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On the coefficients of the expansion of elements from $C[0,1]$ space by the Faber-Schauder system

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Abstract. Elements (functions) of continuous on $[0,1]$ functions space ($C[0,1]$) are described which have $\frac{1}{2}$-monotone coefficients of the expansion by the Faber-Schauder system.

1. Introduction

At first we remind the definition of the Faber-Schauder system. It is a system of functions $\Phi = \{\varphi_n(x)\}_{n=0}^{\infty}$, $x \in [0,1]$, in which $\varphi_0(x) \equiv 1$, $\varphi_1(x) = x$ and for $n = 2^k + i$; $k = 0, 1, \dots$; $i = 1, 2, \dots, 2^k$

(1.1)

$$\varphi_n(x) = \varphi_k^{(i)}(x) = \begin{cases} 0, \text{ if } x \notin (\frac{i-1}{2^k}, \frac{i}{2^k}) \ , \\ \\ 1, \text{ if } x = x_n = x_k^{(i)} = \frac{2i-1}{2^{k+1}}, \\ \\ \text{ is linear and continuous in } [\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}}], [\frac{2i-1}{2^{k+1}}, \frac{i}{2^k}]. \end{cases}$$

By $\Delta_n = \Delta_k^{(i)}$, $n = 2^k + i$ $(n \geq 2)$, we denote the support of function $\varphi_n(x) = = \varphi_k^{(i)}(x)$. Remind that the Faber-Schauder system is a basis in

space $C[0,1]$ (the space of continuous on $[0,1]$ functions), i.e. for every function $f(x) \in C[0,1]$ there exists a unique series of the form

$$\sum_{n=0}^{\infty} A_n(f)\varphi_n(x) \ , \tag{1.2}$$

where

$$A_0(f) = f(0) \ , \ A_1(f) = f(1) - f(0) \ ,$$

$$A_n(f) = A_{k,i}(f) = f\left(\frac{2i-1}{2^{k+1}}\right) - \frac{1}{2}\left[f\left(\frac{i-1}{2^k}\right) + f\left(\frac{i}{2^k}\right)\right] \ , \tag{1.3}$$

which uniformly converges to $f(x)$ on $[0,1]$ (see [19]).

Definition 1. Let $t \in (0,1]$. We say that coefficients of $f(x) \in C[0,1]$ function's expansion by the Faber-Schauder system are t**-monotone** if

$$|A_{n_k}(f)| \geq t \cdot |A_{n_{k+i}}(f)| \ , \ k = 1, 2, \dots \ ; \ i \geq 1 \ ,$$

where $\{n_k\}_{k=1}^{\infty}$ is the spectrum of function $f(x)$, i.e. the set of integers where $A_n(f)$ is non zero.

Obviously, if coefficients of the expansion by the Faber-Schauder system are t-monotone for some $t = t_0 \in (0,1]$ then they are t-monotone for all $t \in (0, t_0]$.

Definition 2. Let $t \in (0,1]$ be a fixed parameter. For a given function $f(x) \in C[0,1]$ denote Λ_m any set of m indices (which is not unique in general) such that

$$\min_{n \in \Lambda_m} |A_n(f)| \geq t \cdot \max_{n \notin \Lambda_m} |A_n(f)|$$

and define

$$G_m^t(x, f, \Lambda_m) \equiv \sum_{n \in \Lambda_m} A_n(f)\varphi_n(x) \ ,$$

which is called **Weak Thresholding Greedy Algorithm** with weakness parameter t.

Note that $\{G_m^t(x,f)\}_{m=1}^{\infty}$, $t \in (0,1]$ nonlinear operators have been considered in various Banach spaces with respect to various normalized bases (see for example [1-5], [12-13], [20]).

We consider the following question: let $t \in (0,1]$. Is it possible to modify a continuous on $[0,1]$ function on a "small" set to make coefficients of expansion (1.2) t-monotone?

In the present work we were able to prove the following

Theorem 1. *For every $\epsilon \in (0,1)$ there exists a measurable set $E \subset [0,1]$ with measure $|E| > 1-\epsilon$, such that to each function $f(x) \in C[0,1]$ one can find a function $g(x) \in C[0,1]$ that coincides with $f(x)$ on E and coefficients of which's expansion by the Faber-Schauder system are t-monotone for all $t \in (0, \frac{1}{2}]$.*

From Theorem 1 follows

Theorem 2. *For every $\epsilon \in (0,1)$ there exists a measurable set $E \subset [0,1]$ with measure $|E| > 1-\epsilon$, such that to each function $f(x) \in C[0,1]$ one can find a function $g(x) \in C[0,1]$ that coincides with $f(x)$ on E, sets of indices $\{\Lambda_m\}_{m=1}^{\infty}$ (see Definition 2) and a sequence of integers $\{N_m\}_{m=1}^{\infty}$ so that for all $m \in \mathbb{N}$*

$$G_m^{\frac{1}{2}}(x, g, \Lambda_m) = S_{N_m}(x, g) \ , x \in [0,1] \ ,$$

where $S_N(x,g) = \sum_{n=0}^{N} A_n(g)\varphi_n(x)$, $N = 0, 1, ...$ are partial sums of the series (1.2).

Note that from this theorem immediately follows that the sequence $\{G_m^{\frac{1}{2}}(x, g, \Lambda_m)\}_{m=1}^{\infty}$ uniformly converges to $g(x)$.

Analogous questions are interesting to consider in various functional spaces with respect to other bases. Note that in particular in [6] it is proved the following theorem for the Walsh system:

Theorem (Grigoryan). *For every $\epsilon \in (0,1)$, $p \geq 1$ and each function $f(x) \in L^p[0,1]$ one can find a function $g(x) \in L^p[0,1]$, $mes\{x \in [0,1]; g \neq f\} < \epsilon$, coefficients of which's expansion by the Walsh system are 1-monotone.*

Note that the idea of modification of a function improving its properties belongs to N.N. Luzin. The following famous result was obtained by him in 1912 (see [14]).

Theorem (N.N. Luzin's C-property). *Each measurable and a.e. finite function can be made continuous after a modification on a subset of arbitrarily small Lebesgue measure.*

In 1939, Men'shov [15] proved the following fundamental theorem.

Theorem (Men'shov's C-strong property). *Let $f(x)$ be an a.e. finite measurable function on $[0, 2\pi]$. Then for each $\epsilon > 0$ one can define a continuous function $g(x)$ coinciding with $f(x)$ on a subset E of measure $|E| > 2\pi - \epsilon$ such that its Fourier series with respect to the trigonometric system converges uniformly on $[0, 2\pi]$.*

Further interesting results in this direction were obtained by Men'shov [16], Kheladze [11], Price [18], Oskolkov [17], Grigoryan [7-10] and other authors.

The following questions are open:

Question 1. *Is Theorem 1 true for* $t \in (\frac{1}{2}, 1]$ *?*

Question 2. *Do Theorems 1 and 2 hold for other bases in the space* $C[0,1]$ *(in particular, for the Franklin system)?*

Question 3. *Is Theorem 1 true for the trigonometric system for any* $t \in (0,1]$ *?*

Question 4. *Let* $t \in (0,1)$. *Is there a continuous function on* $[0,1]$ *such that Weak Thresholding Greedy Algorithm by the Faber-Schauder system with weakness parameter* t *diverges by measure (or at least almost everywhere) in* $[0,1]$ *?*

Concerning with Question 4 note that in [5] it is proved that the answer is positive for $t = 1$.

2. Proofs of basic lemmas

Lemma 1. *Let dyadic interval* $\Delta = \Delta_p^{(i)} = (\frac{i-1}{2^p}, \frac{i}{2^p})$ $(i \in [1, 2^p])$ *and numbers* $\gamma \neq 0$, $N_0 \in \mathbb{N}$, $0 < \epsilon < |\Delta|$ *be given. Then there exist a measurable set* $E \subset \Delta$ *of measure* $|E| > |\Delta| - \epsilon$ *and a polynomial*

$$Q(x) = \sum_{n=N_0}^{N} A_n \varphi_n(x) \ ,$$

by the system (1.1), such that

1) $$Q(x) = \begin{cases} \gamma, & \text{if } x \in E, \\ 0, & \text{if } x \in [0,1] \setminus \Delta \end{cases} \quad ; \quad \| Q(x) \|_C = |\gamma| \ ,$$

2) $$A_n = \gamma \ , \ \frac{\gamma}{2} \ \text{or } 0 \ , \ n \in [N_0, N].$$

Proof. At first we consider the case when $N_0 \leq 2^p + i$. We take natural number $q > log_2 \frac{1}{\epsilon} + 1$ $(0 < \epsilon < |\Delta|)$ and put

$$E = \Delta \setminus \left\{ \left(\frac{i-1}{2^p} \ , \frac{i-1}{2^p} + \frac{1}{2^q} \right) \bigcup \left(\frac{i}{2^p} - \frac{1}{2^q} \ , \frac{i}{2^p} \right) \right\} \ ,$$

$$g(x)=\begin{cases}\gamma, \text{ if } x\in E\ ,\\ \\ 0, \text{ if } x\in[0,1]\setminus\Delta\ ,\\ \\ \text{is linear and continuous on } [\frac{i-1}{2^p},\frac{i-1}{2^p}+\frac{1}{2^q}] \text{ and } [\frac{i}{2^p}-\frac{1}{2^q},\frac{i}{2^p}]\ .\end{cases}$$

It is clear that $|E|>|\Delta|-\epsilon$. Taking into account (1.3), it is not hard to see that

$$g(x)=\gamma\cdot\varphi_p^{(i)}(x)+\frac{\gamma}{2}\cdot\left\{\sum_{k=1}^{q-p-1}\varphi_{p+k}^{(2t_k-1)}(x)+\sum_{k=1}^{q-p-1}\varphi_{p+k}^{(2h_k)}(x)\right\}\ ,$$

where $t_1=h_1=i$; $t_{k+1}=2t_k-1$, $h_{k+1}=2h_k$, so that the function $g(x)$ presents a polynomial by Faber-Schauder system, satisfying the conditions of the lemma.

Now we consider the case $N_0>2^p+i$. In this case some points x_n , $n<N_0$(see (1.1)) belong to Δ. We denote them by x_{n_l}, $l=1,2,...L$ $(x_{n_1}=x_p^{(i)}=\frac{2i-1}{2^{p+1}})$.

We take a natural number $q>log_2\frac{L+1}{\epsilon}+1$ $(0<\epsilon<|\Delta|)$ and put

$$E=\Delta\backslash\left\{\bigcup_{l=1}^{L}\left(x_{n_l}-\frac{1}{2^q},x_{n_l}+\frac{1}{2^q}\right)\bigcup\left(\frac{i-1}{2^p}\ ,\frac{i-1}{2^p}+\frac{1}{2^q}\right)\bigcup\left(\frac{i}{2^p}-\frac{1}{2^q}\ ,\frac{i}{2^p}\right)\right\},$$

(2.1)

$$g(x)=\begin{cases}\gamma, \text{ if } x\in E\ ,\\ \\ 0, \text{ if } x\in([0,1]\setminus\Delta)\ \bigcup\{x_{n_l}\}_{l=1}^{L}\ ,\\ \\ \text{is linear and continuous on } [x_{n_l}-\frac{1}{2^q},x_{n_l}]\text{ and}\\ [x_{n_l},x_{n_l}+\frac{1}{2^q}]\ (l\in[1,L]) \text{ and on } [\frac{i-1}{2^p},\frac{i-1}{2^p}+\frac{1}{2^q}]\ ,\ [\frac{i}{2^p}-\frac{1}{2^q},\frac{i}{2^p}]\ .\end{cases}$$

It is clear that $2^q>N_0$, $\|\ g(x)\ \|_C=|\gamma|$ and $|E|>|\Delta|-\epsilon$. Taking into account (1.3) we obtain that in the expansion of the function $g(x)\in C_{[0,1]}$ by the system (1.1), coefficients of functions $\{\varphi_n(x)\}$ with numbers $n<N_0$ and $n>2^q$ and coefficients of functions $\{\varphi_n(x)\}$ with numbers $N_0\le n\le 2^q$, whichs supports don't belong to Δ or $\Delta_n\subset E$, are equal to 0. From (1.3) and (2.1) it follows that coefficients of functions $\varphi_n(x)$, $N_0\le n\le 2^q$ $(\Delta_n\subset\Delta\ ,\ \Delta_n\nsubseteq E)$, are equal to either γ or $\frac{\gamma}{2}$, depending on the fact that function $g(x)$ takes value 0 in both endpoints of Δ_n or only in one endpoint. So we obtain that the function $g(x)$ presents a polynomial by the system (1.1) satisfying the conditions of the lemma.$\square$

Let $\|\cdot\|_C = \max\limits_{x\in[0,1]} |\cdot|$ is the norm of the space $C_{[0,1]}$.

Lemma 2. *Let dyadic interval* $\Delta = \Delta_p^{(i)} = (\frac{i-1}{2^p}, \frac{i}{2^p})$ $(i \in [1, 2^p])$ *and numbers* $\gamma \neq 0$, $N_0 \in \mathbb{N}$, $0 < \epsilon < |\Delta|$ *be given. Then one can find a measurable set* $E \subset \Delta$ *and a polynomial* $Q(x)$ *by the Faber-Schauder system of the form*

$$Q(x) = \sum_{k=k_0}^{K} A_k \varphi_{n_k}(x) \ , \ n_{k_0} \geq N_0 \ ,$$

satisfying the following conditions:

1) $$|E| > |\Delta| - \epsilon \ ,$$

2) $$Q(x) = \begin{cases} \gamma, \text{ when } x \in E, \\ 0, \text{ when } x \in [0,1] \setminus \Delta \end{cases} \quad ; \quad \| Q(x) \|_C = |\gamma| \ ,$$

3) $$\epsilon > |A_k| \geq \frac{1}{2}|A_{k+1}| > 0 \ , \ \forall k \in [k_0, K) \ ,$$

4) $$\max_{k_0 \leq m \leq K} \left\| \sum_{k=k_0}^{m} A_k \varphi_{n_k}(x) \right\|_C = \| Q(x) \|_C = |\gamma| \ .$$

Proof. Let $\nu_0 > \frac{|\gamma|}{\epsilon}$ be some natural number. By virtue of Lemma 1, for every natural number $\nu \in [1, \nu_0]$ one can find a measurable set $E_\nu \subset \Delta$ and a polynomial of the form

$$Q_\nu(x) = \sum_{n=N_{\nu-1}}^{N_\nu - 1} A_n^{(\nu)} \varphi_n(x) \ \ , \ N_\nu > N_{\nu-1} \ ,$$

satisfying the conditions:

1′) $$|E_\nu| > |\Delta| - \frac{\epsilon}{\nu_0} \ ,$$

2′) $$Q_\nu(x) = \begin{cases} \frac{\gamma}{\nu_0}, \text{ if } x \in E_\nu, \\ 0, \text{ if } x \in [0,1] \setminus \Delta \end{cases} \quad ; \quad \| Q_\nu(x) \|_C = \frac{|\gamma|}{\nu_0} \ ,$$

$3')$ $$A_n^{(\nu)} = \frac{\gamma}{\nu_0}\ ,\ \frac{\gamma}{2\nu_0}\ \text{ or }\ 0\ ,\ n \in [N_{\nu-1}, N_\nu).$$

We put

$$E = \bigcap_{\nu=1}^{\nu_0} E_\nu$$

and

$$Q(x) = \sum_{\nu=1}^{\nu_0} Q_\nu(x) = \sum_{k=k_0}^{K} A_k \varphi_{n_k}(x)\ , \tag{2.2}$$

where $\{A_k\}_{k=k_0}^K$ are nonzero coefficients from $\{A_n^{(\nu)}\ ,\ n \in [N_{\nu-1}, N_\nu)\ ,\ \nu \in [1, \nu_0]\}$. From $1')$, $2')$, $3')$ it follows that the set E and the polynomial $Q(x)$ satisfy conditions 1)-3) of the lemma and that all $\{A_k\}_{k=k_0}^K$ have the same sign.

Hence, taking into account (2.2) we obtain

$$\max_{k_0 \le m \le K} \left\| \sum_{k=k_0}^{m} A_k \varphi_{n_k}(x) \right\|_C = \left\| \sum_{k=k_0}^{K} A_k \varphi_{n_k}(x) \right\|_C = \| Q(x) \|_C = |\gamma|$$

and the assertion follows. $\square$.

Lemma 3. *Let dyadic intervals* $\{\Delta^{(\nu)}\}_{\nu=1}^{2^p} = \{\Delta_p^{(\nu)}\}_{\nu=1}^{2^p}$ $(p \ge 1)$, *numbers* $\gamma_\nu \ne 0$, $\nu = 1, 2, ..., 2^p$; $0 < \epsilon < 1$; $N_0 \in \mathbb{N}$ *and a step function of the form* $f(x) = \sum_{\nu=1}^{2^p} \gamma_\nu \chi_{\tilde{\Delta}^{(\nu)}}$, *where* $\tilde{\Delta}^{(\nu)} = [\frac{\nu-1}{2^p}, \frac{\nu}{2^p})$, $\nu = 1, ..., 2^p - 1$, *and* $\tilde{\Delta}^{(2^p)} = [\frac{2^p-1}{2^p}, 1]$, *be given. Then one can find a measurable set* $E \subset [0, 1]$ *and a polynomial* $Q(x)$ *by the Faber-Schauder system of the form*

$$Q(x) = \sum_{k=k_0}^{K} A_k \varphi_{n_k}(x)\ ,\ n_{k_0} \ge N_0\ ,$$

satisfying the following conditions:

1) $| E | > 1 - \epsilon$.
2) $Q(x) = f(x)$ for all $x \in E$.
3) $\| Q(x) \|_C = \| f(x) \|_\infty = \max\{|\gamma_\nu|\ ,\ \nu \in [1, 2^p]\}$.
4) $\epsilon > |A_k| \ge \frac{1}{2}|A_{k+1}| > 0, \quad \forall k \in [k_0, K),$.
5) $\max_{k_0 \le m \le K} \left\| \sum_{k=k_0}^{m} A_k \varphi_{n_k}(x) \right\|_C = \| f(x) \|_\infty$,

where we put $\| f(x) \|_\infty = \sup\limits_{x\in[0,1]} \{|f(x)|\}$.

Proof. By virtue of Lemma 2, there are a measurable set $E_1 \subset \Delta^{(1)}$ and a polynomial of the form

$$Q_1(x) = \sum_{k=k_0}^{k_1-1} A_k^{(1)} \varphi_{n_k}(x) \ , \ n_{k_0} \geq N_0 \ ,$$

satisfying the conditions:

$$|E_1| > |\Delta^{(1)}| - \frac{\epsilon}{2^p}$$

$$Q_1(x) = \begin{cases} \gamma_1, & \text{if } x \in E_1 \ , \\ 0, & \text{if } x \in [0,1] \setminus \Delta^{(1)} \end{cases} \quad ; \quad \| Q_1(x) \|_C = |\gamma_1|$$

$$\epsilon > |A_k^{(1)}| \geq \frac{1}{2}|A_{k+1}^{(1)}| > 0, \quad \forall k \in [k_0, k_1 - 1)$$

$$\max_{k_0 \leq m < k_1} \left\| \sum_{k=k_0}^{m} A_k^{(1)} \varphi_{n_k}(x) \right\|_C = \| Q_1(x) \|_C = |\gamma_1| \ .$$

Similarly, there are a measurable set $E_2 \subset \Delta^{(2)}$ and a polynomial of the form

$$Q_2(x) = \sum_{k=k_1}^{k_2-1} A_k^{(2)} \varphi_{n_k}(x) \ ,$$

such that

$$|E_2| > |\Delta^{(2)}| - \frac{\epsilon}{2^p} \ ,$$

$$Q_2(x) = \begin{cases} \gamma_2, & \text{if } x \in E_2 \ , \\ 0, & \text{if } x \in [0,1] \setminus \Delta^{(2)} \end{cases} \quad ; \quad \| Q_2(x) \|_C = |\gamma_2| \ ,$$

$$A_{k_1-1}^{(1)} > |A_k^{(2)}| \geq \frac{1}{2}|A_{k+1}^{(2)}| > 0, \quad \forall k \in [k_1, k_2 - 1),$$

$$\max_{k_1 \leq m < k_2} \left\| \sum_{k=k_1}^{m} A_k^{(2)} \varphi_{n_k}(x) \right\|_C = \| Q_2(x) \|_C = |\gamma_2| \ .$$

Proceeding thus, inductively, for each $\nu \in [1, 2^p]$ one determines a measurable set $E_\nu \subset \Delta^{(\nu)}$ and a polynomial of the form

$$Q_\nu(x) = \sum_{k=k_{\nu-1}}^{k_\nu-1} A_k^{(\nu)} \varphi_{n_k}(x) \ , \tag{2.3}$$

satisfying the conditions:

$$|E_\nu| > |\Delta^{(\nu)}| - \frac{\epsilon}{2^p} \ , \tag{2.4}$$

$$Q_\nu(x) = \begin{cases} \gamma_\nu, \ \text{if } x \in E_\nu \ , \\ 0, \ \ \text{if } x \in [0,1] \setminus \Delta^{(\nu)} \end{cases} \quad ; \ \ \| \ Q_\nu(x) \ \|_C = |\gamma_\nu| \ , \tag{2.5}$$

(2.6)
$$\begin{cases} \epsilon > |A_k^{(1)}| \geq \frac{1}{2}|A_{k+1}^{(1)}| > 0, \ \ \forall k \in [k_0, k_1 - 1), \\ \\ A_{k_{\nu-1}-1}^{(\nu-1)} > |A_k^{(\nu)}| \geq \frac{1}{2}|A_{k+1}^{(\nu)}| > 0 \ , \ \ \forall k \in [k_{\nu-1}, k_\nu - 1) \ , \ \nu \in [2, 2^p] \ , \end{cases}$$

$$\max_{k_{\nu-1} \leq m < k_\nu} \left\| \sum_{k=k_{\nu-1}}^{m} A_k^{(\nu)} \varphi_{n_k}(x) \right\|_C = \| \ Q_\nu(x) \ \|_C = |\gamma_\nu| \ . \tag{2.7}$$

We put

$$E = \bigcup_{\nu=1}^{2^p} E_\nu, \tag{2.8}$$

$$Q(x) = \sum_{\nu=1}^{2^p} Q_\nu(x) = \sum_{\nu=1}^{2^p} \left(\sum_{k=k_{\nu-1}}^{k_\nu-1} A_k^{(\nu)} \varphi_{n_k}(x) \right) = \sum_{k=k_0}^{K} A_k \varphi_{n_k}(x) \tag{2.9}$$

(see (2.3)), where $A_k = A_k^{(\nu)}$ for $k \in [k_{\nu-1}, k_\nu - 1]$, $\nu \in [1, 2^p]$. From (2.4) and (2.8) it follows that

$$|E| = \sum_{\nu=1}^{2^p} |E_\nu| > \sum_{\nu=1}^{2^p} |\Delta^{(\nu)}| - 2^p \cdot \frac{\epsilon}{2^p} = 1 - \epsilon \ .$$

So conditions 1) and 4) (see (2.6)) of Lemma 3 hold.

Now let $m \in [k_0, K]$. Then there is a unique $\nu \in [1, 2^p]$ such that $k_{\nu-1} \le m < k_\nu$. So we have

$$\sum_{k=k_0}^{m} A_k \varphi_{n_k}(x) = \sum_{j=1}^{\nu-1} Q_j(x) + \sum_{k=k_{\nu-1}}^{m} A_k^{(\nu)} \varphi_{n_k}(x)$$

(see (2.9)). From here , from (2.5), (2.7) and from the fact that supports of different intervals from $\{\Delta^{(\nu)}\}_{\nu=1}^{2^p}$ are pairwise-disjoint, we have

$$\begin{aligned} \max_{k_0 \le m < K} \left\| \sum_{n=k_0}^{m} A_k \varphi_{n_k}(x) \right\|_C &= \left\| \sum_{\nu=1}^{2^p} Q_\nu(x) \right\|_C \\ &= \| Q(x) \|_C \\ &= \max_{\nu \in [1,2^p]} \{|\gamma_\nu|\} \\ &= \| f(x) \|_\infty , \end{aligned}$$

i.e. conditions 3) and 5) of Lemma 3 hold. Similarly, using (2.5), (2.8) and (2.9), we obtain $Q(x) = f(x)$, if $x \in E$. □

3. Proof of the Theorem 1.

Assume that $0 < \epsilon < 1$. By numbering all step functions of the form

$$S(x) = \sum_{\nu=1}^{2^p} \gamma_\nu \chi_{\tilde{\Delta}^{(\nu)}} ,$$

where γ_ν, $\nu = 1, 2, ..., 2^p$ are non-zero rational numbers, and $\tilde{\Delta}^{(\nu)} = [\frac{\nu-1}{2^p}, \frac{\nu}{2^p})$, $\nu = 1, 2, ..., 2^p - 1$ and $\tilde{\Delta}^{(2^p)} = [\frac{2^p-1}{2^p}, 1]$, $p \in \mathbb{N}$, we can represent them as a sequence

$$\{f_m(x)\}_{m=1}^{\infty} . \tag{3.1}$$

Using Lemma 3 repeatedly we can find sequences of sets $\{E_m\}_{m=1}^{\infty}$ and polynomials

$$Q_m(x) = \sum_{k=k_{m-1}}^{k_m-1} a_k^{(m)} \varphi_{n_k} , \ m \in \mathbb{N} , \tag{3.2}$$

that satisfy the following conditions for all $m \in \mathbb{N}$:

$$|E_m| > 1 - \epsilon \cdot 2^{-m}, \tag{3.3}$$

$$Q_m(x) = f_m(x), \quad x \in E_m \ , \tag{3.4}$$

$$|a^{(m-1)}_{k_{m-1}-1}| > |a^{(m)}_k| \geq \frac{1}{2}|a^{(m)}_{k+i}| \ , \ i \in [1, k_m - k) \ , \ k \in [k_{m-1}, k_m - 1) \ , \tag{3.5}$$

$$\max_{k_{m-1} \leq N < k_m} \left\| \sum_{k=k_{m-1}}^{N} a^{(m)}_k \varphi_{n_k}(x) \right\|_C = \| \ Q_m(x) \ \|_C = \| \ f_m(x) \ \|_\infty \ . \tag{3.6}$$

Recall that by $\| \ f(x) \ \|_\infty$ we mean $\sup\limits_{x \in [0,1]} \{|f(x)|\}$. Let

$$E = \bigcap_{m=1}^{\infty} E_m \ . \tag{3.7}$$

Obviously (see (3.3))

$$|E| > 1 - \epsilon.$$

Let $f(x) \in C[0,1]$. It can be easily seen that there exists a subsequence $\{f_{m_l}(x)\}_{l=1}^{\infty}$ of (3.1) such that

$$\lim_{N \to \infty} \left\| \sum_{n=1}^{N} f_{m_l}(x) - f(x) \right\|_\infty = 0 \ , \tag{3.8}$$

$$\| \ f_{m_l}(x) \ \|_\infty \leq 2^{-l}, \quad l \geq 2 \ . \tag{3.9}$$

Consider the function $g(x)$ defined as follows:

$$g(x) = \sum_{k=1}^{\infty} c_k \varphi_{N_k}(x) \ , \tag{3.10}$$

where $\{c_k\}_{k=1}^{\infty}$ is the sequence $a^{(m_1)}_{k_{m_1-1}}, ..., a^{(m_1)}_{k_{m_1}-1}, ..., a^{(m_l)}_{k_{m_l-1}}, ..., a^{(m_l)}_{k_{m_l}-1}, ...$ and $\{N_k\}_{k=1}^{\infty} = \{n_k \ , \ k \in [k_{m_l-1}, k_{m_l} - 1] \ , \ l \in \mathbb{N}\}$ (see (3.2)).

Hence it follows from conditions (3.4)-(3.9) that the series (3.10) converges uniformly and we have

$$|c_k| \geq \frac{1}{2}|c_{k+1}|, \quad k \geq 1 \ ,$$

$$g(x) \in C_{[0,1]}, \quad g(x) = f(x), \qquad x \in E$$

and the result is proved. □

References

[1] R.A. DeVore, V. N. Temlyakov, *Some remarks on greedy algorithms*, Advances in Comp. Math., 5 (1996), 173–187.

[2] S.J. Dilworth, N.J. Kalton, D. Kutzarova, *On the existence of almost greedy bases in Banach spaces*, Studia Math., 159 (2003), 67–101.

[3] M.G. Grigoryan, *On convergence of greedy algorithm by trigonometric system in the metric of L^p*, J. Contemp. Math. Anal., 39 (2004), 35–52.

[4] M.G. Grigoryan , R.E. Zink, *Greedy approximation with respect to certain subsystems of the Walsh orthonormal system*, Proc. Amer. Math. Soc. 134 (2006), 3495–3505.

[5] M.G. Grigorian and A.A. Sargsyan, *Non-linear approximation of continuous functions by the Faber-Schauder system*, Sbornik Mathematics, 199 (2008), 629–653.

[6] M.G. Grigoryan, On the Fourier-Walsh Coefficients, Real Analysis Exchange, 34 (2008/2009), 47–55.

[7] M. G. Grigoryan (Grigorian), *On convergence of Fourier series in complete orthonormal systems in the L1-metric and almost everywhere*, Mat. Sb. 181 (1990), 1011–1030; English transl. in Math. USSR-Sb. 70 (1991), 445–466.

[8] M. G. Grigorian, *On the representation of functions by orthogonal series in weighted L^p spaces*, Studia Math.,134 (1999), 207–216.

[9] M. G. Grigorian, K. S. Kazarian and F. Soria, *Mean convergence of orthogonal Fourier series of modified functions*, Trans. Amer. Math. Soc., 352 (2000), 3777–3798.

[10] M. G. Grigorian, *On the L^p_μ-strong property of orthonormal systems*, Mat. Sb. 194 (2003), 77–106; English transl. in Sb. Math. 194 (2003), 1503–1532.

[11] S. V. Kheladze, *Convergence of Fourier series almost everywhere and in the L-metric*, Mat. Sb. 107 (1978), 245.-258; English transl. in Math. USSR-Sb. 35 (1979), 527–539.

[12] S.V. Konyagin and V.N. Emlyakov, *A remark on Greedy approximation in Banach spaces*, East J. on Approx., 5 (1999), 1–15.

[13] T.W. Korner, *Divergence of decreasing rearranged Fourier series*, Ann. of Math., 144 (1996), 167–180.

[14] N.N. Lusin, *On a central theorem of integral calculus*, Mat. Sb., 28 (1912), 266–294.

[15] D.E. Menchoff, *Sur la convergence uniforme des series de Fourier*, Mat. Sb., 11 (1942), 67–96.

[16] D.E. Menchoff, *On Fourier series of continuous functions*, Uchen. Zapiski Mosk. Gos. Univ., 148 (1951), 108–132.

[17] K. I. Oskolkov, *Uniform modulus of continuity of summable functions on sets of positive measure*, Dokl. Akad. Nauk SSSR, 229 (1976), 304–306; English transl. in Soviet Math. Dokl. 17 (1976), 1028–1030.

[18] J. J. Price, *Walsh series and adjustment of functions on small sets*, Illinois J. Math.,13 (1969), 131–136.

[19] J. Schauder, *Zur Theorie stetiger Abbildungen in Funktionalrä umen*, Math. Z., 26 (1927), 47–65.

[20] V.N. Temlyakov, *The best m-term approximation and Greedy Algorithms*, Advanced in Comp. Math., 8 (1998), 249–265.

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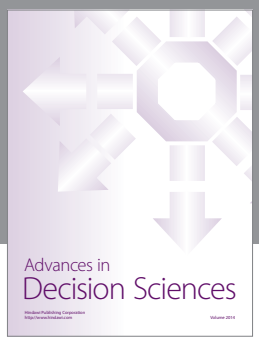

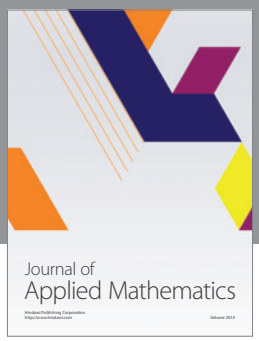

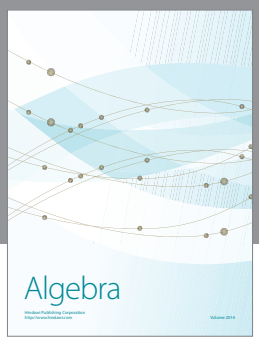

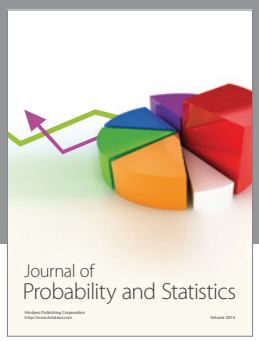

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