On the coefficients of the expansion of elements from $C[0,1]$ space by the Faber-Schauder system

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Abstract. Elements (functions) of continuous on $[0,1]$ functions space $(C[0,1])$ are described which have $\frac{1}{2}$-monotone coefficients of the expansion by the Faber-Schauder system.

1. Introduction

At first we remind the definition of the Faber-Schauder system. It is a system of functions $\Phi = \{\varphi_n(x)\}_{n=0}^{\infty}$, $x \in [0,1]$, in which $\varphi_0(x) \equiv 1$, $\varphi_1(x) = x$ and for $n = 2^k + i$; $k = 0, 1, \ldots$; $i = 1, 2, \ldots, 2^k$

\begin{equation}
\varphi_n(x) = \varphi_k^{(i)}(x) = \begin{cases} 
0, & \text{if } x \notin \left(\frac{i-1}{2^k}, \frac{i}{2^k}\right), \\
1, & \text{if } x = x_n = x_k^{(i)} = \frac{2i-1}{2^{k+1}}, \\
is linear and continuous in \left[\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}}\right], \left[\frac{2i-1}{2^{k+1}}, \frac{i}{2^k}\right].
\end{cases}
\end{equation}

By $\Delta_n = \Delta_k^{(i)}$, $n = 2^k + i$ ($n \geq 2$), we denote the support of function $\varphi_n(x) = \varphi_k^{(i)}(x)$. Remind that the Faber-Schauder system is a basis in
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space \( C[0,1] \) (the space of continuous on \([0,1]\) functions), i.e. for every function \( f(x) \in C[0,1] \) there exists a unique series of the form

\[
\sum_{n=0}^{\infty} A_n(f) \varphi_n(x),
\]

where

\[
A_0(f) = f(0), \quad A_1(f) = f(1) - f(0),
\]

\[
A_n(f) = A_{k,i}(f) = f \left( \frac{2i - 1}{2^{k+1}} \right) - \frac{1}{2} \left[ f \left( \frac{i - 1}{2^k} \right) + f \left( \frac{i}{2^k} \right) \right],
\]

which uniformly converges to \( f(x) \) on \([0,1]\) (see [19]).

**Definition 1.** Let \( t \in (0,1] \). We say that coefficients of \( f(x) \in C[0,1] \) function's expansion by the Faber-Schauder system are \( t \)-**monotone** if

\[
|A_{n_k}(f)| \geq t \cdot |A_{n_{k+i}}(f)|, \quad k = 1, 2, ... ; \quad i \geq 1,
\]

where \( \{n_k\}_{k=1}^{\infty} \) is the spectrum of function \( f(x) \), i.e. the set of integers where \( A_n(f) \) is non zero.

Obviously, if coefficients of the expansion by the Faber-Schauder system are \( t \)-monotone for some \( t = t_0 \in (0,1] \) then they are \( t \)-monotone for all \( t \in (0,t_0] \).

**Definition 2.** Let \( t \in (0,1] \) be a fixed parameter. For a given function \( f(x) \in C[0,1] \) denote \( \Lambda_m \) any set of \( m \) indices (which is not unique in general) such that

\[
\min_{n \in \Lambda_m} |A_n(f)| \geq t \cdot \max_{n \notin \Lambda_m} |A_n(f)|
\]

and define

\[
G^t_m(x, f, \Lambda_m) = \sum_{n \in \Lambda_m} A_n(f) \varphi_n(x),
\]

which is called **Weak Thresholding Greedy Algorithm** with weakness parameter \( t \).

Note that \( \{G^t_m(x, f)\}_{m=1}^{\infty}, \quad t \in (0,1] \) nonlinear operators have been considered in various Banach spaces with respect to various normalized bases (see for example [1-5], [12-13], [20]).

We consider the following question: let \( t \in (0,1] \). Is it possible to modify a continuous on \([0,1]\) function on a "small" set to make coefficients of expansion (1.2) \( t \)-monotone?

In the present work we were able to prove the following
Theorem 1. For every $\epsilon \in (0, 1)$ there exists a measurable set $E \subset [0, 1]$ with measure $|E| > 1 - \epsilon$, such that to each function $f(x) \in C[0, 1]$ one can find a function $g(x) \in C[0, 1]$ that coincides with $f(x)$ on $E$ and coefficients of which’s expansion by the Faber-Schauder system are $t$-monotone for all $t \in (0, \frac{1}{2})$.

From Theorem 1 follows

Theorem 2. For every $\epsilon \in (0, 1)$ there exists a measurable set $E \subset [0, 1]$ with measure $|E| > 1 - \epsilon$, such that to each function $f(x) \in C[0, 1]$ one can find a function $g(x) \in C[0, 1]$ that coincides with $f(x)$ on $E$, set so of indices $\{\Lambda_m\}_{m=1}^{\infty}$ (see Definition 2) and a sequence of integers $\{N_m\}_{m=1}^{\infty}$ so that for all $m \in \mathbb{N}$

$$G_{N_m}^F(x,g,\Lambda_m) = S_{N_m}(x,g), x \in [0, 1],$$

where $S_N(x,g) = \sum_{n=0}^{N} A_n(g)\varphi_n(x),\quad N = 0, 1, \ldots$ are partial sums of the series (1.2).

Note that from this theorem immediately follows that the sequence $\{G_{N_m}^F(x,g,\Lambda_m)\}_{m=1}^{\infty}$ uniformly converges to $g(x)$.

Analogous questions are interesting to consider in various functional spaces with respect to other bases. Note that in particular in [6] it is proved the following theorem for the Walsh system:

Theorem (Grigoryan). For every $\epsilon \in (0, 1)$, $p \geq 1$ and each function $f(x) \in L^p[0, 1]$ one can find a function $g(x) \in L^p[0, 1]$, mes\{x \in [0, 1]; g \neq f\} < \epsilon, coefficients of which’s expansion by the Walsh system are 1-monotone.

Note that the idea of modification of a function improving its properties belongs to N.N. Luzin. The following famous result was obtained by him in 1912 (see [14]).

Theorem (N.N. Luzin’s C-property). Each measurable and a.e. finite function can be made continuous after a modification on a subset of arbitrarily small Lebesgue measure.

In 1939, Men’shov [15] proved the following fundamental theorem.

Theorem (Men’shov’s C-strong property). Let $f(x)$ be an a.e. finite measurable function on $[0, 2\pi]$. Then for each $\epsilon > 0$ one can define a continuous function $g(x)$ coinciding with $f(x)$ on a subset $E$ of measure $|E| > 2\pi - \epsilon$ such that its Fourier series with respect to the trigonometric system converges uniformly on $[0, 2\pi]$. 
Further interesting results in this direction were obtained by Men’nov [16], Kheladze [11], Price [18], Oskolkov [17], Grigoryan [7-10] and other authors.

The following questions are open:

**Question 1.** Is Theorem 1 true for \( t \in \left( \frac{1}{2}, 1 \right) \)?

**Question 2.** Do Theorems 1 and 2 hold for other bases in the space \( C[0,1] \) (in particular, for the Franklin system)?

**Question 3.** Is Theorem 1 true for the trigonometric system for any \( t \in (0,1] \)?

**Question 4.** Let \( t \in (0,1) \). Is there a continuous function on \([0,1]\) such that Weak Thresholding Greedy Algorithm by the Faber-Schauder system with weakness parameter \( t \) diverges by measure (or at least almost everywhere) in \([0,1]\)?

Concerning with Question 4 note that in [5] it is proved that the answer is positive for \( t = 1 \).

2. Proofs of basic lemmas

**Lemma 1.** Let dyadic interval \( \Delta = \Delta^{(i)} = (\frac{i-1}{2^p}, \frac{i}{2^p}) \) \((i \in [1,2^p])\) and numbers \( \gamma \neq 0, N_0 \in \mathbb{N}, 0 < \epsilon < |\Delta| \) be given. Then there exist a measurable set \( E \subset \Delta \) of measure \(|E| > |\Delta| - \epsilon \) and a polynomial

\[
Q(x) = \sum_{n=N_0}^{N} A_n \varphi_n(x),
\]

by the system (1.1), such that

1) \[
Q(x) = \begin{cases} 
\gamma, & \text{if } x \in E, \\
0, & \text{if } x \in [0,1] \setminus \Delta
\end{cases}; \quad \|Q(x)\|_C = |\gamma|,
\]

2) \[
A_n = \gamma, \quad \frac{\gamma}{2} \text{ or } 0, \quad n \in [N_0,N].
\]

**Proof.** At first we consider the case when \( N_0 \leq 2^p + i \). We take natural number \( q > \log_2 \frac{1}{\epsilon} + 1 \) \((0 < \epsilon < |\Delta|)\) and put

\[
E = \Delta \setminus \left\{ \left( \frac{i-1}{2^p}, \frac{i-1}{2^p} + \frac{1}{2^q} \right) \cup \left( \frac{i}{2^p} - \frac{1}{2^q}, \frac{i}{2^p} \right) \right\},
\]
a polynomial by the system (1.1) satisfying the conditions of the lemma. 

\[ g(x) = \begin{cases} 
\gamma, & \text{if } x \in E, \\
0, & \text{if } x \in [0, 1] \setminus \Delta, 
\end{cases} \]

is linear and continuous on \([\frac{i-1}{2^q}, \frac{i-1}{2^q} + \frac{1}{2^q}]\) and \([\frac{i}{2^q} - \frac{1}{2^q}, \frac{i}{2^q}]\).

It is clear that \(|E| > |\Delta| - \epsilon\). Taking into account (1.3), it is not hard to see that

\[ g(x) = \gamma \cdot \varphi_p^{(i)}(x) + \frac{\gamma}{2} \left( \sum_{k=1}^{q-p-1} \varphi_{p+k}^{(2h_k-1)}(x) + \sum_{k=1}^{q-p-1} \varphi_{p+k}^{(2h_k)}(x) \right), \]

where \(t_1 = h_1 = i; t_{k+1} = 2t_k - 1, \ h_{k+1} = 2h_k\), so that the function \(g(x)\) presents a polynomial by Faber-Schauder system, satisfying the conditions of the lemma.

Now we consider the case \(N_0 > 2^q + i\). In this case some points \(x_n, n < N_0\) (see (1.1)) belong to \(\Delta\). We denote them by \(x_{n_i}, i = 1, 2, \ldots L\) \((x_{n_i} = x_p^{(i)} = \frac{2^q - 1}{2^{q+1}})\).

We take a natural number \(q > \log_2 \frac{L+i}{\epsilon} + 1\) \((0 < \epsilon < |\Delta|)\) and put

\[ E = \Delta \setminus \left\{ \bigcup_{l=1}^{L} \left( x_{n_l} - \frac{1}{2^q}, x_{n_l} + \frac{1}{2^q} \right) \bigcup \left( \frac{i-1}{2^q}, \frac{i-1}{2^q} + \frac{1}{2^q} \right) \bigcup \left( \frac{i}{2^q} - \frac{1}{2^q}, \frac{i}{2^q} \right) \right\}, \]

(2.1)

\[ g(x) = \begin{cases} 
\gamma, & \text{if } x \in E, \\
0, & \text{if } x \in ([0, 1] \setminus \Delta) \cup \{x_{n_i}\}_{i=1}^{L}, 
\end{cases} \]

is linear and continuous on \([x_{n_i} - \frac{1}{2^q}, x_{n_i}]\) and \([x_{n_i}, x_{n_i} + \frac{1}{2q} \ (l \in [1, L])\) and on \([\frac{i-1}{2^q}, \frac{i-1}{2^q} + \frac{1}{2^q}]\) and \([\frac{i}{2^q} - \frac{1}{2^q}, \frac{i}{2^q}]\).

It is clear that \(2^q > N_0\), \(\|g(x)\|_C = |\gamma|\) and \(|E| > |\Delta| - \epsilon\). Taking into account (1.3) we obtain that in the expansion of the function \(g(x) \in C_{[0, 1]}\) by the system (1.1), coefficients of functions \(\{\varphi_n(x)\}\) with numbers \(n < N_0\) and \(n > 2^q\) and coefficients of functions \(\{\varphi_n(x)\}\) with numbers \(N_0 \leq n \leq 2^q\), whichs supports don’t belong to \(\Delta\) or \(\Delta_n \subset E\), are equal to 0. From (1.3) and (2.1) it follows that coefficients of functions \(\varphi_n(x), N_0 \leq n \leq 2^q\) \((\Delta_n \subset \Delta, \Delta_n \not\subseteq E)\), are equal to either \(\gamma\) or \(\frac{\gamma}{2}\), depending on the fact that function \(g(x)\) takes value 0 in both endpoints of \(\Delta_n\) or only in one endpoint. So we obtain that the function \(g(x)\) presents a polynomial by the system (1.1) satisfying the conditions of the lemma. \(\square\)
Let $\| \cdot \| = \max_{x \in [0,1]} | \cdot |$ is the norm of the space $C_{[0,1]}$.

**Lemma 2.** Let dyadic interval $\Delta = \Delta_p(i) = \left( \frac{i-1}{2^p}, \frac{i}{2^p} \right)$ ($i \in [1,2^p]$) and numbers $\gamma \neq 0$, $N_0 \in \mathbb{N}$, $0 < \epsilon < |\Delta|$ be given. Then one can find a measurable set $E \subset \Delta$ and a polynomial $Q(x)$ by the Faber-Schauder system of the form

$$Q(x) = \sum_{k=k_0}^{K} A_k \varphi_n(x), \ n_k \geq N_0,$$

satisfying the following conditions:

1) $|E| > |\Delta| - \epsilon$,

2) $Q(x) = \begin{cases} \gamma, & \text{when } x \in E, \\ 0, & \text{when } x \in [0,1] \setminus \Delta \end{cases} \quad \| Q(x) \|_C = |\gamma|$, 

3) $\epsilon > |A_k| \geq \frac{1}{2} |A_{k+1}| > 0, \ \forall k \in [k_0, K]$, 

4) $\max_{k_0 \leq m \leq K} \left\| \sum_{k=k_0}^{m} A_k \varphi_n(x) \right\|_C = \| Q(x) \|_C = |\gamma|$.

**Proof.** Let $\nu_0 > \frac{|\gamma|}{\epsilon}$ be some natural number. By virtue of Lemma 1, for every natural number $\nu \in [1, \nu_0]$ one can find a measurable set $E_\nu \subset \Delta$ and a polynomial of the form

$$Q_\nu(x) = \sum_{n=N_{\nu-1}}^{N_{\nu}-1} A_n^{(\nu)} \varphi_n(x), \ N_\nu > N_{\nu-1},$$

satisfying the conditions:

1) $|E_\nu| > |\Delta| - \frac{\epsilon}{\nu_0}$,

2) $Q_\nu(x) = \begin{cases} \gamma_{\nu_0}, & \text{if } x \in E_\nu, \\ 0, & \text{if } x \in [0,1] \setminus \Delta \end{cases} \quad \| Q_\nu(x) \|_C = \frac{|\gamma|}{\nu_0}$
3') \quad A_n^{(\nu)} = \frac{\gamma}{\nu_0}, \frac{\gamma}{2\nu_0} \text{ or } 0, \ n \in [N_{\nu-1}, N_{\nu}).

We put

\[ E = \bigcap_{\nu=1}^{\nu_0} E_{\nu} \]

and

\[ Q(x) = \sum_{\nu=1}^{\nu_0} Q_{\nu}(x) = \sum_{k=k_0}^{K} A_k \varphi_{n_k}(x), \]

where \( \{A_k\}_{k=k_0}^{K} \) are nonzero coefficients from \( \{A_n^{(\nu)} \}, n \in [N_{\nu-1}, N_{\nu}), \nu \in [1, \nu_0]. \) From 1'), 2'), 3') it follows that the set \( E \) and the polynomial \( Q(x) \) satisfy conditions 1)-3) of the lemma and that all \( \{A_k\}_{k=k_0}^{K} \) have the same sign.

Hence, taking into account (2.2) we obtain

\[
\max_{k_0 \leq m \leq K} \left\| \sum_{k=k_0}^{m} A_k \varphi_{n_k}(x) \right\|_C = \left\| \sum_{k=k_0}^{K} A_k \varphi_{n_k}(x) \right\|_C = \| Q(x) \|_C = |\gamma|
\]

and the assertion follows. \( \square. \)

**Lemma 3.** Let dyadic intervals \( \{\Delta^{(\nu)}_k\}_{\nu=1}^{2^p} = \{\Delta^{(\nu)}_k\}_{\nu=1}^{2^p} \) \( (p \geq 1), \) numbers \( \gamma_{\nu} \neq 0, \nu = 1, 2, \ldots, 2^p ; 0 < \epsilon < 1 ; N_0 \in \mathbb{N} \) and a step function of the form \( f(x) = \sum_{\nu=1}^{2^p} \gamma_{\nu} \chi_{\Delta^{(\nu)}}, \) where \( \Delta^{(\nu)} = \left[ \frac{\nu}{2^p}, \frac{\nu + 1}{2^p} \right), \nu = 1, \ldots, 2^p - 1, \) and \( \Delta^{(2^p)} = \left[ \frac{2^p}{2^p}, 1 \right], \) be given. Then one can find a measurable set \( E \subset [0, 1] \) and a polynomial \( Q(x) \) by the Faber-Schauder system of the form

\[ Q(x) = \sum_{k=k_0}^{K} A_k \varphi_{n_k}(x), \ n_{k_0} \geq N_0, \]

satisfying the following conditions:

1) \( |E| > 1 - \epsilon. \)

2) \( Q(x) = f(x) \) for all \( x \in E. \)

3) \( ||Q(x)||_C = ||f(x)||_\infty = \max\{|\gamma_{\nu}|, \nu \in [1, 2^p]\}. \)

4) \( \epsilon > |A_k| \geq \frac{1}{2}|A_{k+1}| > 0, \ \forall k \in [k_0, K]. \)

5) \( \max_{k_0 \leq m \leq K} \left\| \sum_{k=k_0}^{m} A_k \varphi_{n_k}(x) \right\|_C = ||f(x)||_\infty. \)
where we put \( \| f(x) \|_\infty = \sup_{x \in [0,1]} \{|f(x)|\} \).

**Proof.** By virtue of Lemma 2, there are a measurable set \( E_1 \subset \Delta^{(1)} \) and a polynomial of the form
\[
Q_1(x) = \sum_{k=k_0}^{k_1-1} A_k^{(1)} \varphi_n^k(x), \quad n_{k_0} \geq N_0,
\]
satisfying the conditions:
\[
|E_1| > |\Delta^{(1)}| - \frac{\epsilon}{2^p}
\]
\[
Q_1(x) = \begin{cases} 
\gamma_1, & \text{if } x \in E_1, \\
0, & \text{if } x \in [0,1] \setminus \Delta^{(1)}.
\end{cases}
\quad ; \quad \|Q_1(x)\|_C = |\gamma_1|
\]
\[
\epsilon > |A_k^{(1)}| \geq \frac{1}{2} |A_{k_1+1}^{(1)}| > 0, \quad \forall k \in [k_0, k_1 - 1)
\]
\[
\max_{k_0 \leq m < k_1} \left\| \sum_{k=k_0}^{m} A_k^{(1)} \varphi_n^k(x) \right\|_C = \|Q_1(x)\|_C = |\gamma_1|.
\]

Similarly, there are a measurable set \( E_2 \subset \Delta^{(2)} \) and a polynomial of the form
\[
Q_2(x) = \sum_{k=k_1}^{k_2-1} A_k^{(2)} \varphi_n^k(x),
\]
such that
\[
|E_2| > |\Delta^{(2)}| - \frac{\epsilon}{2^p},
\]
\[
Q_2(x) = \begin{cases} 
\gamma_2, & \text{if } x \in E_2, \\
0, & \text{if } x \in [0,1] \setminus \Delta^{(2)}.
\end{cases}
\quad ; \quad \|Q_2(x)\|_C = |\gamma_2|,
\]
\[
A_{k_1}^{(1)} > |A_k^{(2)}| \geq \frac{1}{2} |A_{k_1+1}^{(2)}| > 0, \quad \forall k \in [k_1, k_2 - 1),
\]
\[
\max_{k_1 \leq m < k_2} \left\| \sum_{k=k_1}^{m} A_k^{(2)} \varphi_n^k(x) \right\|_C = \|Q_2(x)\|_C = |\gamma_2|.
\]

Proceeding thus, inductively, for each \( \nu \in [1,2^p] \) one determines a measurable set \( E_\nu \subset \Delta^{(\nu)} \) and a polynomial of the form...
\( Q_\nu(x) = \sum_{k=k_{\nu-1}}^{k_\nu-1} A_k^{(\nu)} \varphi_{n_k}(x) \),

satisfying the conditions:

\( |E_\nu| > |\Delta^{(\nu)}| - \frac{\epsilon}{2^p} \),

\( Q_\nu(x) = \begin{cases} \gamma_\nu, & \text{if } x \in E_\nu, \\ 0, & \text{if } x \in [0,1] \setminus \Delta^{(\nu)} \end{cases} ; \| Q_\nu(x) \|_C = |\gamma_\nu| .

\( \epsilon > |A_k^{(1)}| \geq \frac{1}{2} |A_{k+1}^{(1)}| > 0, \ \forall k \in [k_0, k_1 - 1], \)

\( A_{k_{\nu-1} - 1}^{(\nu-1)} > |A_k^{(\nu)}| \geq \frac{1}{2} |A_{k+1}^{(\nu)}| > 0, \ \forall k \in [k_{\nu-1}, k_\nu - 1), \ \nu \in [2, 2^p] , \)

\( \max_{k_{\nu-1} \leq m < k_\nu} \left\| \sum_{k=k_{\nu-1}}^{m} A_k^{(\nu)} \varphi_{n_k}(x) \right\|_C = \| Q_\nu(x) \|_C = |\gamma_\nu| .

We put

\( E = \bigcup_{\nu=1}^{2^p} E_\nu, \)

\( Q(x) = \sum_{\nu=1}^{2^p} Q_\nu(x) = \sum_{\nu=1}^{2^p} \left( \sum_{k=k_{\nu-1}}^{k_\nu-1} A_k^{(\nu)} \varphi_{n_k}(x) \right) = \sum_{k=k_0}^{K} A_k \varphi_{n_k}(x) \)

(see (2.3)), where \( A_k = A_k^{(\nu)} \) for \( k \in [k_{\nu-1}, k_\nu - 1] , \ \nu \in [1, 2^p] . \) From (2.4) and (2.8) it follows that

\( |E| = \sum_{\nu=1}^{2^p} |E_\nu| > \sum_{\nu=1}^{2^p} |\Delta^{(\nu)}| - 2^p \cdot \frac{\epsilon}{2^p} = 1 - \epsilon . \)

So conditions 1) and 4) (see (2.6)) of Lemma 3 hold.
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Now let $m \in [k_0, K]$. Then there is a unique $\nu \in [1, 2^p]$ such that $k_{\nu-1} \leq m < k_\nu$. So we have

$$\sum_{k=k_0}^{m} A_k \varphi_{n_k}(x) = \sum_{j=1}^{\nu-1} Q_j(x) + \sum_{k=k_{\nu-1}}^{m} A_k(\nu) \varphi_{n_k}(x)$$

(see (2.9)). From here, from (2.5), (2.7) and from the fact that supports of different intervals from $\{\Delta(\nu)\}_{\nu=1}^{2^p}$ are pairwise-disjoint, we have

$$\max_{k_0 \leq m < K} \left\| \sum_{n=k_0}^{m} A_k \varphi_{n_k}(x) \right\|_C = \left\| \sum_{\nu=1}^{2^p} Q_\nu(x) \right\|_C = \| Q(x) \|_C = \max_{\nu \in [1, 2^p]} \{|\gamma_\nu|\} = \| f(x) \|_\infty,$$

i.e. conditions 3) and 5) of Lemma 3 hold. Similarly, using (2.5), (2.8) and (2.9), we obtain $Q(x) = f(x)$, if $x \in E$. \qed

3. Proof of the Theorem 1.

Assume that $0 < \epsilon < 1$. By numbering all step functions of the form

$$S(x) = \sum_{\nu=1}^{2^p} \gamma_\nu \chi_{\hat{\Delta}(\nu)},$$

where $\gamma_\nu, \nu = 1, 2, ..., 2^p$ are non-zero rational numbers, and $\hat{\Delta}(\nu) = \left[ \frac{\nu-1}{2^p}, \frac{\nu}{2^p} \right), \nu = 1, 2, ..., 2^p - 1$ and $\hat{\Delta}(2^p) = \left[ \frac{2^p-1}{2^p}, 1 \right], p \in \mathbb{N}$, we can represent them as a sequence

$$\{f_m(x)\}_{m=1}^{\infty}.\tag{3.1}$$

Using Lemma 3 repeatedly we can find sequences of sets $\{E_m\}_{m=1}^{\infty}$ and polynomials

$$Q_m(x) = \sum_{k=k_{m-1}}^{k_m-1} a_k^{(m)} \varphi_{n_k}, \text{ } m \in \mathbb{N}, \tag{3.2}$$

that satisfy the following conditions for all $m \in \mathbb{N}$:

$$|E_m| > 1 - \epsilon \cdot 2^{-m}, \tag{3.3}$$
\[Q_m(x) = f_m(x), \quad x \in E_m,\]

\[|a_{k_{m-1}}^{(m-1)}| > |a_k^{(m)}| \geq \frac{1}{2} |a_{k+i}^{(m)}|, \quad i \in [1, k_m - k], \quad k \in [k_{m-1}, k_m - 1],\]

\[
\max_{k_{m-1} \leq N < k_m} \left\| \sum_{k=k_{m-1}}^{N} a_k^{(m)} \varphi_{n_k}(x) \right\| \leq \| Q_m(x) \|_C = \| f_m(x) \|_\infty.
\]

Recall that by \( \| f(x) \|_\infty \) we mean \( \sup_{x \in [0,1]} \{|f(x)|\} \). Let

\[
E = \bigcap_{m=1}^{\infty} E_m.
\]

Obviously (see (3.3)) \( |E| > 1 - \epsilon \).

Let \( f(x) \in C[0,1] \). It can be easily seen that there exists a subsequence \( \{f_{m_l}(x)\}_{l=1}^\infty \) of (3.1) such that

\[
\lim_{N \to \infty} \left\| \sum_{n=1}^{N} f_{m_l}(x) - f(x) \right\|_\infty = 0,
\]

\[
\| f_{m_l}(x) \|_\infty \leq 2^{-l}, \quad l \geq 2.
\]

Consider the function \( g(x) \) defined as follows:

\[
g(x) = \sum_{k=1}^{\infty} c_k \varphi_{n_k}(x),
\]

where \( \{c_k\}_{k=1}^{\infty} \) is the sequence \( a_{k_{m_l}}^{(m_1)}, a_{k_{m_l}}^{(m_2)}, \ldots a_{k_{m_l}}^{(m_l)}, \ldots, a_{k_{m_l}}^{(m_l)} \)

and \( \{N_k\}_{k=1}^{\infty} = \{n_k, k \in [k_{m_l}, k_{m_l} - 1], l \in \mathbb{N}\} \) (see (3.2)).

Hence it follows from conditions (3.4)-(3.9) that the series (3.10) converges uniformly and we have

\[|c_k| \geq \frac{1}{2} |c_{k+1}|, \quad k \geq 1,\]

\[
g(x) \in C[0,1], \quad g(x) = f(x), \quad x \in E
\]

and the result is proved. \( \square \)
Coefficients of the expansion of elements from $C[0,1]$ space

References


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