Fractional integral operators on Herz spaces for supercritical indices

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Abstract. We consider the boundedness of fractional integral operators I_β on Herz spaces $K_q^{\alpha,p}(\mathbb{R}^n)$, where $q \geq n/\beta$. We introduce a new function space that is a variant of Lipschitz space. Our results are optimal.

1. Introduction

Since Beurling [1] introduced the Beurling algebras and Herz [5] generalized these spaces, many studies have been done for Herz spaces (see, for example, [3], [4] and [9]). Li and Yang [7] considered the boundedness of fractional integral operators on Herz spaces. In this paper we consider the boundedness of fractional integral operators on Herz spaces for supercritical indices and we shall show our results are optimal.

The following notation is used: For a set $E \subset \mathbb{R}^n$ we denote the Lebesgue measure of E by $|E|$. We denote the characteristic function of E by χ_E . We write a ball of radius R centered at x_0 by $B(x_0, R) = \{x; |x - x_0| < R\}$, and write $A_k = \{x \in \mathbb{R}^n; 2^{k-1} < |x| \leq 2^k\}$ where $k \in \mathbb{Z}$.

First we define homogeneous Herz spaces and fractional integral. Let $0 < p < \infty$, $1 \leq q < \infty$ and $\alpha \in \mathbb{R}^1$.

Definition 1. $\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}); \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\}$, where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\int_{A_k} |f(x)|^q dx \right)^{p/q} \right\}^{1/p}.$$

Remark . $K_q^{0,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$.

Definition 2.

$$I_\beta f(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\beta}} f(y) dy \quad \text{for } 0 < \beta < n.$$

The next proposition is well-known (see, for example, [14]).

Proposition 1. I_β is bounded from $L^{q_1}(\mathbb{R}^n)$ to $L^{q_2}(\mathbb{R}^n)$ where $1/q_2 = 1/q_1 - \beta/n > 0$.

Li and Yang [7] proved the following.

Proposition 2. I_β is bounded from $\dot{K}_{q_1}^{\alpha,p}(\mathbb{R}^n)$ to $\dot{K}_{q_2}^{\alpha,p}(\mathbb{R}^n)$ where

(1) $1/q_2 = 1/q_1 - \beta/n > 0$, $\beta - n/q_1 < \alpha < n(1 - 1/q_1)$ and $0 < p < \infty$.

When $q_1 \geq n/\beta$, we know the following proposition (see [12] and [14]). We define Lipschitz spaces and modified fractional integral. Let $0 \leq \varepsilon < 1$.

Definition 3.

$$Lip_\varepsilon(\mathbb{R}^n) = \left\{ f; \|f\|_{Lip_\varepsilon} = \sup_Q \inf_c \frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |f(x) - c| dx < \infty \right\},$$

where the supremum is taken over all balls $Q \subset \mathbb{R}^n$.

We denote $Lip_0(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

Proposition 3. When $0 < \varepsilon < 1$,

$$\|f\|_{Lip_\varepsilon} \approx \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\varepsilon}.$$

Definition 4.

$$\tilde{I}_\beta f(x) = \int_{\mathbb{R}^n} \left\{ \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|y|^{n-\beta}} \chi_{\{|y| \geq 1\}} \right\} f(y) dy \quad \text{for } 0 < \beta < n.$$

Proposition 4. $\tilde{I}_\beta$ is bounded from $L^q(\mathbb{R}^n)$ to $Lip_{\beta-n/q}(\mathbb{R}^n)$ where

$$(2) \quad 0 \leq \beta - n/q < 1.$$

Epecially $\tilde{I}_\beta$ is bounded from $L^{n/\beta}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.

We consider the boudedness of I_β on $\dot{K}_q^{\alpha,p}$ where $q \geq n/\beta$. Lu and Yang [8] (see also [6]) proved the following.

Proposition 5. Assume that $q_1 \geq n/\beta$ and $0 < p < n/\beta$. Then I_β is bounded from $K_{q_1}^{\alpha_1,p}(\mathbb{R}^n)$ to $K_{q_2}^{\alpha_2,p}(\mathbb{R}^n)$ for appropriate α_1, α_2 and q_2 .

However this result does not imply Proposition 4, because Proposition 5 is not applicable to the case $p = q_1$. Note that $K_q^{0,q} = L^q$. In this paper we consider the theorem that is an extension of Proposition 4. By a simple observation we obtain the following result. We define CMO (central mean oscillation) introduced by Garcia-Cuerva [3]. Let $0 \leq \varepsilon < 1$.

Definition 5.

$$CMO_\varepsilon(\mathbb{R}^n) = \left\{ f; \|f\|_{CMO_\varepsilon} = \sup_{R>0} \inf_c \frac{1}{|B(0,R)|^{1+\varepsilon/n}} \int_{B(0,R)} |f(x) - c| dx < \infty \right\}.$$

Theorem 1. $\tilde{I}_\beta$ is bounded from $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ to $CMO_{\beta-n/q-\alpha}(\mathbb{R}^n)$ where

$$(3) \quad q \geq n/\beta, \beta - n/q - 1 < \alpha \leq \beta - n/q \text{ and } 0 < p < \infty.$$

The spaces $CMO_{\beta-n/q-\alpha}$ are independent of p , but the operator norms $\|\tilde{I}_\beta\|_{\dot{K}_q^{\alpha,p} \rightarrow CMO_{\beta-n/q-\alpha}}$ depend on p .

Note that the modified fractional integrals $\tilde{I}_\beta$ are well-defined on $\dot{K}_q^{\alpha,p}$ where $\beta - n/q - 1 < \alpha$. Compare with the condition (1).

However this result is insufficient, because when $\alpha = 0$ and $p = q = n/\beta$ it says that

$$\tilde{I}_\beta : L^{n/\beta} \rightarrow CMO_0.$$

Since $BMO \subsetneq CMO_0$, it does not imply Proposition 4.

In the following, we consider the theorem that is consistent with Proposition 4 when $\alpha = 0$ and $p = q$.

In Section 4, we shall show that $\tilde{I}_\beta$ is not bounded from $K_q^{\alpha,p}$ to $Lip_{\beta-n/q}$ in general, therefore we need to consider other function spaces.

2. Main Theorem

For our purpose we consider a variant of Lipschitz space. Our definition is the following. Let $0 \leq \varepsilon < 1$ and $\lambda \in \mathbb{R}^1$.

Definition 6.

$$\begin{aligned} & Lip_\varepsilon^\lambda(\mathbb{R}^n) \\ &= \left\{ f; \|f\|_{Lip_\varepsilon^\lambda} = \sup_{\substack{x \in \mathbb{R}^n \\ R > 0}} \inf_c \frac{1}{(|x| + R)^\lambda} \frac{1}{|B(x, R)|^{1+\varepsilon/n}} \int_{B(x, R)} |f(y) - c| dy < \infty \right\}. \end{aligned}$$

Remarks . $Lip_\varepsilon^0(\mathbb{R}^n) = Lip_\varepsilon(\mathbb{R}^n)$ and $Lip_0^0(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

$$\|f\|_{Lip_\varepsilon^\lambda} \approx \sup_{\substack{x \in \mathbb{R}^n \\ R > 0}} \frac{1}{(|x| + R)^\lambda} \frac{1}{|B(x, R)|^{1+\varepsilon/n}} \int_{B(x, R)} |f(y) - f_{B(x, R)}| dy < \infty,$$

where $f_{B(x, R)} = \frac{1}{|B(x, R)|} \int_{B(x, R)} f(y) dy$.

Such function spaces are introduced by Nakai et al. [10], [11], [2]. They consider more general function spaces, that is, generalized Campanato spaces:

$$\begin{aligned} & Lip_\varepsilon^\varphi(\mathbb{R}^n) \\ &= \left\{ f; \|f\|_{Lip_\varepsilon^\varphi} = \sup_{\substack{x \in \mathbb{R}^n \\ R > 0}} \inf_c \frac{1}{\varphi(x, R)} \frac{1}{|B(x, R)|^{1+\varepsilon/n}} \int_{B(x, R)} |f(y) - c| dy < \infty \right\}. \end{aligned}$$

Before we state our results, we investigate some properties of $Lip_\varepsilon^\lambda(\mathbb{R}^n)$. Since $Lip_\varepsilon^{\beta-n/q-\alpha-\varepsilon} \subset CMO_{\beta-n/q-\alpha}$, Theorem 1 is obtained from Theorems 2 and 3 below. The following property is easily obtained from the definition.

$$(4) \quad Lip_{\varepsilon'}^{\lambda'}(\mathbb{R}^n) \subset Lip_\varepsilon^\lambda(\mathbb{R}^n) \quad \text{where} \quad \varepsilon' - \varepsilon = \lambda - \lambda' \geq 0.$$

Especially

$$(5) \quad \begin{aligned} Lip_{\beta-n/q}^{-\alpha}(\mathbb{R}^n) &\subset Lip_{\beta-n/q-\alpha}(\mathbb{R}^n) & \text{if } \alpha > 0, \\ Lip_{\beta-n/q-\alpha}(\mathbb{R}^n) &\subset Lip_{\beta-n/q}^{-\alpha}(\mathbb{R}^n) & \text{if } \alpha < 0. \end{aligned}$$

This observation is important for our theorems (see Remark below). We give some examples.

$$\begin{aligned}
& |x|^{1+\varepsilon} \in Lip_\varepsilon^1(\mathbb{R}^n), \quad |x|^{-\varepsilon} \in Lip_0^{-\varepsilon}(\mathbb{R}^n) \quad \text{where } 0 \leq \varepsilon < 1. \\
& (\log |x|)^2 \chi_{\{|x| \geq 1\}} \in Lip_0^1(\mathbb{R}^n) \setminus BMO(\mathbb{R}^n), \\
& (\log |x|)^2 \chi_{\{|x| \leq 1\}} \in Lip_0^{-1}(\mathbb{R}^n) \setminus BMO(\mathbb{R}^n).
\end{aligned}$$

Our main result is the following.

Theorem 2. *Let $q \geq n/\beta$, $0 < p < \infty$ and $\beta - n/q - 1 < \alpha < n - n/q$. If $0 \leq \beta - n/q < 1 + \min(0, \alpha)$, then $\tilde{I}_\beta$ is bounded from $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ to $Lip_{\beta-n/q}^{-\alpha}(\mathbb{R}^n)$.*

Remark . The spaces $Lip_{\beta-n/q}^{-\alpha}$ are defined only for $\beta - n/q < 1$. We shall show that the index $\beta - n/q$ is optimal in Section 4, and also show that if $\alpha < 0$ then $\tilde{I}_\beta$ is not bounded from $\dot{K}_q^{\alpha,p}$ to $Lip_{\beta-n/q-\alpha}$ (see (5)).

When $\alpha > 0$ and $1 \leq \beta - n/q < 1 + \alpha$, we have the following result.

Theorem 3. *Let $q \geq n/\beta$ and $0 < p < \infty$. If $\alpha \geq 0$ and $0 \leq \beta - n/q - \alpha < 1$, then $\tilde{I}_\beta$ is bounded from $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ to $Lip_{\beta-n/q-\alpha}(\mathbb{R}^n)$.*

Remark . The case $\beta - n/q < 1$ is contained in Theorem 2.

Corollary .

$$\begin{aligned}
& \tilde{I}_\beta : \dot{K}_q^{0,p}(\mathbb{R}^n) \rightarrow Lip_{\beta-n/q}(\mathbb{R}^n) \quad \text{if} \quad 0 < \beta - n/q < 1, \\
& \tilde{I}_\beta : \dot{K}_{n/\beta}^{0,p}(\mathbb{R}^n) \rightarrow BMO(\mathbb{R}^n).
\end{aligned}$$

Since $L^q \subset K_q^{0,p}$ when $q \leq p$, Corollary is an extension of Proposition 4.

3. Proofs

In order to prove Theorem 2 we prepare two lemmas.

Lemma 1. *Assume that $1 \geq 1/s > \max(1/q + \alpha/n, 1/q)$. Then*

$$(6) \quad \left(\int_{B(0,2^k)} |f(x)|^s dx \right)^{1/s} \leq C 2^{k(n(1/s-1/q)-\alpha)} \|f\|_{\dot{K}_q^{\alpha,p}}.$$

Throughout this paper, C is a positive constant which is independent of essential parameters and not necessarily same at each occurrence.

Proof. We write

$$\begin{aligned} \left(\int_{B(0,2^k)} |f(x)|^s dx \right)^{1/s} &= \left(\sum_{j \leq k} \int_{A_j} |f(x)|^s dx \right)^{1/s} \\ &\leq C \left\{ \sum_{j \leq k} 2^{j(n(1-s/q)-\alpha s)} \cdot 2^{j\alpha s} \left(\int_{A_j} |f(x)|^q dx \right)^{s/q} \right\}^{1/s} =: X. \end{aligned}$$

If $p > s$. Let $1/t + s/p = 1$. Then we have

$$\begin{aligned} X &\leq C \left\{ \sum_{j \leq k} 2^{j(n(1-s/q)-\alpha s)t} \right\}^{1/ts} \left\{ \sum_{j \leq k} 2^{j\alpha p} \left(\int_{A_j} |f(x)|^q dx \right)^{p/q} \right\}^{1/p} \\ &\leq C 2^{k(n(1/s-1/q)-\alpha)} \|f\|_{\dot{K}_q^{\alpha,p}} \end{aligned}$$

since $n(1/s - 1/q) - \alpha > 0$. If $p \leq s$, then we have

$$\begin{aligned} X &\leq \left\{ \sum_{j \leq k} 2^{j(n(1-s/q)-\alpha s)p/s} \cdot 2^{j\alpha p} \left(\int_{A_j} |f(x)|^q dx \right)^{p/q} \right\}^{1/p} \\ &\leq C 2^{k(n(1/s-1/q)-\alpha)} \|f\|_{\dot{K}_q^{\alpha,p}}. \end{aligned}$$

□

Lemma 2. Assume that $\beta - n/q - 1 < \alpha$. Then

$$(7) \quad \int_{|y| \geq 2^k} \frac{|f(y)|}{|y|^{n+1-\beta}} dy \leq C 2^{k(\beta-n/q-\alpha-1)} \|f\|_{\dot{K}_q^{\alpha,p}}.$$

Proof. We write

$$\begin{aligned} \int_{|y| \geq 2^k} \frac{|f(y)|}{|y|^{n+1-\beta}} dy &= \sum_{j \geq k} \int_{A_j} \frac{|f(y)|}{|y|^{n+1-\beta}} dy \\ &\leq C \sum_{j \geq k} 2^{j(\beta-n/q-\alpha-1)} \cdot 2^{j\alpha} \left(\int_{A_j} |f(y)|^q dy \right)^{1/q} =: X. \end{aligned}$$

If $p > 1$, then we have

$$\begin{aligned} X &\leq C \left\{ \sum_{j \geq k} 2^{j(\beta-n/q-\alpha-1)p'} \right\}^{1/p'} \left\{ \sum_{j \geq k} 2^{j\alpha p} \left(\int_{A_j} |f(y)|^q dy \right)^{p/q} \right\}^{1/p} \\ &\leq C 2^{k(\beta-n/q-\alpha-1)} \|f\|_{\dot{K}_q^{\alpha,p}} \end{aligned}$$

since $\beta - n/q - \alpha - 1 < 0$.

If $p \leq 1$, then we have

$$\begin{aligned} X &\leq C \left\{ \sum_{j \geq k} 2^{j(\beta-n/q-\alpha-1)p} \cdot 2^{j\alpha p} \left(\int_{A_j} |f(y)|^q dy \right)^{p/q} \right\}^{1/p} \\ &\leq C 2^{k(\beta-n/q-\alpha-1)} \|f\|_{\dot{K}_q^{\alpha,p}}. \end{aligned}$$

□

Now we are in a position to prove Theorem 2.

Proof. [Proof of Theorem 2.] Let $\varepsilon = \beta - n/q$. Fix a ball $Q = B(x_0, R)$ and we estimate

$$\inf_c \frac{(|x_0| + R)^\alpha}{|Q|^{1+\varepsilon/n}} \int_Q |\tilde{I}_\beta f(x) - c| dx.$$

Let k be the least integer such that $Q \subset B(0, 2^k)$. Note that

$$(8) \quad |x_0| + R \approx 2^k.$$

We consider three cases:

- (i) $Q \cap B(0, 2^{k-2}) \neq \emptyset$,
- (ii) $Q \cap B(0, 2^{k-2}) = \emptyset$ and $R \geq 2^{k-4}$,
- (iii) $Q \cap B(0, 2^{k-2}) = \emptyset$ and $R < 2^{k-4}$.

The case (i) or (ii). Note that $|Q| \geq C 2^{kn}$ in both cases. We write

$$f(x) = f(x) \chi_{B(0, 2^{k+1})} + f(x) \chi_{\mathbb{R}^n \setminus B(0, 2^{k+1})} =: f_1(x) + f_2(x).$$

First we estimate $\tilde{I}_\beta f_1$. Let $c_1 = - \int_{|y| \geq 1} f_1(y) / |y|^{n-\beta} dy$. Then $\tilde{I}_\beta f_1(x) - c_1 = I_\beta f_1(x)$. Applying Lemma 1 for $s = 1$, we have

$$\begin{aligned} (9) \quad \int_Q |I_\beta f_1(x)| dx &\leq \int_{B(0, 2^{k+1})} |f(y)| \left(\int_Q \frac{1}{|x-y|^{n-\beta}} dx \right) dy \\ &\leq C 2^{k(n(1-1/q)-\alpha)} |Q|^{\beta/n} \|f\|_{\dot{K}_q^{\alpha,p}}, \end{aligned}$$

and obtain

$$\frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |\tilde{I}_\beta f_1 - c_1| dx \leq C (|x_0| + R)^{-\alpha} \|f\|_{\dot{K}_q^{\alpha,p}}.$$

Next we estimate $\tilde{I}_\beta f_2$. Let $c_2 = \tilde{I}_\beta f(x_0)$. For any $x \in Q$,

$$\begin{aligned}
 (10) \quad |\tilde{I}_\beta f_2(x) - c_2| &\leq \int \left| \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x_0-y|^{n-\beta}} \right| |f_2(y)| dy \\
 &\leq CR \int_{|y| \geq 2^{k+1}} \frac{|f(y)|}{|x_0-y|^{n+1-\beta}} dy \\
 &\leq CR 2^{k(\beta-n/q-\alpha-1)} \|f\|_{\dot{K}_q^{\alpha,p}}
 \end{aligned}$$

by Lemma 2, and we have

$$\frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |\tilde{I}_\beta f_2 - c_2| dx \leq C(|x_0| + R)^{-\alpha} \|f\|_{\dot{K}_q^{\alpha,p}}.$$

The case (iii). We write

$$\begin{aligned}
 f(x) &= f(x) \chi_{B(x_0, 2R)} + f(x) \chi_{\mathbb{C}\{2^{k-3} < |x| \leq 2^{k+1}\}} + f(x) \chi_{\{2^{k-3} < |x| \leq 2^{k+1}\} \setminus B(x_0, 2R)} \\
 &=: f_1(x) + f_2(x) + f_3(x).
 \end{aligned}$$

First we estimate $\tilde{I}_\beta f_1$. Let $c_1 = -\int_{|y| \geq 1} f_1(y)/|y|^{n-\beta} dy$. By the same estimate as (9), we have

$$\begin{aligned}
 \int_Q |\tilde{I}_\beta f_1(x) - c_1| dx &\leq C|Q|^{\beta/n} \int_{B(x_0, 2R)} |f(y)| dy \\
 &\leq C \left(\int_{B(x_0, 2R)} |f(y)|^q dy \right)^{1/q} |Q|^{1-1/q+\beta/n} \\
 &\leq C 2^{-k\alpha} |Q|^{1-1/q+\beta/n} \|f\|_{\dot{K}_q^{\alpha,p}},
 \end{aligned}$$

because $B(x_0, 2R) \subset \cup_{j=k-2}^{k+1} A_j$. Therefore we obtain

$$\frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |I_\beta f_1(x) - c_1| dx \leq C(|x_0| + R)^{-\alpha} \|f\|_{\dot{K}_q^{\alpha,p}}.$$

Next we estimate $\tilde{I}_\beta f_2$. Let $c_2 = \tilde{I}_\beta f_2(x_0)$. By the same estimate as (10) and Lemma 1 for $s = 1$, it follows that for any $x \in Q$,

$$\begin{aligned}
 |\tilde{I}_\beta f_2(x) - c_2| &\leq CR \left(\int_{|y| \geq 2^{k+1}} \frac{|f(y)|}{|x_0-y|^{n+1-\beta}} dy + \int_{|y| \leq 2^{k-3}} \frac{|f(y)|}{|x_0-y|^{n+1-\beta}} dy \right) \\
 &\leq CR \left(2^{k(\beta-n/q-\alpha-1)} \|f\|_{\dot{K}_q^{\alpha,p}} + 2^{-k(n+1-\beta)} \int_{|y| \leq 2^{k-3}} |f(y)| dy \right) \\
 &\leq CR (2^{k(\beta-n/q-\alpha-1)} \|f\|_{\dot{K}_q^{\alpha,p}} + 2^{-k(n+1-\beta)} \cdot 2^{k(n-n/q-\alpha)} \|f\|_{\dot{K}_q^{\alpha,p}}) \\
 &\leq CR 2^{k(\beta-n/q-\alpha-1)} \|f\|_{\dot{K}_q^{\alpha,p}}.
 \end{aligned}$$

Since $\beta - n/q - 1 < 0$, we obtain

$$\frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |I_\beta f_2(x) - c_2| dx \leq C(|x_0| + R)^{-\alpha} \|f\|_{\dot{K}_q^{\alpha,p}}.$$

Finally we estimate $\tilde{I}_\beta f_3$. Let $c_3 = \tilde{I}_\beta f_3(x_0)$. By the same estimate as f_2 , it follows that for any $x \in Q$,

$$\begin{aligned} |\tilde{I}_\beta f_3(x) - c_3| &\leq CR \int_{|x_0-y|>2R} \frac{|f_3(y)|}{|x_0-y|^{n+1-\beta}} dy \\ &\leq CR \left(\int_{|x_0-y|>2R} \frac{dy}{|x_0-y|^{(n+1-\beta)q'}} \right)^{1/q'} \left(\int_{2^{k-3}<|y|\leq 2^{k+1}} |f(y)|^q dy \right)^{1/q} \\ &\leq CR \cdot R^{-n-1+\beta+n/q'} 2^{-k\alpha} \|f\|_{\dot{K}_q^{\alpha,p}} \\ &\leq CR^{\beta-n/q} 2^{-k\alpha} \|f\|_{\dot{K}_q^{\alpha,p}}, \end{aligned}$$

and we obtain

$$\frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |I_\beta f_3(x) - c_2| dx \leq C(|x_0| + R)^{-\alpha} \|f\|_{\dot{K}_q^{\alpha,p}}.$$

□

The following result is known, see e.g., [2], [11] and [13].

Proposition 6.

$$\|\tilde{I}_\beta f\|_{Lip_{\beta-n/t}} \leq C\|f\|_{M_s^t} \quad \text{where } 0 \leq \beta - n/t < 1 \quad \text{and } 1 < s \leq t.$$

Also we need the following

Definition 7.

$$M_s^t(\mathbb{R}^n) = \left\{ f; \|f\|_{M_s^t} = \sup_Q \frac{1}{|Q|^{1/s-1/t}} \left(\int_Q |f(x)|^s dx \right)^{1/s} < \infty \right\},$$

where the supremum is taken over all balls $Q \subset \mathbb{R}^n$.

Now, we prove Theorem 3.

Proof. [Proof of Theorem 3.] It suffices to show that for $\alpha \geq 0$, $1/t = 1/q + \alpha/n$ and $1 < s < t$,

$$K_q^{\alpha,p}(\mathbb{R}^n) \subset M_s^t(\mathbb{R}^n)$$

since then by Proposition 6 we obtain

$$\|\tilde{I}_\beta f\|_{Lip_{\beta-n/q-\alpha}} = \|\tilde{I}_\beta f\|_{Lip_{\beta-n/t}} \leq C\|f\|_{M_s^t} \leq C\|f\|_{K_q^{\alpha,p}}.$$

For any ball Q , let k be the least integer such that $Q \subset B(0, 2^k)$.

First we consider the case $Q \cap B(0, 2^{k-2}) \neq \emptyset$. Note that $|Q| \geq C2^{kn}$.

By Lemma 1, we have

$$(11) \quad \frac{1}{|Q|^{1/s-1/t}} \left(\int_Q |f(x)|^s dx \right)^{1/s} \leq C \frac{2^{k(n/s-n/q-\alpha)}}{|Q|^{1/s-1/t}} \|f\|_{K_q^{\alpha,p}} \leq C \|f\|_{K_q^{\alpha,p}}.$$

Next we consider the case $Q \cap B(0, 2^{k-2}) = \emptyset$. Note that $Q \subset A_{k-1} \cup A_k$. By Hölder's inequality we have

$$(12) \quad \begin{aligned} \frac{1}{|Q|^{1/s-1/t}} \left(\int_Q |f(x)|^s dx \right)^{1/s} &\leq |Q|^{1/t-1/q} \left(\int_Q |f(x)|^q dx \right)^{1/q} \\ &\leq C 2^{k\alpha} \left(\int_{A_{k-1}} |f(x)|^q dx + \int_{A_k} |f(x)|^q dx \right)^{1/q} \leq C \|f\|_{K_q^{\alpha,p}}. \end{aligned}$$

By (11) and (12), we obtain the desired result. $\square$

4. Counterexample

We show that $\tilde{I}_\beta$ is not bounded from $K_q^{\alpha,p}(\mathbb{R}^n)$ to $Lip_{\beta-n/q}(\mathbb{R}^n)$ when $\alpha \neq 0$.

Proposition 7. *If $\tilde{I}_\beta$ is bounded from $K_q^{\alpha,p}(\mathbb{R}^n)$ to $Lip_\varepsilon^\lambda(\mathbb{R}^n)$, then $\varepsilon + \lambda = \beta - n/q - \alpha$.*

Proof. We denote $f_t(x) = f(x/t)$. Then $\|f_t\|_{Lip_\varepsilon^\lambda} t^{-\varepsilon-\lambda} \|f\|_{Lip_\varepsilon^\lambda}$ and $I_\beta f_t(x) = t^\beta (I_\beta f)(x/t)$. Let $f_0(x) = \chi_{\{1 < |x| < 2\}}$ and $\tilde{f}_j(x) = 2^{-j(\alpha+n/q)} f_0(x/2^j)$. Then $\|\tilde{f}_j\|_{K_q^{\alpha,p}} = C$. On the other hand we have

$$\begin{aligned} \|\tilde{I}_\beta \tilde{f}_j\|_{Lip_\varepsilon^\lambda} &= \|I_\beta \tilde{f}_j\|_{Lip_\varepsilon^\lambda} = 2^{-j(\alpha+n/q)} 2^{j\beta} 2^{-j(\varepsilon+\lambda)} \|I_\beta f_0\|_{Lip_\varepsilon^\lambda} \\ &= 2^{j(\beta-\alpha-n/q-\varepsilon-\lambda)} \|I_\beta f_0\|_{Lip_\varepsilon^\lambda}. \end{aligned}$$

Therefore, if $\sup_j \|\tilde{I}_\beta \tilde{f}_j\|_{Lip_\varepsilon^\lambda} / \|\tilde{f}_j\|_{K_q^{\alpha,p}} < \infty$, then $\varepsilon + \lambda = \beta - n/q - \alpha$. $\square$

By this proposition, if $\tilde{I}_\beta : K_q^{\alpha,p} \rightarrow Lip_\varepsilon$ then $\varepsilon = \beta - n/q - \alpha$. Therefore, if $\tilde{I}_\beta : K_q^{\alpha,p} \rightarrow Lip_{\beta-n/q}$ then $\alpha = 0$. Furthermore we give the following counterexample on $\mathbb{R}^1$.

Counterexample. *If $\varepsilon > 0$, then $\tilde{I}_\beta$ is not bounded from $K_q^{\alpha,p}(\mathbb{R}^1)$ to $Lip_{\beta-1/q+\varepsilon}^{-\alpha-\varepsilon}(\mathbb{R}^1)$.*

Since $Lip_{\beta-1/q+\varepsilon}^{-\alpha-\varepsilon}(\mathbb{R}^1) \subset Lip_{\beta-1/q}^{-\alpha}(\mathbb{R}^1)$ by (4), this counterexample shows that Theorem 2 is optimal.

Proof. Let $f_j(x) = 2^{-j\alpha} \chi_{[2^j, 2^{j+1}]}(x)$ for $j \geq 1$. Then $\|f_j\|_{K_q^{\alpha,p}} = 1$. We calculate the mean oscillation of $\tilde{I}_\beta f_j$ on intervals $B_j = (2^j - 1, 2^j)$: the center is $2^j - 1/2$ and the radius is $1/2$. Since

$$\tilde{I}_\beta f(x_1) - \tilde{I}_\beta f(x_2) = \int_{\mathbb{R}^1} \left\{ \frac{1}{|x_1 - y|^{1-\beta}} - \frac{1}{|x_2 - y|^{1-\beta}} \right\} f(y) dy,$$

it follows that for $x \in B_j$,

$$\tilde{I}_\beta f_j(x) - \tilde{I}_\beta f_j(2^j) = \frac{2^{-j\alpha}}{\beta} \left\{ (2^j + 1 - x)^\beta - (2^j - x)^\beta - 1 \right\},$$

and we have

$$(\tilde{I}_\beta f_j)_{B_j} := \frac{1}{|B_j|} \int_{B_j} \tilde{I}_\beta f_j(x) dx = \frac{2^{-j\alpha}(2^{\beta+1} - \beta - 3)}{\beta(\beta + 1)} + \tilde{I}_\beta f_j(2^j).$$

Therefore

$$\begin{aligned} & \frac{1}{|B_j|} \int_{B_j} \left| \tilde{I}_\beta f_j(x) - (\tilde{I}_\beta f_j)_{B_j} \right| dx \\ &= \frac{2^{-j\alpha}}{\beta} \int_{2^j-1}^{2^j} \left| (2^j + 1 - x)^\beta - (2^j - x)^\beta - 1 - \frac{2^{\beta+1} - \beta - 3}{\beta + 1} \right| dx \\ &= \frac{2^{-j\alpha}}{\beta} \int_0^1 \left| (2 - x)^\beta - (1 - x)^\beta - 1 - \frac{2^{\beta+1} - \beta - 3}{\beta + 1} \right| dx = C_\beta 2^{-j\alpha}, \end{aligned}$$

where $C_\beta = \beta^{-1} \int_0^1 |(2 - x)^\beta - (1 - x)^\beta - 1 - (2^{\beta+1} - \beta - 3)/(\beta + 1)| dx \neq 0$. We obtain

$$\begin{aligned} & \lim_{j \rightarrow \infty} \frac{1}{(|2^j - 1/2| + 1/2)^{-\alpha-\varepsilon} |B_j|^{1+\beta-1/q+\varepsilon}} \int_{B_j} \left| \tilde{I}_\beta f_j(x) - (\tilde{I}_\beta f_j)_{B_j} \right| dx \\ &= \lim_{j \rightarrow \infty} C_\beta 2^{j\varepsilon} = \infty. \end{aligned}$$

□

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