{B(x, r)} |f(y) - f_{B(x, r)}|^p d\mu(y) \\
&\sim \sup_{x \in \mathcal{X}, \rho(x)/2 \leq r < \rho(x)} \int_{B(x, r)} |f(y) - f_{B(x, r)}|^p d\mu(y) \\
&\sim \sup_{x \in \mathcal{X}, 0 < r < \rho(x)} \int_{B(x, r)} |f(y) - f_{B(x, r)}|^p d\mu(y),
\end{aligned}$$

which imply that $\tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X}) = \tilde{\mathcal{E}}_\rho^{-1/p, p}(\mathcal{X})$, and $\mathcal{E}_\rho^{\alpha, p}(\mathcal{X}) = \mathcal{E}_\rho^{-1/p, p}(\mathcal{X})$.

Similarly,

$$0 < \inf_{x \in \mathcal{X}, \rho(x) \leq r} \mu(B(x, r)) \leq \sup_{x \in \mathcal{X}, \rho(x) \leq r} \mu(B(x, r)) \leq \mu(\mathcal{X}) < \infty,$$

and moreover,

$$\begin{aligned}
& \sup_{x \in \mathcal{X}, r > 0} \frac{1}{[\mu(B(x, r))]^{1+\alpha p}} \int_{B(x, r)} |f(y)|^p d\mu(y) \\
&= \sup_{x \in \mathcal{X}, \rho(x) \leq r} \frac{1}{[\mu(B(x, r))]^{1+\alpha p}} \int_{B(x, r)} |f(y)|^p d\mu(y) \\
&\sim \sup_{x \in \mathcal{X}, \rho(x) \leq r} \int_{B(x, r)} |f(y)|^p d\mu(y) \sim \sup_{x \in \mathcal{X}, r > 0} \int_{B(x, r)} |f(y)|^p d\mu(y) \sim \|f\|_{L^p(\mathcal{X})},
\end{aligned}$$

which leads to that $\mathcal{L}^{\alpha, p}(\mathcal{X}) = \mathcal{L}^{-1/p, p}(\mathcal{X}) = L^p(\mathcal{X})$. $\square$

Theorem 2.2. *Let $\mathcal{X}$ be a doubling metric measure space and ρ an admissible function on $\mathcal{X}$. If $p \in [1, \infty)$ and $\alpha \in [-1/p, 0)$, then the followings hold.*

- (i) $\mathcal{E}_\rho^{\alpha, p}(\mathcal{X}) = \mathcal{L}^{\alpha, p}(\mathcal{X})$ with equivalent norms.
- (ii) For all $f \geq 0$, $f \in \mathcal{E}_\rho^{\alpha, p}(\mathcal{X})$ if and only if $f \in \tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X})$ and moreover, $\|f\|_{\tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X})} \sim \|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})}$.
- (iii) If $M \equiv \sup_{x \in \mathcal{X}} \mu(B(x, \rho(x))) < \infty$, then there exists a positive constant C such that for all f satisfying $-\infty < \text{essinf}_\mathcal{X} f < 0$,

$$\|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})} \leq \|f\|_{\tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X})} \leq C \left[\|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})} + M^{-\alpha} (-\text{essinf}_\mathcal{X} f) \right].$$

- (iv) If $\sup_{x \in \mathcal{X}} \mu(B(x, \rho(x))) = \infty$, then there exists a function $f \in \mathcal{E}_\rho^{\alpha, p}(\mathcal{X})$ such that $-\infty < \text{essinf}_\mathcal{X} f < 0$ and $f \notin \tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X})$.

Remark 2.2. (i) It turns out that Theorem 2.2(i), (ii) and (iii) hold for $\alpha \in (-\infty, 0)$ and $p \in [1, \infty)$ by Theorem 2.1.

(ii) If $\mathcal{X}$ is an RD-space, Theorem 2.2(i) & (ii) are already obtained in [34], which are used to prove Theorem 2.2(i) & (ii). Also we show Theorem 2.2(iii) by first assuming that it is true for RD-space $\mathcal{X}$, which is proved in Proposition 2.1 below. Recall that the space $\mathcal{X}$ is said to have the reverse doubling property if there exist constants $\kappa \in (0, n]$ and $K_1 \in (0, 1]$ such that for all $x \in \mathcal{X}$, $r \in (0, 2 \text{diam}(\mathcal{X}))$ and $\lambda \in (1, 2 \text{diam}(\mathcal{X})/r)$,

$$(2.5) \quad K_1 \lambda^\kappa \mu(B(x, r)) \leq \mu(B(x, \lambda r)).$$

If $(\mathcal{X}, d, \mu)$ satisfies the conditions (2.2) and (2.5), then $(\mathcal{X}, d, \mu)$ is called an RD-space, which was first introduced in [12] (see also [12, 36] for some equivalent characterizations of RD-spaces).

(iii) By an argument similar to that used in the proof of Theorem 2.2(i) & (ii) when $(\mathcal{X}, d, \mu)$ is an RD-space with d being a metric in [34], it is easy to see that if $(\mathcal{X}, d, \mu)$ is an RD-space with d being a quasi-metric, Theorem 2.2(i) & (ii) are also true. Moreover, a slight modification of the proof below shows that the whole Theorem 2.2 holds for $\mathcal{X}$ with d being a quasi-metric.

- (iv) It was proved in [19] that Theorem 2.2(ii) is not true when $\alpha = 0$.

To prove Theorem 2.2, we need some technical lemmas. Following Macías and Segovia [20], we call a doubling metric measure space to be normal if there exist positive constants K_2 and K_3 such that for all $x \in \mathcal{X}$ and $\mu(\{x\}) < r < \mu(\mathcal{X})$,

$$(2.6) \quad K_2 r \leq \mu(B(x, r)) \leq K_3 r.$$

For a doubling metric measure space $(\mathcal{X}, d, \mu)$, let

$$(2.7) \quad \delta(x, y) \equiv \begin{cases} \inf\{\mu(B) : B \text{ is a ball containing } x \text{ and } y\} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Macías and Segovia [20] showed that $(\mathcal{X}, \delta, \mu)$ is a normal space of homogeneous type, namely, δ is a quasi-metric and μ satisfies (2.1) and (2.6). Moreover, the topologies induced on $\mathcal{X}$ by d and δ coincide.

In this section, set $B^d(x, r) \equiv \{y \in \mathcal{X} : d(x, y) < r\}$ and $B^\delta(x, r) \equiv \{y \in \mathcal{X} : \delta(x, y) < r\}$ for all $x \in \mathcal{X}$ and $r > 0$. For all $x \in \mathcal{X}$, let

$$(2.8) \quad \rho_\delta(x) \equiv \mu(B^d(x, \rho(x))).$$

Lemma 2.1. *Let $\mathcal{X}$ be a doubling metric measure space and ρ an admissible function on $\mathcal{X}$, and let ρ_δ be as in (2.8). If $\alpha \in (-\infty, 0)$ and $p \in [1, \infty)$, then $\mathcal{L}^{\alpha, p}(\mathcal{X}, d, \mu) = \mathcal{L}^{\alpha, p}(\mathcal{X}, \delta, \mu)$ and $\mathcal{E}_\rho^{\alpha, p}(\mathcal{X}, d, \mu) = \mathcal{E}_{\rho_\delta}^{\alpha, p}(\mathcal{X}, \delta, \mu)$ with equivalent norms, respectively.*

Lemma 2.2. *Let $\mathcal{X}$ be a doubling metric measure space and ρ an admissible function on $\mathcal{X}$, and let ρ_δ be as in (2.8). Let $\alpha \in (-\infty, 0)$ and $p \in [1, \infty)$. If $f \geq 0$, then $\|f\|_{\tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X}, d, \mu)}$ and $\|f\|_{\tilde{\mathcal{E}}_{\rho_\delta}^{\alpha, p}(\mathcal{X}, \delta, \mu)}$ are equivalent with equivalent constants independent of f .*

Lemma 2.3. *Let $\mathcal{X}$ be a doubling metric measure space and ρ an admissible function on $\mathcal{X}$. Let ρ_δ be as in (2.8), $\alpha \in (-\infty, 0)$ and $p \in [1, \infty)$. If $M \equiv \sup_{x \in \mathcal{X}} \mu(B^d(x, \rho(x))) < \infty$ and $-\infty < \text{essinf }_\mathcal{X} f < 0$, then $\|f\|_{\tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X}, d, \mu)} + M^{-\alpha}(-\text{essinf }_\mathcal{X} f)$ and $\|f\|_{\tilde{\mathcal{E}}_{\rho_\delta}^{\alpha, p}(\mathcal{X}, \delta, \mu)} + M^{-\alpha}(-\text{essinf }_\mathcal{X} f)$ are equivalent with equivalent constants independent of f .*

To prove Lemmas 2.1, 2.2 and 2.3, we first state some basic facts. For any d -ball $B^d(x, r)$, let $\tilde{r} \equiv \mu(B^d(x, r))$. By the definition of δ , we have that

$$(2.9) \quad B^d(x, r) \subset B^\delta(x, \tilde{r}) \quad \text{and} \quad \mu(B^\delta(x, \tilde{r})) \leq K_3 \mu(B^d(x, r)).$$

Moreover,

$$(2.10) \quad \begin{cases} r < \rho(x) & \Rightarrow & \tilde{r} \leq \rho_\delta(x), \\ r = \rho(x) & \Rightarrow & \tilde{r} = \rho_\delta(x), \\ r > \rho(x) & \Rightarrow & \tilde{r} \geq \rho_\delta(x). \end{cases}$$

Conversely, by [22, Lemma 3.9] or [14, Proposition 2.1], for any δ -ball $B^\delta(x, r)$, there exists a positive constant $\tilde{r}$, which may depend on x and

r , such that

$$(2.11) \quad B^\delta(x, r) \subset B^d(x, \tilde{r}) \quad \text{and} \quad \mu(B^d(x, \tilde{r})) \leq C_3 \mu(B^\delta(x, r))$$

for some constant $C_3 \in [1, \infty)$, which is independent of x , r and $\tilde{r}$. In this case, if $r < \rho_\delta(x)/(C_3 K_3)$, then

$$\mu(B^d(x, \tilde{r})) \leq C_3 \mu(B^\delta(x, r)) \leq C_3 K_3 r < \rho_\delta(x) = \mu(B^d(x, \rho(x))).$$

If $r > \rho_\delta(x)/K_2$, then

$$\mu(B^d(x, \tilde{r})) \geq \mu(B^\delta(x, r)) \geq K_2 r > \rho_\delta(x) = \mu(B^d(x, \rho(x))).$$

That is,

$$(2.12) \quad \begin{cases} r < \rho_\delta(x)/(C_3 K_3) & \Rightarrow \tilde{r} < \rho(x), \\ r > \rho_\delta(x)/K_2 & \Rightarrow \tilde{r} > \rho(x). \end{cases}$$

Proof of Lemma 2.1. By (2.9) and (2.11), it is easy to see $\mathcal{L}^{\alpha, p}(\mathcal{X}, d, \mu) = \mathcal{L}^{\alpha, p}(\mathcal{X}, \delta, \mu)$ with equivalent norms.

Now, we prove that $\mathcal{E}_\rho^{\alpha, p}(\mathcal{X}, d, \mu) = \mathcal{E}_{\rho_\delta}^{\alpha, p}(\mathcal{X}, \delta, \mu)$ with equivalent norms.

(Part 1) For any d -ball $B_1 = B^d(x, r)$, let $B_2 = B^\delta(x, \tilde{r})$, where $\tilde{r} = \mu(B^d(x, r))$. From (2.9), it follows that $\mu(B_1) \sim \mu(B_2)$.

Case 1. $r < \rho(x)$. In this case, by (2.10), $\tilde{r} < \rho_\delta(x)$ or $\tilde{r} = \rho_\delta(x)$, which together with (2.9) implies that

$$(2.13) \quad \begin{aligned} & \frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y) - f_{B_1}|^p d\mu(y) \right\}^{1/p} \\ & \leq \frac{2}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y) - f_{B_2}|^p d\mu(y) \right\}^{1/p} \\ & \lesssim \frac{1}{[\mu(B_2)]^\alpha} \left\{ \frac{1}{\mu(B_2)} \int_{B_2} |f(y) - f_{B_2}|^p d\mu(y) \right\}^{1/p} \lesssim \|f\|_{\mathcal{E}_{\rho_\delta}^{\alpha, p}(\mathcal{X}, \delta, \mu)}. \end{aligned}$$

Case 2. $r \geq \rho(x)$. In this case, by (2.10), $\tilde{r} \geq \rho_\delta(x)$, which together with (2.9) leads to that

$$(2.14) \quad \begin{aligned} & \frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y)|^p d\mu(y) \right\}^{1/p} \\ & \lesssim \frac{1}{[\mu(B_2)]^\alpha} \left\{ \frac{1}{\mu(B_2)} \int_{B_2} |f(y)|^p d\mu(y) \right\}^{1/p} \lesssim \|f\|_{\mathcal{E}_{\rho_\delta}^{\alpha, p}(\mathcal{X}, \delta, \mu)}. \end{aligned}$$

Therefore, $\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X},d,\mu)} \lesssim \|f\|_{\mathcal{E}_{\rho_\delta}^{\alpha,p}(\mathcal{X},\delta,\mu)}$.

(Part 2) For any δ -ball $B_1 \equiv B^\delta(x, r)$, let $B_2 \equiv B^d(x, \tilde{r})$, where $\tilde{r}$ is as in (2.11). From (2.11), it follows that $\mu(B_1) \sim \mu(B_2)$.

Case 1. $r < \rho_\delta(x)/(C_3 K_3)$. In this case, by (2.12), $\tilde{r} < \rho(x)$. By an argument similar to the estimate of (2.13), we have

$$\frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y) - f_{B_1}|^p d\mu(y) \right\}^{1/p} \lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X},d,\mu)}.$$

Case 2. $r > \rho_\delta(x)/K_2$. In this case, by (2.12), $\tilde{r} > \rho(x)$. By an argument similar to the estimate of (2.14), we have

$$\frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y)|^p d\mu(y) \right\}^{1/p} \lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X},d,\mu)}.$$

Case 3. $\rho_\delta(x)/(C_3 K_3) \leq r \leq \rho_\delta(x)/K_2$. In this case, let $B_1 \equiv B^\delta(x, r)$ and $B'_1 \equiv B^\delta(x, (2C_3 K_3/K_2)r)$. Then $\mu(B_1) \sim \mu(B'_1)$. Hence

$$\begin{aligned} & \frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y) - f_{B_1}|^p d\mu(y) \right\}^{1/p} \\ & \leq \frac{2}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y) - f_{B'_1}|^p d\mu(y) \right\}^{1/p} \\ & \lesssim \frac{1}{[\mu(B'_1)]^\alpha} \left\{ \frac{1}{\mu(B'_1)} \int_{B'_1} |f(y) - f_{B'_1}|^p d\mu(y) \right\}^{1/p} \\ & \lesssim \frac{1}{[\mu(B'_1)]^\alpha} \left\{ \frac{1}{\mu(B'_1)} \int_{B'_1} |f(y)|^p d\mu(y) \right\}^{1/p}, \end{aligned}$$

and

$$\frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y)|^p d\mu(y) \right\}^{1/p} \lesssim \frac{1}{[\mu(B'_1)]^\alpha} \left\{ \frac{1}{\mu(B'_1)} \int_{B'_1} |f(y)|^p d\mu(y) \right\}^{1/p}.$$

Since $(2C_3 K_3/K_2)r > \rho_\delta(x)/K_2$, using Case 2, we have

$$\frac{1}{[\mu(B'_1)]^\alpha} \left\{ \frac{1}{\mu(B'_1)} \int_{B'_1} |f(y)|^p d\mu(y) \right\}^{1/p} \lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X},d,\mu)}.$$

Therefore, $\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X},\delta,\mu)} \lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X},d,\mu)}$ and we are done. $\square$

Proof of Lemma 2.2. (Part 1) For any d -ball $B_1 \equiv B^d(x, r)$, let $B_2 \equiv B^\delta(x, \tilde{r})$, where $\tilde{r} \equiv \mu(B^d(x, r))$. From (2.9), it follows that $\mu(B_1) \sim \mu(B_2)$.

Case 1. $r < \rho(x)$. In this case, by (2.10), $\tilde{r} < \rho_\delta(x)$ or $\tilde{r} = \rho_\delta(x)$, which implies that

$$\begin{aligned}
 (2.15) \quad & \frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} \left[f(y) - \operatorname{essinf}_{B_1} f \right]^p d\mu(y) \right\}^{1/p} \\
 & \leq \frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} \left[f(y) - \operatorname{essinf}_{B_2} f \right]^p d\mu(y) \right\}^{1/p} \\
 & \lesssim \frac{1}{[\mu(B_2)]^\alpha} \left\{ \frac{1}{\mu(B_2)} \int_{B_2} \left[f(y) - \operatorname{essinf}_{B_2} f \right]^p d\mu(y) \right\}^{1/p} \lesssim \|f\|_{\mathcal{E}_{\rho_\delta}^{\alpha,p}(\mathcal{X}, \delta, \mu)}.
 \end{aligned}$$

Case 2. $r \geq \rho(x)$. In this case, by (2.10), $\tilde{r} \geq \rho_\delta(x)$. By an argument similar to the estimate of (2.14), we have

$$\frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y)|^p d\mu(y) \right\}^{1/p} \lesssim \|f\|_{\mathcal{E}_{\rho_\delta}^{\alpha,p}(\mathcal{X}, \delta, \mu)}.$$

Therefore, $\|f\|_{\mathcal{E}_{\rho}^{\alpha,p}(\mathcal{X}, d, \mu)} \lesssim \|f\|_{\mathcal{E}_{\rho_\delta}^{\alpha,p}(\mathcal{X}, \delta, \mu)}$.

(Part 2) For any δ -ball $B_1 = B^\delta(x, r)$, let $B_2 = B^d(x, \tilde{r})$, where $\tilde{r}$ is as in (2.11). From (2.11), it follows that $\mu(B_1) \sim \mu(B_2)$.

Case 1. $r < \rho_\delta(x)/(C_3 K_3)$. In this case, by (2.12), $\tilde{r} < \rho(x)$. By an argument similar to the estimate of (2.15), we have

$$\frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} \left[f(y) - \operatorname{essinf}_{B_1} f \right]^p d\mu(y) \right\}^{1/p} \lesssim \|f\|_{\mathcal{E}_{\rho}^{\alpha,p}(\mathcal{X}, d, \mu)}.$$

Case 2. $r > \rho_\delta(x)/K_2$. In this case, by (2.12), $\tilde{r} > \rho(x)$. By an argument similar to the estimate of (2.14), we have

$$\frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y)|^p d\mu(y) \right\}^{1/p} \lesssim \|f\|_{\mathcal{E}_{\rho}^{\alpha,p}(\mathcal{X}, d, \mu)}.$$

Case 3. $\rho_\delta(x)/(C_3 K_3) \leq r \leq \rho_\delta(x)/K_2$. In this case, let $B_1 \equiv B^\delta(x, r)$ and $B'_1 \equiv B^\delta(x, (2C_3 K_3/K_2)r)$. Then $\mu(B_1) \sim \mu(B'_1)$. Hence, if $f \geq 0$,

then

$$\begin{aligned} & \frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} \left[f(y) - \operatorname{essinf}_{B_1} f \right]^p d\mu(y) \right\}^{1/p} \\ & \leq \frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y)|^p d\mu(y) \right\}^{1/p} \\ & \lesssim \frac{1}{[\mu(B'_1)]^\alpha} \left\{ \frac{1}{\mu(B'_1)} \int_{B'_1} |f(y)|^p d\mu(y) \right\}^{1/p}, \end{aligned}$$

and

$$\frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y)|^p d\mu(y) \right\}^{1/p} \lesssim \frac{1}{[\mu(B'_1)]^\alpha} \left\{ \frac{1}{\mu(B'_1)} \int_{B'_1} |f(y)|^p d\mu(y) \right\}^{1/p}.$$

Since $(2C_3K_3/K_2)r > \rho_\delta(x)/K_2$, using Case 2, we have

$$\frac{1}{[\mu(B'_1)]^\alpha} \left\{ \frac{1}{\mu(B'_1)} \int_{B'_1} |f(y)|^p d\mu(y) \right\}^{1/p} \lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X},d,\mu)}.$$

Therefore, if $f \geq 0$, then $\|f\|_{\mathcal{E}_{\rho_\delta}^{\alpha,p}(\mathcal{X},\delta,\mu)} \lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X},d,\mu)}$. $\square$

Proof of Lemma 2.3. Let $M \equiv \sup_{x \in \mathcal{X}} \mu(B^d(x, \rho(x))) = \sup_{x \in \mathcal{X}} \rho_\delta(x) < \infty$. By the same way as in the proof of Lemma 2.2, we divide the proof into (Part 1) and (Part 2). Then we have the same conclusions as in Case 2 of (Part 1) and in Cases 1 and 2 of (Part 2) of the proof of Lemma 2.2. So we only need to consider Case 1 of (Part 1) and Case 3 of (Part 2) therein.

(Part 1) For any d -ball $B_1 \equiv B^d(x, r)$, let $B_2 \equiv B^\delta(x, \tilde{r})$, where $\tilde{r} \equiv \mu(B^d(x, r))$. From (2.9), it follows that $\mu(B_1) \sim \mu(B_2)$.

Case 1. $r < \rho(x)$. In this case, by (2.10), $\tilde{r} < \rho_\delta(x)$ or $\tilde{r} = \rho_\delta(x)$. If $\tilde{r} < \rho_\delta(x)$, then we have the same inequality as (2.15). If $\tilde{r} = \rho_\delta(x)$, then $\mu(B_2) \sim \rho_\delta(x)$. Hence, if $-\infty < \operatorname{essinf}_{\mathcal{X}} f < 0$, then

$$\begin{aligned} (2.16) \quad & \frac{1}{[\mu(B_1)]^\alpha} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} \left[f(y) - \operatorname{essinf}_{B_1} f \right]^p d\mu(y) \right\}^{1/p} \\ & \lesssim \frac{1}{[\mu(B_2)]^\alpha} \left\{ \frac{1}{\mu(B_2)} \int_{B_2} \left[f(y) - \operatorname{essinf}_{B_2} f \right]^p d\mu(y) \right\}^{1/p} \\ & \lesssim \frac{1}{[\mu(B_2)]^\alpha} \left\{ \frac{1}{\mu(B_2)} \int_{B_2} |f(y)|^p d\mu(y) \right\}^{1/p} + \frac{-\operatorname{essinf}_{\mathcal{X}} f}{[\rho_\delta(x)]^\alpha} \\ & \lesssim \|f\|_{\tilde{\mathcal{E}}_{\rho_\delta}^{\alpha,p}(\mathcal{X},\delta,\mu)} + M^{-\alpha} \left(-\operatorname{essinf}_{\mathcal{X}} f \right). \end{aligned}$$

Therefore, if $-\infty < \operatorname{essinf}_{\mathcal{X}} f < 0$, then

$$\|f\|_{\tilde{\mathcal{E}}_{\rho}^{\alpha,p}(\mathcal{X},d,\mu)} \lesssim \|f\|_{\tilde{\mathcal{E}}_{\delta}^{\alpha,p}(\mathcal{X},\delta,\mu)} + M^{-\alpha} \left(-\operatorname{essinf}_{\mathcal{X}} f \right).$$

(Part 2) For any δ -ball $B_1 \equiv B^{\delta}(x, r)$, let $B_2 \equiv B^d(x, \tilde{r})$, where $\tilde{r}$ is as in (2.11). From (2.11), it follows that $\mu(B_1) \sim \mu(B_2)$.

Case 3. $\rho_{\delta}(x)/(C_3 K_3) \leq r \leq \rho_{\delta}(x)/K_2$. In this case, let $B_1 \equiv B^{\delta}(x, r)$ and $B'_1 \equiv B^{\delta}(x, (2C_3 K_3/K_2)r)$. Then $\mu(B_1) \sim \mu(B'_1) \sim \rho_{\delta}(x)$. Hence, if $-\infty < \operatorname{essinf}_{\mathcal{X}} f < 0$, then

$$\begin{aligned} & \frac{1}{[\mu(B_1)]^{\alpha}} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} \left[f(y) - \operatorname{essinf}_{B_1} f \right]^p d\mu(y) \right\}^{1/p} \\ & \lesssim \frac{1}{[\mu(B'_1)]^{\alpha}} \left\{ \frac{1}{\mu(B'_1)} \int_{B'_1} \left[f(y) - \operatorname{essinf}_{B'_1} f \right]^p d\mu(y) \right\}^{1/p} \\ & \lesssim \frac{1}{[\mu(B'_1)]^{\alpha}} \left\{ \frac{1}{\mu(B'_1)} \int_{B'_1} |f(y)|^p d\mu(y) \right\}^{1/p} + \frac{-\operatorname{essinf}_{\mathcal{X}} f}{[\rho_{\delta}(x)]^{\alpha}} \\ & \lesssim \frac{1}{[\mu(B'_1)]^{\alpha}} \left\{ \frac{1}{\mu(B'_1)} \int_{B'_1} |f(y)|^p d\mu(y) \right\}^{1/p} + M^{-\alpha} \left(-\operatorname{essinf}_{\mathcal{X}} f \right), \end{aligned}$$

and

$$\frac{1}{[\mu(B_1)]^{\alpha}} \left\{ \frac{1}{\mu(B_1)} \int_{B_1} |f(y)|^p d\mu(y) \right\}^{1/p} \lesssim \frac{1}{[\mu(B'_1)]^{\alpha}} \left\{ \frac{1}{\mu(B'_1)} \int_{B'_1} |f(y)|^p d\mu(y) \right\}^{1/p}.$$

Since $(2C_3 K_3/K_2)r > \rho_{\delta}(x)/K_2$, using Case 2, we have

$$\frac{1}{[\mu(B'_1)]^{\alpha}} \left\{ \frac{1}{\mu(B'_1)} \int_{B'_1} |f(y)|^p d\mu(y) \right\}^{1/p} \lesssim \|f\|_{\tilde{\mathcal{E}}_{\rho}^{\alpha,p}(\mathcal{X},d,\mu)} + M^{-\alpha} \left(-\operatorname{essinf}_{\mathcal{X}} f \right).$$

Therefore, if $-\infty < \operatorname{essinf}_{\mathcal{X}} f < 0$, then

$$\|f\|_{\tilde{\mathcal{E}}_{\rho_{\delta}}^{\alpha,p}(\mathcal{X},\delta,\mu)} \lesssim \|f\|_{\tilde{\mathcal{E}}_{\rho}^{\alpha,p}(\mathcal{X},d,\mu)} + M^{-\alpha} \left(-\operatorname{essinf}_{\mathcal{X}} f \right).$$

□

Lemma 2.4 (c.f. [21, Lemma 3.3]). *Let $\alpha \in [-1/p, 0)$. Then for all $B \subset \mathcal{X}$,*

$$\|\chi_B\|_{\mathcal{L}^{\alpha,p}(\mathcal{X})} = [\mu(B)]^{-\alpha}.$$

Proof. From the equality

$$\frac{1}{[\mu(B)]^\alpha} \left\{ \frac{1}{\mu(B)} \int_B |\chi_B(x)|^p d\mu(x) \right\}^{1/p} = [\mu(B)]^{-\alpha},$$

it follows that $\|\chi_B\|_{\mathcal{L}^{\alpha,p}(\mathcal{X})} \geq [\mu(B)]^{-\alpha}$. For any balls $B(z, r)$, if $\mu(B(z, r)) < \mu(B)$, then

$$\frac{1}{[\mu(B(z, r))]^\alpha} \left\{ \frac{1}{\mu(B(z, r))} \int_{B(z, r)} |\chi_B(x)|^p d\mu(x) \right\}^{1/p} \leq \mu(B(z, r))^{-\alpha} \leq [\mu(B)]^{-\alpha}.$$

If $\mu(B(z, r)) \geq \mu(B)$, then

$$\begin{aligned} & \frac{1}{[\mu(B(z, r))]^\alpha} \left\{ \frac{1}{\mu(B(z, r))} \int_{B(z, r)} |\chi_B(x)|^p d\mu(x) \right\}^{1/p} \\ &= \frac{1}{[\mu(B(z, r))]^\alpha} \left\{ \frac{\mu(B \cap B(z, r))}{\mu(B(z, r))} \right\}^{1/p} \\ &= \left(\frac{\mu(B \cap B(z, r))}{\mu(B(z, r))} \right)^{\alpha+1/p} [\mu(B \cap B(z, r))]^{-\alpha} \leq [\mu(B)]^{-\alpha}. \end{aligned}$$

Therefore, $\|\chi_B\|_{\mathcal{L}^{\alpha,p}(\mathcal{X})} \leq [\mu(B)]^{-\alpha}$, which implies that $\|\chi_B\|_{\mathcal{L}^{\alpha,p}(\mathcal{X})} = [\mu(B)]^{-\alpha}$. $\square$

Proof of Theorem 2.2. Since $(\mathcal{X}, \delta, \mu)$ is normal, $(\mathcal{X}, \delta, \mu)$ is also an RD-space.

(i) By Lemma 2.1 and Remark 2.2(iii), we have

$$\mathcal{E}_\rho^{\alpha,p}(\mathcal{X}, d, \mu) = \mathcal{E}_{\rho_\delta}^{\alpha,p}(\mathcal{X}, \delta, \mu) = \mathcal{L}^{\alpha,p}(\mathcal{X}, \delta, \mu) = \mathcal{L}^{\alpha,p}(\mathcal{X}, d, \mu).$$

(ii) If $f \geq 0$, then, by Remark 2.2(iii), we obtain that $\|f\|_{\mathcal{E}_{\rho_\delta}^{\alpha,p}(\mathcal{X}, \delta, \mu)} \sim \|f\|_{\tilde{\mathcal{E}}_{\rho_\delta}^{\alpha,p}(\mathcal{X}, \delta, \mu)}$, which together with Lemmas 2.1 and 2.2 yields that

$$\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X}, d, \mu)} \sim \|f\|_{\mathcal{E}_{\rho_\delta}^{\alpha,p}(\mathcal{X}, \delta, \mu)} \sim \|f\|_{\tilde{\mathcal{E}}_{\rho_\delta}^{\alpha,p}(\mathcal{X}, \delta, \mu)} \sim \|f\|_{\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X}, d, \mu)}.$$

(iii) In the case $M \equiv \sup_{x \in \mathcal{X}} \mu(B(x, \rho(x))) < \infty$, if $-\infty < \operatorname{essinf}_{\mathcal{X}} f < 0$, then, by Proposition 2.1 below, we obtain that

$$\|f\|_{\mathcal{E}_{\rho_\delta}^{\alpha,p}(\mathcal{X}, \delta, \mu)} \leq \|f\|_{\tilde{\mathcal{E}}_{\rho_\delta}^{\alpha,p}(\mathcal{X}, \delta, \mu)} \lesssim \|f\|_{\mathcal{E}_{\rho_\delta}^{\alpha,p}(\mathcal{X}, \delta, \mu)} + M^{-\alpha} \left(-\operatorname{essinf}_{\mathcal{X}} f \right),$$

which together with Lemmas 2.1 and 2.3 yields that

$$\begin{aligned} \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X},d,\mu)} &\leq \|f\|_{\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X},d,\mu)} \lesssim \|f\|_{\tilde{\mathcal{E}}_{\rho_\delta}^{\alpha,p}(\mathcal{X},\delta,\mu)} + M^{-\alpha} \left(-\operatorname{essinf}_{\mathcal{X}} f \right) \\ &\sim \|f\|_{\mathcal{E}_{\rho_\delta}^{\alpha,p}(\mathcal{X},\delta,\mu)} + M^{-\alpha} \left(-\operatorname{essinf}_{\mathcal{X}} f \right) \\ &\sim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X},d,\mu)} + M^{-\alpha} \left(-\operatorname{essinf}_{\mathcal{X}} f \right). \end{aligned}$$

(iv) Since $\sup_{x \in \mathcal{X}} \mu(B(x, \rho(x))) = \infty$, we choose $B_j \equiv B(z_j, \rho(z_j)/2)$, $j \in \mathbb{N}$, so that $\mu(B_j) \rightarrow \infty$ as $j \rightarrow \infty$. Then, we have two situations that

- (I) for all j , $(\cup_{i=1}^j B_i) \cap B_{j+1} = \emptyset$, or
- (II) there exists $j_0 \in \mathbb{N}$ such that $B_{j_1} \cap B_{j_2} = \emptyset$ for $1 \leq j_1 < j_2 \leq j_0$, and that $(\cup_{i=1}^{j_0} B_i) \cap B_j \neq \emptyset$ for all $j > j_0$.

Let $b > 0$.

Case (I). For each j , choose $t_j > 0$ so that $t_j < \rho(z_j)/2$ and $[\mu(B(z_j, t_j))]^{-\alpha} < 1/2^j$. In this case, $\mu(B(z_j, t_j)) < (1/2^j)^{-1/\alpha} < 1$. Let $f \equiv -b \sum_j f_j$ and $f_j \equiv \chi_{B(z_j, t_j)}$. Then $\operatorname{essinf}_{\mathcal{X}} f = -b$ and by Lemma 2.4, we have

$$\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})} \leq 2\|f\|_{\mathcal{L}^{\alpha,p}(\mathcal{X})} \leq 2b \sum_{j=1}^{\infty} \|f_j\|_{\mathcal{L}^{\alpha,p}(\mathcal{X})} = 2b \sum_{j=1}^{\infty} [\mu(B(z_j, t_j))]^{-\alpha} \leq 2b.$$

On the other hand,

$$\begin{aligned} &\frac{1}{[\mu(B_j)]^\alpha} \left(\frac{1}{\mu(B_j)} \int_{B_j} \left[f(x) - \operatorname{essinf}_{B_j} f \right]^p d\mu(x) \right)^{1/p} \\ &= \frac{1}{[\mu(B_j)]^\alpha} \left(\frac{1}{\mu(B_j)} \int_{B_j} \left[-bf_j(x) - (-b) \right]^p d\mu(x) \right)^{1/p} \\ &\geq \frac{b}{[\mu(B_j)]^\alpha} \left(1 - \frac{1}{\mu(B_j)} \right)^{1/p} \rightarrow \infty \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Case (II). Let $f \equiv -b \sum_{j=1}^{j_0} f_j$ and $f_j(x) \equiv \chi_{B_j}(x)$. Then $\operatorname{essinf}_{\mathcal{X}} f = -b$ and by Lemma 2.4, we have

$$\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})} \leq 2\|f\|_{\mathcal{L}^{\alpha,p}(\mathcal{X})} \leq 2b \sum_{j=1}^{j_0} \|f_j\|_{\mathcal{L}^{\alpha,p}(\mathcal{X})} = 2b \sum_{j=1}^{j_0} [\mu(B_j)]^{-\alpha} < \infty.$$

On the other hand, for $j > j_0$,

$$\begin{aligned}
& \frac{1}{[\mu(B_j)]^\alpha} \left(\frac{1}{\mu(B_j)} \int_{B_j} \left[f(x) - \operatorname{ess\,inf}_{B_j} f \right]^p d\mu(x) \right)^{1/p} \\
&= \frac{1}{[\mu(B_j)]^\alpha} \left(\frac{1}{\mu(B_j)} \int_{B_j} \left[-b \sum_{i=1}^{j_0} f_i(x) - (-b) \right]^p d\mu(x) \right)^{1/p} \\
&= \frac{b}{[\mu(B_j)]^\alpha} \left(\frac{1}{\mu(B_j)} \left[\mu(B_j) - \mu \left(\left(\bigcup_{i=1}^{j_0} B_i \right) \cap B_j \right) \right] \right)^{1/p} \\
&\geq \frac{b}{[\mu(B_j)]^\alpha} \left(1 - \frac{\sum_{i=1}^{j_0} \mu(B_i)}{\mu(B_j)} \right)^{1/p} \rightarrow \infty \quad \text{as } j \rightarrow \infty.
\end{aligned}$$

Combining the estimates for Cases (I) and (II) yields (vi), which completes the proof of Theorem 2.2. $\square$

In the proof of Theorem 2.2(iii) above, we used the following proposition.

Proposition 2.1. *Theorem 2.2 (iii) holds if $\mathcal{X}$ is an RD-space.*

To prove Proposition 2.1, we begin with some technical lemmas. A straightforward computation via (2.5) leads to the following technical lemma.

Lemma 2.5. *Let $\mathcal{X}$ be an RD-space and $\theta \in (0, \infty)$. Then, there exists a positive constant C such that for all $z \in \mathcal{X}$ and $0 < r < s < \infty$,*

$$\int_r^s \frac{dt}{t [\mu(B(z, t))]^\theta} \leq \frac{C}{[\mu(B(z, r))]^\theta}.$$

Let

$$MO(f, B) \equiv \frac{1}{\mu(B)} \int_B |f(y) - f_B| d\mu(y).$$

Then, by Lemma 2.4 in [24], there exists a positive constant C such that for all $z \in \mathcal{X}$ and $0 < r < s < \infty$,

$$(2.17) \quad |f_{B(z, r)} - f_{B(z, s)}| \leq C \int_r^{2s} \frac{MO(f, B(z, t))}{t} dt.$$

Lemma 2.6. *Let $\mathcal{X}$ be an RD-space and $\alpha \in [-1/p, 0)$. Then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha, p}(\mathcal{X})$, $z \in \mathcal{X}$ and $0 < r < \rho(z)$,*

$$|f_{B(z, r)} - f_{B(z, \rho(z))}| \leq C \|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})} [\mu(B(z, r))]^\alpha.$$

Proof. **Case 1.** $\rho(z)/2 \leq r < \rho(z)$. By (2.1) and the Hölder inequality, we have

$$\begin{aligned} |f_{B(z,r)} - f_{B(z,\rho(z))}| &\leq \frac{1}{\mu(B(z,r))} \int_{B(z,r)} |f(x)| d\mu(x) + |f_{B(z,\rho(z))}| \\ &\leq (C_1 + 1) \frac{1}{\mu(B(z,\rho(z)))} \int_{B(z,\rho(z))} |f(x)| d\mu(x) \\ &\leq (C_1 + 1) \left(\frac{1}{\mu(B(z,\rho(z)))} \int_{B(z,\rho(z))} |f(x)|^p d\mu(x) \right)^{1/p} \\ &\leq (C_1 + 1) \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})} [\mu(B(z,r))]^\alpha. \end{aligned}$$

Case 2. $0 < r < \rho(z)/2$. Using (2.17) and Lemma 2.5, we have

$$\begin{aligned} |f_{B(z,r)} - f_{B(z,\rho(z)/2)}| &\lesssim \int_r^{\rho(z)} \frac{MO(f, B(z,t))}{t} dt \\ &\lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})} \int_r^{\rho(z)} \frac{[\mu(B(z,t))]^\alpha}{t} dt \lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})} [\mu(B(z,r))]^\alpha. \end{aligned}$$

Combining the estimates for Case 1 and Case 2 completes the proof of Lemma 2.6. $\square$

Proof of Proposition 2.1. Let $M \equiv \sup_{x \in \mathcal{X}} \mu(B(x, \rho(x))) < \infty$. By Lemma 2.6, we have that if $0 < r < \rho(z)$, then

$$|f_{B(z,r)} - f_{B(z,\rho(z))}| \lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})} [\mu(B(z,r))]^\alpha.$$

Now, for $B \equiv B(z, r)$, if $0 < r < \rho(z)$, then

$$\begin{aligned} &\frac{1}{[\mu(B)]^\alpha} \left(\frac{1}{\mu(B)} \int_B \left[f(x) - \operatorname{essinf}_B f \right]^p d\mu(x) \right)^{1/p} \\ &\leq \frac{1}{[\mu(B)]^\alpha} \left(\frac{1}{\mu(B)} \int_B \left[f(x) - \operatorname{essinf}_{B(z,\rho(z))} f \right]^p d\mu(x) \right)^{1/p} \\ &\leq \frac{1}{[\mu(B)]^\alpha} \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p d\mu(x) \right)^{1/p} \\ &\quad + \frac{1}{[\mu(B)]^\alpha} |f_{B(z,r)} - f_{B(z,\rho(z))}| + \frac{1}{[\mu(B)]^\alpha} \left| f_{B(z,\rho(z))} - \operatorname{essinf}_{B(z,\rho(z))} f \right| \\ &\lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})} + \frac{1}{[\mu(B)]^\alpha} \left| f_{B(z,\rho(z))} - \operatorname{essinf}_{B(z,\rho(z))} f \right|. \end{aligned}$$

If $-\infty < \operatorname{essinf}_{\mathcal{X}} f < 0$, then

$$\begin{aligned}
& \frac{1}{[\mu(B)]^\alpha} \left| f_{B(z, \rho(z))} - \operatorname{essinf}_{B(z, \rho(z))} f \right| \\
& \leq \frac{1}{[\mu(B)]^\alpha} |f_{B(z, \rho(z))}| + \frac{1}{[\mu(B)]^\alpha} \left| \operatorname{essinf}_{\mathcal{X}} f \right| \\
& \leq \frac{[\mu(B(z, \rho(z))))^\alpha}{[\mu(B)]^\alpha} \|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})} + \frac{[\mu(B(z, \rho(z))))^\alpha}{[\mu(B)]^\alpha} \frac{|\operatorname{essinf}_{\mathcal{X}} f|}{[\mu(B(z, \rho(z))))^\alpha} \\
& \leq \|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})} + M^{-\alpha} \left| \operatorname{essinf}_{\mathcal{X}} f \right|.
\end{aligned}$$

Therefore, if $-\infty < \operatorname{essinf}_{\mathcal{X}} f < 0$, then

$$\|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})} \leq \|f\|_{\tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X})} \lesssim \|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})} + M^{-\alpha} (-\operatorname{essinf}_{\mathcal{X}} f),$$

which completes the proof of Proposition 2.1. $\square$

3. Boundedness of Lusin-area and g_λ^* functions

Let $\mathcal{X}$ be a doubling metric measure space and ρ an admissible function. In this section, we consider the boundedness of certain variants of Lusin-area and g_λ^* functions from $\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})$ to $\tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X})$. The boundedness from $\operatorname{BMO}_\rho(\mathcal{X})$ to $\operatorname{BLO}_\rho(\mathcal{X})$ of these operators were obtained in [19]. We remark that unlike the boundedness of the g_λ^* function, to obtain the boundedness of the Lusin-area function, we need to assume that $\mathcal{X}$ has the following volume regularity Property (P), which was introduced in [19]; see also [4, 30].

Definition 3.1 ([19]). A doubling metric measure space $(\mathcal{X}, d, \mu)$ is said to have *Property (P)*, if there exist positive constants δ and C such that for all $x \in \mathcal{X}$, $s \in (0, \infty)$ and $r \in (s, \infty)$,

$$(3.1) \quad \mu(B(x, r+s)) - \mu(B(x, r)) \leq C \left(\frac{s}{r} \right)^\delta \mu(B(x, r)).$$

Remark 3.1. There are many examples of doubling metric measure spaces having Property (P), such as $(\mathbb{R}^d, |\cdot|, dx)$, the d -dimensional Euclidean space endowed with the Euclidean norm $|\cdot|$ and the Lebesgue measure dx ; $(\mathbb{R}^d, |\cdot|, w(x)dx)$, the d -dimensional Euclidean space endowed with the Euclidean norm $|\cdot|$ and the measure $w(x)dx$, where w is an $A_2(\mathbb{R}^d)$ weight and dx is the Lebesgue measure; $(\mathbb{H}^n, d, dx)$, the $(2n+1)$ -dimensional Heisenberg group $\mathbb{H}^n$ with a left-invariant metric d and the

Lebesgue measure dx ; $(\mathbb{G}, d, \mu)$, the nilpotent Lie group $\mathbb{G}$ with a Carnot-Carathéodory (control) distance d and a left invariant Haar measure μ and so on; see [19] for more details.

In what follows, we always set $V_r(x) \equiv \mu(B(x, r))$ and $V(x, y) \equiv \mu(B(x, d(x, y)))$ for all $x, y \in \mathcal{X}$ and $r \in (0, \infty)$.

Let ρ be an admissible function on $\mathcal{X}$ and $\{Q_t\}_{t>0}$ a family of operators bounded on $L^2(\mathcal{X})$ with integral kernels $\{Q_t(x, y)\}_{t>0}$ satisfying that there exist constants $C, \delta_1 \in (0, \infty), \delta_2 \in (0, 1)$ and $\gamma \in (0, \infty)$ such that for all $t \in (0, \infty)$ and $x, y \in \mathcal{X}$,

$$(Q)_i \quad |Q_t(x, y)| \leq C \frac{1}{V_t(x)+V_t(y)} \left(\frac{t}{t+d(x, y)}\right)^\gamma \left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1};$$

$$(Q)_{ii} \quad \left|\int_{\mathcal{X}} Q_t(x, z) d\mu(z)\right| \leq C \left(\frac{t}{t+\rho(x)}\right)^{\delta_2}.$$

For all $f \in L^1_{\text{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$, define the Littlewood-Paley g -function by setting

$$(3.2) \quad g(f)(x) \equiv \left\{ \int_0^\infty |Q_t(f)(x)|^2 \frac{dt}{t} \right\}^{1/2},$$

and Lusin-area and g_λ^* functions, respectively, by setting

$$(3.3) \quad S(f)(x) \equiv \left\{ \int_0^\infty \int_{d(x, y) < t} |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \right\}^{1/2},$$

and

$$(3.4) \quad g_\lambda^*(f)(x) \equiv \left\{ \iint_{\mathcal{X} \times (0, \infty)} \left(\frac{t}{t+d(x, y)}\right)^\lambda |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \right\}^{1/2},$$

where $\lambda \in (0, \infty)$.

Theorem 3.1. *Let $\mathcal{X}$ be a doubling metric measure space having Property (P). Let $p \in (1, \infty)$, ρ be an admissible function on $\mathcal{X}$, the Lusin-area function $S(f)$ as in (3.3) and $\alpha \in (-\infty, \min\{\gamma/n, \delta_1/[(1+k_0)n], \delta_2/n, \delta/(2n)\})$. If $S(f)$ is bounded on $L^p(\mathcal{X})$, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha, p}(\mathcal{X})$, $[S(f)]^2 \in \tilde{\mathcal{E}}_\rho^{2\alpha, p/2}(\mathcal{X})$ and $\|[S(f)]^2\|_{\tilde{\mathcal{E}}_\rho^{2\alpha, p/2}(\mathcal{X})} \leq C \|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})}^2$.*

To prove Theorem 3.1, we begin with the following two technical lemmas, which were obtained in [34].

Lemma 3.1 ([34, Lemma 2.4]). *Let $\alpha \in \mathbb{R}$, $p \in [1, \infty)$, ρ be an admissible function on $\mathcal{X}$ and $\mathcal{D}$ as in Definition 2.3. Then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha, p}(\mathcal{X})$,*

(i) for all balls $B \equiv B(x_0, r) \notin \mathcal{D}$,

$$\frac{1}{\mu(B)} \int_B |f(z)| d\mu(z) \leq \begin{cases} C \left(\frac{\rho(x_0)}{r} \right)^{\alpha n} [\mu(B)]^\alpha \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}, & \alpha > 0, \\ C \left(1 + \log \frac{\rho(x_0)}{r} \right) [\mu(B)]^\alpha \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}, & \alpha \leq 0; \end{cases}$$

(ii) for all $x \in \mathcal{X}$ and $0 < r_1 < r_2$,

$$|f_{B(x, r_1)} - f_{B(x, r_2)}| \leq \begin{cases} C \left(\frac{r_2}{r_1} \right)^{\alpha n} [\mu(B(x, r_1))]^\alpha \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}, & \alpha > 0, \\ C \left(1 + \log \frac{r_2}{r_1} \right) [\mu(B(x, r_1))]^\alpha \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}, & \alpha \leq 0. \end{cases}$$

Lemma 3.2 ([34, Lemma 4.1]). *Let $\alpha \in (-\infty, \min\{\gamma/n, \delta_2/n\})$, $p \in (1, \infty)$ and ρ be an admissible function on $\mathcal{X}$. Then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$, $x \in \mathcal{X}$ and $t > 0$,*

$$|Q_t(f)(x)| \leq C \left(\frac{\rho(x)}{t + \rho(x)} \right)^{\delta_1} [\mu(B(x, t))]^\alpha \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}.$$

Proof of Theorem 3.1. By similarity, we only prove the case when $\alpha > 0$.

Let $f \in \mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$. By the homogeneity of $\|\cdot\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}$ and $\|\cdot\|_{\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})}$, we may assume that $\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})} = 1$. Let $B \equiv B(x_0, r)$. We prove Theorem 3.1 by considering the following two cases.

Case I. $B \equiv B(x_0, r) \in \mathcal{D}$. In this case, $r \geq \rho(x_0)$. We need to prove that

$$(3.5) \quad \int_B [S(f)(x)]^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

For all $x \in \mathcal{X}$, write

$$\begin{aligned} [S(f)(x)]^2 &= \int_0^{8r} \int_{d(x,y) < t} |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} + \int_{8r}^\infty \int_{d(x,y) < t} \dots \\ &\equiv [S_1(f)(x)]^2 + [S_2(f)(x)]^2. \end{aligned}$$

By the $L^p(\mathcal{X})$ -boundedness of $S(f)$ and (2.1), we have

$$(3.6) \quad \int_B [S_1(f\chi_{2B})(x)]^p d\mu(x) \lesssim \int_{2B} |f(x)|^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

Fix $x \in B$. Notice that if $d(x, y) < t$, then for all $z \in \mathcal{X}$,

$$(3.7) \quad t + d(y, z) \sim t + d(x, z) \quad \text{and} \quad V_t(y) + V(y, z) \sim V_t(x) + V(x, z).$$

From $(Q)_i$, (3.7), (2.2), the Hölder inequality and $\gamma > \alpha n$, it follows that for all $t < 8r$ and $y \in \mathcal{X}$ with $d(x, y) < t$, we have

(3.8)

$$\begin{aligned} |Q_t(f\chi_{(2B)^c})(y)| &\lesssim \int_{(2B)^c} \frac{1}{V_t(y) + V(y, z)} \left(\frac{t}{t + d(y, z)} \right)^\gamma |f(z)| d\mu(z) \\ &\lesssim \int_{(2B)^c} \frac{1}{V_t(x) + V(x, z)} \left(\frac{t}{t + d(x, z)} \right)^\gamma |f(z)| d\mu(z) \\ &\lesssim \left(\frac{t}{r} \right)^\gamma \sum_{j=1}^{\infty} \frac{2^{-j\gamma}}{\mu(2^{j+1}B)} \int_{2^{j+1}B} |f(z)| d\mu(z) \\ &\lesssim \left(\frac{t}{r} \right)^\gamma [\mu(B)]^\alpha \sum_{j=1}^{\infty} 2^{-j(\gamma - \alpha n)} \lesssim \left(\frac{t}{r} \right)^\gamma [\mu(B)]^\alpha. \end{aligned}$$

Notice that for all $x, y \in \mathcal{X}$ satisfying $d(x, y) < t$, we have

$$(3.9) \quad V_t(x) \sim V_t(y).$$

It then follows from (3.8) and (3.9) together with $\gamma \in (0, \infty)$ that

(3.10)

$$\int_B [S_1(f\chi_{(2B)^c})(x)]^p d\mu(x) \lesssim \left[\int_0^{8r} \left(\frac{t}{r} \right)^{2\gamma} \frac{dt}{t} \right]^{p/2} [\mu(B)]^{1+\alpha p} \lesssim [\mu(B)]^{1+\alpha p},$$

which together with (3.6) tells us that

$$(3.11) \quad \int_B [S_1(f)(x)]^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

Observe that for all $y \in \mathcal{X}$ with $d(x, y) < t$, by (2.3), we have

$$(3.12) \quad \frac{\rho(y)}{t + \rho(y)} \lesssim \left(\frac{\rho(x)}{t} \right)^{\frac{1}{1+k_0}},$$

and that for all $x \in B$ with $r \geq \rho(x_0)$, by (2.3), we also have that $\rho(x) \lesssim r$. Combining these two observations yields that for all $x \in B$ and $y \in \mathcal{X}$ with $d(x, y) < t$,

$$(3.13) \quad \frac{\rho(y)}{t + \rho(y)} \lesssim \left(\frac{r}{t} \right)^{\frac{1}{1+k_0}}.$$

It then follows from Lemma 3.2, (3.13) and (2.2) that for all $x \in B$, $t \geq 8r$ and $y \in \mathcal{X}$ with $d(x, y) < t$,

$$\begin{aligned}
(3.14) \quad |Q_t(f)(y)| &\lesssim \left(\frac{\rho(y)}{t + \rho(y)} \right)^{\delta_1} [\mu(B(y, t))]^\alpha \\
&\lesssim \left(\frac{r}{t} \right)^{\frac{\delta_1}{1+k_0}} [\mu(B(x, t))]^\alpha \\
&\lesssim \left(\frac{r}{t} \right)^{\frac{\delta_1}{1+k_0} - \alpha n} [\mu(B)]^\alpha,
\end{aligned}$$

which together with the assumption that $\delta_1 > (1 + k_0)\alpha n$ implies that

$$\int_B [S_2(f)(x)]^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p} \left[\int_{8r}^\infty \left(\frac{r}{t} \right)^{\frac{2\delta_1}{1+k_0} - 2\alpha n} \frac{dt}{t} \right]^{p/2} \lesssim [\mu(B)]^{1+\alpha p}.$$

By this and (3.11), we obtain (3.5). Moreover, it follows from (3.5) that $S(f)(x) < \infty$ for almost every $x \in \mathcal{X}$.

Case II. $B \equiv B(x_0, r) \notin \mathcal{D}$. In this case, $r < \rho(x_0)$. We need to prove that

$$(3.15) \quad \int_B \left\{ [S(f)(x)]^2 - \operatorname{essinf}_B [S(f)]^2 \right\}^{p/2} d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

To this end, for all $x \in \mathcal{X}$, write

$$\begin{aligned}
[S(f)(x)]^2 &= \int_0^{8r} \int_{d(x, y) < t} |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} + \int_{8r}^{8\rho(x_0)} \cdots + \int_{8\rho(x_0)}^\infty \cdots \\
&\equiv [S_r(f)(x)]^2 + [S_{r, x_0}(f)(x)]^2 + [S_\infty(f)(x)]^2.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_B \left\{ [S(f)(x)]^2 - \operatorname{essinf}_B [S(f)]^2 \right\}^{p/2} d\mu(x) \\
&\lesssim \int_B [S_r(f)(x)]^p d\mu(x) + \int_B \left\{ [S_{r, x_0}(f)(x)]^2 - \operatorname{essinf}_B [S_{r, x_0}(f)]^2 \right\}^{p/2} d\mu(x) \\
&\quad + \int_B \left\{ [S_\infty(f)(x)]^2 - \operatorname{essinf}_B [S_\infty(f)]^2 \right\}^{p/2} d\mu(x) \\
&\leq \int_B [S_r(f)(x)]^p d\mu(x) + \mu(B) \sup_{x, x' \in B} |[S_{r, x_0}(f)(x)]^2 - [S_{r, x_0}(f)(x')]^2|^{p/2} \\
&\quad + \mu(B) \sup_{x, x' \in B} |[S_\infty(f)(x)]^2 - [S_\infty(f)(x')]^2|^{p/2} \equiv \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3.
\end{aligned}$$

Write $f \equiv f_1 + f_2 + f_B$, where $f_1 \equiv (f - f_B)\chi_{2B}$ and $f_2 \equiv (f - f_B)\chi_{(2B)^c}$. By the $L^p(\mathcal{X})$ -boundedness of $S(f)$ and (2.2), we have

$$(3.16) \quad \int_B [S_r(f_1)(x)]^p d\mu(x) \lesssim \int_{2B} |f - f_B|^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

It follows from $(Q)_i$, (3.7), (2.2), the Hölder inequality, Lemma 3.1(ii) and $\gamma > \alpha n$ that for all $x \in B$ and $y \in \mathcal{X}$ with $d(x, y) < t \leq 8r$,

$$\begin{aligned} |Q_t(f_2)(y)| &\leq \int_{(2B)^c} \frac{1}{V_t(y) + V(y, z)} \left(\frac{t}{t + d(y, z)} \right)^\gamma |f(z) - f_B| d\mu(z) \\ &\lesssim \int_{(2B)^c} \frac{1}{V_t(x) + V(x, z)} \left(\frac{t}{t + d(x, z)} \right)^\gamma |f(z) - f_B| d\mu(z) \\ &\lesssim \left(\frac{t}{r} \right)^\gamma \sum_{j=1}^{\infty} 2^{-j\gamma} \frac{1}{\mu(2^{j+1}B)} \int_{2^{j+1}B} [|f(z) - f_{2^{j+1}B}| + |f_{2^{j+1}B} - f_B|] d\mu(z) \\ &\lesssim \left(\frac{t}{r} \right)^\gamma [\mu(B)]^\alpha \sum_{j=1}^{\infty} 2^{-j(\gamma - \alpha n)} \lesssim \left(\frac{t}{r} \right)^\gamma [\mu(B)]^\alpha, \end{aligned}$$

which together with (3.9) leads to that

$$(3.17) \quad \int_B [S_r(f_2)(x)]^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p} \left[\int_0^{8r} \left(\frac{t}{r} \right)^{2\gamma} \frac{dt}{t} \right]^{p/2} \lesssim [\mu(B)]^{1+\alpha p}.$$

Observe that by (2.3), for any $a \in (0, \infty)$, there exists a constant $\tilde{C}_a \in [1, \infty)$ such that for all $x, y \in \mathcal{X}$ with $d(x, y) \leq a\rho(x)$,

$$(3.18) \quad \rho(y)/\tilde{C}_a \leq \rho(x) \leq \tilde{C}_a \rho(y).$$

By this, we obtain that for all $x \in B$ and $y \in \mathcal{X}$ satisfying $d(x, y) < t$ with $t \in (0, 8r)$ and $r < \rho(x_0)$, $\rho(y) \sim \rho(x_0)$. Hence, by $(Q)_{ii}$ and Lemma 3.1(i), we have

$$|Q_t(f_B)(y)| \lesssim \left(\frac{t}{\rho(y)} \right)^{\delta_2} |f_B| \lesssim \left(\frac{t}{\rho(x_0)} \right)^{\delta_2} \left(\frac{\rho(x_0)}{r} \right)^{\alpha n} [\mu(B)]^\alpha,$$

which together with $\delta_2 > \alpha n$, $r < \rho(x_0)$ and (3.9) implies that

$$\begin{aligned} \int_B [S_r(f_B)(x)]^p d\mu(x) &\lesssim [\mu(B)]^{1+\alpha p} \left(\frac{\rho(x_0)}{r} \right)^{\alpha p n} \left[\int_0^{8r} \left(\frac{t}{\rho(x_0)} \right)^{2\delta_2} \frac{dt}{t} \right]^{p/2} \\ &\lesssim [\mu(B)]^{1+\alpha p}. \end{aligned}$$

Combining this, (3.16) and (3.17) yields $I_1 \lesssim [\mu(B)]^{1+\alpha p}$.

Now we turn our attention to prove that for all $x, x' \in B$,

$$|[S_{r, x_0}(f)(x)]^2 - [S_{r, x_0}(f)(x')]^2| \lesssim [\mu(B)]^{2\alpha}.$$

Write

$$\begin{aligned} & |[S_{r, x_0}(f)(x)]^2 - [S_{r, x_0}(f)(x')]^2| \\ &= \left| \int_{8r}^{8\rho(x_0)} \int_{d(x, y) < t} |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} - \int_{8r}^{8\rho(x_0)} \int_{d(x', y) < t} \dots \right| \\ &\leq \int_{8r}^{8\rho(x_0)} \int_{B(x, t) \triangle B(x', t)} |Q_t(f - f_B)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\ &\quad + \int_{8r}^{8\rho(x_0)} \int_{B(x, t) \triangle B(x', t)} |Q_t(f_B)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \equiv J_1 + J_2, \end{aligned}$$

where $B(x, t) \triangle B(x', t) \equiv [B(x, t) \setminus B(x', t)] \cup [B(x', t) \setminus B(x, t)]$.

By the facts that $x, x' \in B$ and $t \geq 8r$, we have $B(x, t - 2r) \subset [B(x, t) \cap B(x', t)]$. Since $\mathcal{X}$ has the volume regularity Property (P), we obtain

$$\mu(B(x, t) \setminus B(x', t)) \leq \mu(B(x, t)) - \mu(B(x, t - 2r)) \lesssim \left(\frac{r}{t}\right)^\delta \mu(B(x, t)).$$

By symmetry, we also have $\mu(B(x', t) \setminus B(x, t)) \lesssim \left(\frac{r}{t}\right)^\delta \mu(B(x', t))$, which together with (2.1) implies that

$$(3.19) \quad \mu(B(x, t) \triangle B(x', t)) \lesssim \left(\frac{r}{t}\right)^\delta \mu(B(x, t)).$$

By (Q)_i, (3.7), (3.19), (3.9), (2.2), the Hölder inequality and Lemma 3.1(ii), we obtain

$$\begin{aligned} J_1 &\lesssim \int_{8r}^{8\rho(x_0)} \left(\frac{r}{t}\right)^\delta \left[\int_{\mathcal{X}} \frac{1}{V_t(x) + V(x, z)} \left(\frac{t}{t + d(x, z)}\right)^\gamma |f(z) - f_B| d\mu(z) \right]^2 \frac{dt}{t} \\ &\lesssim \int_{8r}^{8\rho(x_0)} \left(\frac{r}{t}\right)^\delta \left[\frac{1}{\mu(2B)} \int_{2B} |f(z) - f_B| d\mu(z) \right. \\ &\quad \left. + \sum_{j=1}^{\infty} \frac{t^\gamma}{(t + 2^{j-1}r)^\gamma} \frac{1}{\mu(2^{j+1}B)} \int_{2^{j+1}B} |f(z) - f_B| d\mu(z) \right]^2 \frac{dt}{t} \\ &\lesssim \int_{8r}^{8\rho(x_0)} \left(\frac{r}{t}\right)^\delta \left[1 + \sum_{j=0}^{\infty} \frac{t^\gamma 2^{j\alpha n}}{(t + 2^{j-1}r)^\gamma} \right]^2 \frac{dt}{t} [\mu(B)]^{2\alpha}. \end{aligned}$$

Notice that $\gamma > \alpha n$ and $\delta > 2\alpha n$. Choosing $\epsilon \in (\alpha n, \min\{\gamma, \delta/2\})$, we have

$$\begin{aligned} J_1 &\lesssim \left\{ 1 + \int_{8r}^{8\rho(x_0)} \left(\frac{r}{t}\right)^\delta \left[\sum_{j=0}^{\infty} \frac{t^\gamma 2^{j\alpha n}}{t^{\gamma-\epsilon} (2^{j-1}r)^\epsilon} \right]^2 \frac{dt}{t} \right\} [\mu(B)]^{2\alpha} \\ &\lesssim \left\{ 1 + \int_{8r}^{\infty} \left(\frac{r}{t}\right)^{\delta-2\epsilon} \frac{dt}{t} \right\} [\mu(B)]^{2\alpha} \lesssim [\mu(B)]^{2\alpha}. \end{aligned}$$

Thus, $J_1 \lesssim [\mu(B)]^{2\alpha}$.

Notice that $r < \rho(x_0)$ and $t \in (8r, 8\rho(x_0))$. By (3.18), we have that for any $x \in B$ and $y \in \mathcal{X}$ with $d(x, y) < t$, $\rho(x_0) \sim \rho(x) \sim \rho(y)$. Choosing $\eta \in (0, 1)$ such that $\eta\delta_2 = \alpha n$, then by Lemma 3.1(i), (Q)_{ii} and (3.9), we have

$$\begin{aligned} J_2 &\lesssim \int_{8r}^{8\rho(x_0)} \left(\frac{r}{t}\right)^\delta \left[\left(\frac{t}{t+\rho(x_0)}\right)^{\delta_2} \left(\frac{\rho(x_0)}{r}\right)^{\alpha n} [\mu(B)]^\alpha \right]^2 \frac{dt}{t} \\ &\lesssim \int_{8r}^{8\rho(x_0)} \left(\frac{r}{t}\right)^\delta \left[\left(\frac{t}{\rho(x_0)}\right)^{\eta\delta_2} \left(\frac{\rho(x_0)}{r}\right)^{\alpha n} [\mu(B)]^\alpha \right]^2 \frac{dt}{t} \\ &\lesssim [\mu(B)]^{2\alpha} \int_{8r}^{\infty} \left(\frac{r}{t}\right)^{\delta-2\alpha n} \frac{dt}{t} \lesssim [\mu(B)]^{2\alpha}. \end{aligned}$$

Combining the estimates for J_1 and J_2 yields $I_2 \lesssim [\mu(B)]^{1+\alpha p}$.

To prove Theorem 3.1, it remains to estimate the term I_3 . For all $x, x' \in B$, write

$$\begin{aligned} &| [S_\infty(f)(x)]^2 - [S_\infty(f)(x')]^2 | \\ &= \left| \int_{8\rho(x_0)}^{\infty} \int_{d(x,y)<t} |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} - \int_{8\rho(x_0)}^{\infty} \int_{d(x',y)<t} \dots \right| \\ &\leq \int_{8\rho(x_0)}^{\infty} \int_{B(x,t) \triangle B(x',t)} |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t}. \end{aligned}$$

Notice that $r < \rho(x_0)$. Hence, for $t \in (8\rho(x_0), \infty)$, (3.19) still holds. On the other hand, for all $x \in B$ and $y \in \mathcal{X}$ with $d(x, y) < t$, by (3.12), Lemma 3.2, (2.2) and the fact that $\rho(x_0) \sim \rho(x)$, we obtain that

$$|Q_t(f)(y)| \lesssim \left(\frac{\rho(y)}{t+\rho(y)}\right)^{\delta_1} [\mu(B(y, t))]^\alpha \lesssim \left(\frac{\rho(x_0)}{t}\right)^{\frac{\delta_1}{1+k_0}} \left(\frac{t}{r}\right)^{\alpha n} [\mu(B)]^\alpha.$$

It follows from this, (3.19), (3.9) and $\delta > 2\alpha n$ that for all $x, x' \in B$,

$$\begin{aligned} |[S_\infty(f)(x)]^2 - [S_\infty(f)(x')]^2| &\lesssim \int_{8\rho(x_0)}^\infty \left(\frac{\rho(x_0)}{t} \right)^{\frac{2\delta_1}{1+k_0}} \left(\frac{r}{t} \right)^{\delta-2\alpha n} [\mu(B)]^{2\alpha} \frac{dt}{t} \\ &\lesssim [\mu(B)]^{2\alpha}, \end{aligned}$$

so that $I_3 \lesssim [\mu(B)]^{1+\alpha p}$. This finishes the proof of Theorem 3.1. $\square$

As a consequence of Theorem 3.1, we have the following conclusion, which can be proved by an argument similar to that used in the proof of [34, Corollary 4.1]. We omit the details.

Corollary 3.1. *With the assumptions same as in Theorem 3.1, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$, $S(f) \in \tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$ and $\|S(f)\|_{\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})} \leq C\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}$.*

Remark 3.2. (i) If $\alpha = 0$, Theorem 3.1 and Corollary 3.1 were already obtained in [19].

(ii) If $\alpha > 0$, then by Remark 2.1(iv), the space $\tilde{\mathcal{E}}_\rho^{2\alpha,p/2}(\mathcal{X})$ in Theorem 3.1 and the space $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$ in Corollary 3.1 are exactly the spaces $\mathcal{E}_\rho^{2\alpha,p/2}(\mathcal{X})$ and $\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$, respectively.

(iii) If $\alpha < 0$, then by Theorem 2.2 and the fact that the Lusin-area function is nonnegative, we know that if the space $\tilde{\mathcal{E}}_\rho^{2\alpha,p/2}(\mathcal{X})$ in Theorem 3.1 and the space $\tilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$ in Corollary 3.1 are replaced, respectively, by the spaces $\mathcal{E}_\rho^{2\alpha,p/2}(\mathcal{X})$ and $\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$, we obtain the same results.

Now we study the boundedness of g_λ^* function in localized Morrey-Campanato spaces. In this case, $\mathcal{X}$ is not necessary to have Property (P).

Theorem 3.2. *Let $\mathcal{X}$ be a doubling metric measure space. Let $p \in (1, \infty)$, ρ be an admissible function on $\mathcal{X}$, the g_λ^* function $g_\lambda^*(f)$ as in (3.4) with $\lambda \in (3n, \infty)$ and*

$$\alpha \in (-\infty, \min\{\gamma/n, \delta_1/[(1+k_0)n], \delta_2/n, (\lambda-3n)/[2(1+k_0)n], 1/(2n)\}).$$

If $g_\lambda^(f)$ is bounded on $L^p(\mathcal{X})$, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$, $[g_\lambda^*(f)]^2 \in \tilde{\mathcal{E}}_\rho^{2\alpha,p/2}(\mathcal{X})$ and $\|[g_\lambda^*(f)]^2\|_{\tilde{\mathcal{E}}_\rho^{2\alpha,p/2}(\mathcal{X})} \leq C\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}^2$.*

Proof. By similarity, we only prove the case when $\alpha > 0$. Let $f \in \mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$, by the homogeneity of $\|\cdot\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}$ and $\|\cdot\|_{\tilde{\mathcal{E}}_\rho^{2\alpha,p/2}(\mathcal{X})}$, we may assume that $\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})} = 1$.

Let $B \equiv B(x_0, r)$. For any nonnegative integer k , let

$$J(k) \equiv \{(y, t) \in \mathcal{X} \times (0, \infty) : d(y, x_0) < 2^{k+1}r \text{ and } 0 < t < 2^{k+1}r\}.$$

For any $f \in \mathcal{E}_\rho^{\alpha, p}(\mathcal{X})$ and $x \in \mathcal{X}$, write

$$\begin{aligned} [g_\lambda^*(f)(x)]^2 &= \iint_{J(0)} \left(\frac{t}{t+d(x, y)} \right)^\lambda |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} + \iint_{[\mathcal{X} \times (0, \infty)] \setminus J(0)} \cdots \\ &\equiv [g_{\lambda, 0}^*(f)(x)]^2 + [g_{\lambda, \infty}^*(f)(x)]^2. \end{aligned}$$

We now consider the following two cases.

Case I. $B \equiv B(x_0, r) \in \mathcal{D}$. Here, $r \geq \rho(x_0)$. We first prove that

$$(3.20) \quad \int_B [g_{\lambda, 0}^*(f)(x)]^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

For any $x \in B$, write

$$\begin{aligned} [g_{\lambda, 0}^*(f)(x)]^2 &\leq \iint_{\substack{J(0) \\ d(x, y) < t}} \left(\frac{t}{t+d(x, y)} \right)^\lambda |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\ &\quad + 2 \iint_{\substack{J(0) \\ d(x, y) \geq t}} \left(\frac{t}{t+d(x, y)} \right)^\lambda |Q_t(f\chi_{8B})(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\ &\quad + 2 \iint_{\substack{J(0) \\ d(x, y) \geq t}} \left(\frac{t}{t+d(x, y)} \right)^\lambda |Q_t(f\chi_{(8B)^c})(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\ &\equiv I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

Note that for all $x \in B$, $I_1(x) \leq [S(f)(x)]^2$ and then (3.5) gives

$$(3.21) \quad \int_B [I_1(x)]^{p/2} d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

We remark that in the proof of (3.5), we did not use Property (P) of $\mathcal{X}$.

By the $L^p(\mathcal{X})$ -boundedness of $g_\lambda^*(f)$ and (2.1), we have

$$(3.22) \quad \int_B [I_2(x)]^{p/2} d\mu(x) \lesssim \int_{8B} |f(x)|^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

To deal with $I_3(x)$, noticing that for all $z \in (8B)^c$ and $y \in \mathcal{X}$ with $d(y, x_0) < 2r$, we have that $d(y, z) \sim d(x_0, z)$ and $V(y, z) \sim V(x_0, z)$. Hence, from the assumptions that $\lambda > n$ and $\gamma > \alpha n$, it follows

$$\begin{aligned} I_3(x) &\lesssim \int_0^{2r} \int_{\substack{d(y, x_0) < 2r \\ d(x, y) \geq t}} \left(\frac{t}{t+d(x, y)} \right)^\lambda \\ &\quad \times \left[\int_{(8B)^c} \frac{1}{V_t(y) + V(y, z)} \left(\frac{t}{t+d(y, z)} \right)^\gamma |f(z)| d\mu(z) \right]^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \end{aligned}$$

$$\begin{aligned}
&\lesssim \int_0^{2r} \int_{\substack{d(y, x_0) < 2r \\ d(x, y) \geq t}} \left(\frac{t}{t + d(x, y)} \right)^\lambda \left[\int_{(8B)^c} \frac{1}{V(x_0, z)} \left(\frac{t}{d(x_0, z)} \right)^\gamma |f(z)| d\mu(z) \right]^2 \\
&\quad \times \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\
&\lesssim \int_0^{2r} \int_{\substack{d(y, x_0) < 2r \\ d(x, y) \geq t}} \left(\frac{t}{t + d(x, y)} \right)^\lambda \\
&\quad \times \left[\sum_{j=3}^{\infty} \left(\frac{t}{2^j r} \right)^\gamma \frac{1}{\mu(2^{j+1}B)} \int_{2^{j+1}B} |f(z)| d\mu(z) \right]^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\
&\lesssim \int_0^{2r} \int_{\substack{d(y, x_0) < 2r \\ d(x, y) \geq t}} \left(\frac{t}{t + d(x, y)} \right)^\lambda \left\{ \left(\frac{t}{r} \right)^\gamma \sum_{j=3}^{\infty} 2^{-j(\gamma - \alpha n)} [\mu(B)]^\alpha \right\}^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\
&\lesssim [\mu(B)]^{2\alpha} \int_0^{2r} \left(\frac{t}{r} \right)^{2\gamma} \sum_{k=0}^{\infty} \int_{2^k t \leq d(x, y) < 2^{k+1} t} 2^{-k\lambda} 2^{kn} \frac{d\mu(y)}{V_{2^{k+1}t}(x)} \frac{dt}{t} \lesssim [\mu(B)]^{2\alpha},
\end{aligned}$$

which implies that $\int_B [I_3(x)]^{p/2} d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}$. Combining this, (3.21) and (3.22) proves (3.20).

Now we prove that

$$(3.23) \quad \int_B [g_{\lambda, \infty}^*(f)(x)]^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

Notice that for $(y, t) \in J(k) \setminus J(k-1)$ with $k \in \mathbb{N}$ and $x \in B$, $t + d(x, y) \sim 2^k r$. Thus,

$$\begin{aligned}
&[g_{\lambda, \infty}^*(f)(x)]^2 \\
&\lesssim \sum_{k=1}^{\infty} \iint_{J(k) \setminus J(k-1)} \left(\frac{t}{2^k r} \right)^\lambda \\
&\quad \times \left[\int_{2^{k+4}B} \frac{1}{V_t(y) + V(y, z)} \left(\frac{t}{t + d(y, z)} \right)^\gamma \left(\frac{\rho(y)}{t + \rho(y)} \right)^{\delta_1} |f(z)| d\mu(z) \right]^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\
&\quad + \sum_{k=1}^{\infty} \iint_{J(k) \setminus J(k-1)} \left(\frac{t}{2^k r} \right)^\lambda \left[\int_{(2^{k+4}B)^c} \cdots d\mu(z) \right]^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \equiv E_1(x) + E_2(x).
\end{aligned}$$

The fact that $r \geq \rho(x_0)$ and (2.3) imply that for all $y \in \mathcal{X}$ with $d(y, x_0) < 2^{k+1}r$,

$$(3.24) \quad \rho(y) \lesssim [\rho(x_0)]^{\frac{1}{1+k_0}} (2^k r)^{\frac{k_0}{1+k_0}}.$$

By the assumptions that $\lambda \in (3n, \infty)$, $\delta_1 > (1+k_0)\alpha n$ and $(\lambda-3n) > 2(1+k_0)\alpha n$, we choose $\eta_1 \in [(0, \delta_1) \cap ((1+k_0)\alpha n, (\lambda-3n)/2)]$. Therefore, $\lambda-2\eta_1-3n > 0$. It then follows from (3.24) that

$$\begin{aligned} E_1(x) &\lesssim \sum_{k=1}^{\infty} \int_0^{2^{k+1}r} \int_{d(y, x_0) < 2^{k+1}r} \left(\frac{t}{2^k r}\right)^{\lambda} \left(\frac{2^k r}{t}\right)^{2n} \left(\frac{[\rho(x_0)]^{\frac{1}{1+k_0}} (2^k r)^{\frac{k_0}{1+k_0}}}{t}\right)^{2\eta_1} \\ &\quad \times 2^{2k\alpha n} [\mu(B)]^{2\alpha} \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\ &\lesssim [\mu(B)]^{2\alpha} \sum_{k=1}^{\infty} 2^{2k\alpha n} \int_0^{2^{k+1}r} \left(\frac{t}{2^k r}\right)^{\lambda} \left(\frac{2^k r}{t}\right)^{3n} \left(\frac{[\rho(x_0)]^{\frac{1}{1+k_0}} (2^k r)^{\frac{k_0}{1+k_0}}}{t}\right)^{2\eta_1} \frac{dt}{t} \\ &\lesssim [\mu(B)]^{2\alpha} \sum_{k=1}^{\infty} 2^{2k\alpha n} \left[\frac{\rho(x_0)}{2^k r}\right]^{\frac{2\eta_1}{1+k_0}} \lesssim [\mu(B)]^{2\alpha}. \end{aligned}$$

Choosing $\eta_2 \in [(0, \delta_1) \cap ((1+k_0)\alpha n, (\lambda-3n)/2)]$, then $\lambda+2\gamma-2\eta_2-n > 0$. Notice that for $z \in (2^{k+4}B)^{\mathbb{G}}$ and $y \in \mathcal{X}$ with $d(y, x_0) < 2^{k+1}r$, $d(y, z) \sim d(x_0, z)$ and $V(y, z) \sim V(x_0, z)$. This together with (3.24) and $\gamma > \alpha n$ implies that

$$\begin{aligned} E_2(x) &\lesssim \sum_{k=1}^{\infty} \int_0^{2^{k+1}r} \int_{d(y, x_0) < 2^{k+1}r} \left(\frac{t}{2^k r}\right)^{\lambda} \left[\left(\frac{[\rho(x_0)]^{\frac{1}{1+k_0}} (2^k r)^{\frac{k_0}{1+k_0}}}{t}\right)^{\eta_2}\right. \\ &\quad \times \sum_{j=1}^{\infty} \left(\frac{t}{2^{j+k+1}r}\right)^{\gamma} \frac{1}{\mu(2^{j+k+4}B)} \int_{2^{j+k+4}B} |f(z)| d\mu(z) \left.\right]^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\ &\lesssim [\mu(B)]^{2\alpha} \sum_{k=1}^{\infty} 2^{2k\alpha n} \int_0^{2^{k+1}r} \left(\frac{t}{2^k r}\right)^{\lambda+2\gamma-n} \left(\frac{[\rho(x_0)]^{\frac{1}{1+k_0}} (2^k r)^{\frac{k_0}{1+k_0}}}{t}\right)^{2\eta_2} \frac{dt}{t} \\ &\lesssim [\mu(B)]^{2\alpha} \sum_{k=1}^{\infty} 2^{2k\alpha n} \left[\frac{\rho(x_0)}{2^k r}\right]^{\frac{2\eta_2}{1+k_0}} \lesssim [\mu(B)]^{2\alpha}, \end{aligned}$$

which together with the estimate of $E_1(x)$ yields (3.23).

Combining (3.20) and (3.23) yields that

$$(3.25) \quad \int_B [g_{\lambda}^*(f)(x)]^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

Moreover, from (3.25), it follows that $g_\lambda^*(f)(x) < \infty$ for almost every $x \in \mathcal{X}$.

Case II. $B \equiv B(x_0, r) \notin \mathcal{D}$. In this case, $r < \rho(x_0)$. We need to prove that

$$\int_B \left\{ [g_\lambda^*(f)(x)]^2 - \operatorname{ess\,inf}_B [g_\lambda^*(f)]^2 \right\}^{p/2} d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

To this end, for all $x \in \mathcal{X}$, write

$$\begin{aligned} & \int_B \left\{ [g_\lambda^*(f)(x)]^2 - \operatorname{ess\,inf}_B [g_\lambda^*(f)]^2 \right\}^{p/2} d\mu(x) \\ & \lesssim \int_B [g_{\lambda,0}^*(f)(x)]^p d\mu(x) + \int_B \left\{ [g_{\lambda,\infty}^*(f)(x)]^2 - \operatorname{ess\,inf}_B [g_{\lambda,\infty}^*(f)]^2 \right\}^{p/2} d\mu(x) \\ & \lesssim \int_B [g_{\lambda,0}^*(f)(x)]^p d\mu(x) + \mu(B) \sup_{x, x' \in B} |[g_{\lambda,\infty}^*(f)(x)]^2 - [g_{\lambda,\infty}^*(f)(x')]^2|^{p/2}. \end{aligned}$$

Now we prove that

$$(3.26) \quad \int_B [g_{\lambda,0}^*(f)(x)]^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

To this end, write $f \equiv f_1 + f_2 + f_B$, where $f_1 \equiv (f - f_B)\chi_{8B}$ and $f_2 \equiv (f - f_B)\chi_{(8B)^c}$. By the $L^p(\mathcal{X})$ -boundedness of $g_\lambda^*(f)$, (2.2) and Lemma 3.1(ii), we have

$$(3.27) \quad \int_B [g_{\lambda,0}^*(f_1)(x)]^p d\mu(x) \lesssim \int_{8B} |f - f_B|^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

Notice that for $z \in (8B)^c$ and $y \in \mathcal{X}$ with $d(y, x_0) < 2r$, $d(y, z) \sim d(x_0, z)$ and $V(y, z) \sim V(x_0, z)$. This together with $(Q)_i$, (2.2), the Hölder inequality, Lemma 3.1(ii) and $\gamma > \alpha n$ yields that

$$\begin{aligned} |Q_t(f_2)(y)| & \lesssim \int_{(8B)^c} \frac{1}{V_t(y) + V(y, z)} \left(\frac{t}{t + d(y, z)} \right)^\gamma |f(z) - f_B| d\mu(z) \\ & \lesssim \int_{(8B)^c} \frac{1}{V(x_0, z)} \left(\frac{t}{d(x_0, z)} \right)^\gamma |f(z) - f_B| d\mu(z) \\ & \lesssim \sum_{k=1}^{\infty} \left(\frac{t}{2^{kr}} \right)^\gamma \left[\frac{1}{\mu(2^{k+3}B)} \int_{2^{k+3}B} |f(z) - f_{2^{k+3}B}| d\mu(z) + |f_{2^{k+3}B} - f_B| \right] \\ & \lesssim \left(\frac{t}{r} \right)^\gamma [\mu(B)]^\alpha \sum_{k=1}^{\infty} 2^{-k(\gamma - \alpha n)} \lesssim \left(\frac{t}{r} \right)^\gamma [\mu(B)]^\alpha. \end{aligned}$$

By an argument similar to the estimates of (3.10) and $I_3(x)$, we obtain

(3.28)

$$\begin{aligned}
& \int_B [g_{\lambda,0}^*(f_2)(x)]^p d\mu(x) \\
& \lesssim \int_B \left[\int_0^{2r} \int_{d(x_0,y) < 2r} \left(\frac{t}{t+d(x,y)} \right)^\lambda \left(\frac{t}{r} \right)^{2\gamma} [\mu(B)]^{2\alpha} \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \right]^{p/2} d\mu(x) \\
& \lesssim \int_B \left[\int_0^{2r} \int_{d(x,y) < t} \left(\frac{t}{r} \right)^{2\gamma} [\mu(B)]^{2\alpha} \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \right]^{p/2} d\mu(x) \\
& \quad + \int_B \left[\int_0^{2r} \int_{\substack{d(x_0,y) < 2r \\ d(x,y) \geq t}} \left(\frac{t}{t+d(x,y)} \right)^\lambda \left(\frac{t}{r} \right)^{2\gamma} [\mu(B)]^{2\alpha} \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \right]^{p/2} d\mu(x) \\
& \lesssim [\mu(B)]^{1+\alpha p}.
\end{aligned}$$

For $y \in \mathcal{X}$ with $d(x_0, y) < 2r < 2\rho(x_0)$, by (3.18), we have that $\rho(x_0) \sim \rho(y)$, which together with $(Q)_{ii}$, Lemma 3.1(i) and $\delta_2 > \alpha n$ leads to that

$$|Q_t(f_B)(y)| \lesssim \left(\frac{t}{\rho(y)} \right)^{\delta_2} |f_B| \lesssim \left(\frac{t}{\rho(x_0)} \right)^{\delta_2} \left(\frac{\rho(x_0)}{r} \right)^{\alpha n} [\mu(B)]^\alpha \lesssim \left(\frac{t}{r} \right)^{\delta_2} [\mu(B)]^\alpha.$$

Then, similarly to the estimate of (3.28), we obtain

$$\int_B [g_{\lambda,0}^*(f_B)(x)]^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p},$$

which together with (3.27) and (3.28) yields (3.26).

The proof of Theorem 3.2 is reduced to show that for all $x, x' \in B$,

$$(3.29) \quad |[g_{\lambda,\infty}^*(f)(x)]^2 - [g_{\lambda,\infty}^*(f)(x')]^2| \lesssim [\mu(B)]^{2\alpha}.$$

Write

$$\begin{aligned}
& |[g_{\lambda,\infty}^*(f)(x)]^2 - [g_{\lambda,\infty}^*(f)(x')]^2| \\
& \leq \iint_{\mathcal{X} \times (0, \infty) \setminus J(0)} \left| \left(\frac{t}{t+d(x,y)} \right)^\lambda - \left(\frac{t}{t+d(x',y)} \right)^\lambda \right| |Q_t(f)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\
& \lesssim \sum_{k=1}^{\infty} \iint_{J(k) \setminus J(k-1)} \frac{rt^\lambda}{(2^k r)^{\lambda+1}} |Q_t(f - f_B)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\
& \quad + \sum_{k=1}^{\infty} \iint_{J(k) \setminus J(k-1)} \frac{rt^\lambda}{(2^k r)^{\lambda+1}} |Q_t(f_B)(y)|^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \equiv G_1 + G_2.
\end{aligned}$$

The assumptions that $\lambda \in (3n, \infty)$ and $\gamma > \alpha n$, together with $(Q)_i$ and Lemma 3.1(ii), imply that

$$\begin{aligned}
G_1 &\lesssim \sum_{k=1}^{\infty} \iint_{J(k) \setminus J(k-1)} \frac{rt^\lambda}{(2^k r)^{\lambda+1}} \\
&\quad \times \left[\int_{2^{k+4}B} \frac{1}{V_t(y) + V(y, z)} \left(\frac{t}{t + d(y, z)} \right)^\gamma |f(z) - f_B| d\mu(z) \right]^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\
&\quad + \sum_{k=1}^{\infty} \iint_{J(k) \setminus J(k-1)} \frac{rt^\lambda}{(2^k r)^{\lambda+1}} \left[\int_{(2^{k+4}B)^c} \cdots d\mu(z) \right]^2 \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\
&\lesssim \sum_{k=1}^{\infty} \int_0^{2^{k+1}r} \int_{d(y, x_0) < 2^{k+1}r} \frac{rt^\lambda}{(2^k r)^{\lambda+1}} \left(\frac{2^k r}{t} \right)^{2n} 2^{2k\alpha n} [\mu(B)]^{2\alpha} \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\
&\quad + \sum_{k=1}^{\infty} \int_0^{2^{k+1}r} \int_{d(y, x_0) < 2^{k+1}r} \frac{rt^\lambda}{(2^k r)^{\lambda+1}} \left(\frac{t}{2^k r} \right)^{2\gamma} 2^{2k\alpha n} [\mu(B)]^{2\alpha} \frac{d\mu(y)}{V_t(y)} \frac{dt}{t} \\
&\lesssim [\mu(B)]^{2\alpha} \sum_{k=1}^{\infty} 2^{2k\alpha n} \int_0^{2^{k+1}r} \frac{rt^\lambda}{(2^k r)^{\lambda+1}} \left(\frac{2^k r}{t} \right)^{3n} \frac{dt}{t} \\
&\quad + [\mu(B)]^{2\alpha} \sum_{k=1}^{\infty} 2^{2k\alpha n} \int_0^{2^{k+1}r} \frac{rt^\lambda}{(2^k r)^{\lambda+1}} \left(\frac{2^k r}{t} \right)^{n-2\gamma} \frac{dt}{t} \\
&\lesssim [\mu(B)]^{2\alpha} \sum_{k=1}^{\infty} 2^{2k\alpha n} 2^{-k} \lesssim [\mu(B)]^{2\alpha},
\end{aligned}$$

where in the last inequality, we used the assumption that $2\alpha n < 1$.

Notice that $r < \rho(x_0)$, there exists a positive integer k_* such that $2^{k_*-1}r < \rho(x_0) \leq 2^{k_*}r$. If $k \in \{1, \dots, k_*\}$, then for $y \in \mathcal{X}$ with $d(y, x_0) < 2^{k+1}r$, by (3.18), we have that $\rho(x_0) \sim \rho(y)$; if $k \in \{k_*+1, k_*+2, \dots\}$, then by (2.3), for $t \in (0, 2^{k+1}r)$ and $y \in \mathcal{X}$ with $d(y, x_0) < 2^{k+1}r$, we have

$$\frac{t}{t + \rho(y)} \leq \frac{t}{\rho(y)} \lesssim \frac{t}{\rho(x_0)} \left(1 + \frac{d(y, x_0)}{\rho(x_0)} \right)^{k_0} \lesssim \frac{2^k r}{\rho(x_0)} \left(1 + \frac{2^k r}{\rho(x_0)} \right)^{k_0} \lesssim \left(\frac{2^k r}{\rho(x_0)} \right)^{1+k_0}.$$

From $(Q)_{ii}$, Lemma 3.1(i), $\delta_2 > \alpha n$, $\lambda \in (n, \infty)$ and $2\alpha n < 1$, it then follows that

$$\begin{aligned}
G_2 &\lesssim \sum_{k=1}^{\infty} \int_0^{2^{k+1}r} \int_{d(y, x_0) < 2^{k+1}r} \frac{rt^\lambda}{(2^k r)^{\lambda+1}} \left(\frac{\rho(x_0)}{r} \right)^{2\alpha n} [\mu(B)]^{2\alpha} \\
&\quad \times \left(\frac{t}{t + \rho(y)} \right)^{2\delta_2} \frac{d\mu(y)}{V_t(y)} \frac{dt}{t}
\end{aligned}$$

$$\begin{aligned}
&\lesssim [\mu(B)]^{2\alpha} \left\{ \sum_{k=1}^{k_*} \int_0^{2^{k+1}r} \frac{rt^\lambda}{(2^k r)^{\lambda+1}} \left(\frac{\rho(x_0)}{r} \right)^{2\alpha n} \left(\frac{t}{\rho(x_0)} \right)^{2\alpha n} \left(\frac{2^k r}{t} \right)^n \frac{dt}{t} \right. \\
&\quad \left. + \sum_{k=k_*+1}^{\infty} \int_0^{2^{k+1}r} \frac{rt^\lambda}{(2^k r)^{\lambda+1}} \left(\frac{\rho(x_0)}{r} \right)^{2\alpha n} \left(\frac{2^k r}{\rho(x_0)} \right)^{2\alpha n} \left(\frac{2^k r}{t} \right)^n \frac{dt}{t} \right\} \\
&\lesssim [\mu(B)]^{2\alpha} \sum_{k=1}^{\infty} 2^{-k(1-2\alpha n)} \lesssim [\mu(B)]^{2\alpha}.
\end{aligned}$$

Combining the estimates for G_1 and G_2 yields (3.29), which completes the proof of Theorem 3.2. $\square$

As a consequence of Theorem 3.2, we have the following conclusion.

Corollary 3.2. *With the assumptions same as in Theorem 3.2, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha, p}(\mathcal{X})$, $g_\lambda^*(f) \in \tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X})$ and $\|g_\lambda^*(f)\|_{\tilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X})} \leq C\|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})}$.*

We point out that Remark 3.2 is also suitable to Theorem 3.2 and Corollary 3.2.

The following is a simple corollary of Theorems 3.1 and 3.2, and Corollaries 3.1 and 3.2. We omit the details here; see [34, Section 5].

Proposition 3.1. *Theorems 3.1 and 3.2, and Corollaries 3.1 and 3.2 are true if*

$$Q_t \equiv t^2 \frac{de^{-s\mathcal{L}}}{ds} \Big|_{s=t^2},$$

where $\mathcal{L} = -\Delta + V$ is the Schrödinger operator or the degenerate Schrödinger operator on $\mathbb{R}^d$, or the sub-Laplace Schrödinger operator on Heisenberg groups or connected and simply connected nilpotent Lie groups, and V is a nonnegative function satisfying certain reverse Hölder inequality; see the details in [34, Section 5].

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College of Science, China Agricultural University
 Beijing 100083
 People's Republic of China
 and

School of Mathematical Sciences, Beijing Normal University
 Laboratory of Mathematics and Complex Systems
 Ministry of Education, Beijing 100875
 People's Republic of China
 (E-mail : linhaibo@mail.bnu.edu.cn)

Department of Mathematics, Osaka Kyoiku University
 Kashiwara, Osaka 582-8582
 Japan
 (E-mail : enakai@cc.osaka-kyoiku.ac.jp)

Current Address : Department of Mathematics, Ibaraki University
 Mito Ibaraki 310–5812, Japan, (E-mail : enakai@m.x.ibaraki.ac.jp)

School of Mathematical Sciences, Beijing Normal University
 Laboratory of Mathematics and Complex Systems
 Ministry of Education, Beijing 100875
 People's Republic of China
 (E-mail : dcyang@bnu.edu.cn)

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