On boundedness and compactness of a certain class of kernel operators

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Abstract. New conditions for $L_p[0, \infty) \rightarrow L_q[0, \infty)$ boundedness and compactness ($1 < p, q < \infty$) of the map $f \rightarrow w(x) \int_{a(x)}^{b(x)} k(x, y) f(y) v(y) dy$ with locally integrable weight functions v, w and a positive continuous kernel $k(x, y)$ from the Oinarov's class are obtained.

1. Introduction

Let $1 < p < \infty$ and $\|f\|_p := (\int_0^\infty |f(x)|^p dx)^{1/p}$. Denote L_p the Lebesgue space of all measurable functions on $\mathbb{R}^+ := [0, \infty)$ such that $\|f\|_p < \infty$.

Let $w(x)$ and $v(y)$ be non-negative locally integrable weight functions on $\mathbb{R}^+$. We study $L_p \rightarrow L_q$ boundedness and compactness of an integral operator of the form

$$(1.1) \quad \mathcal{K}f(x) := w(x) \int_{a(x)}^{b(x)} k(x, y) f(y) v(y) dy$$

with border functions $a(x)$ and $b(x)$ satisfying the conditions

(1.2)

- (i) $a(x)$ and $b(x)$ are differentiable and strictly increasing on $(0, \infty)$;
- (ii) $a(0) = b(0) = 0$, $a(x) < b(x)$ for $0 < x < \infty$, $a(\infty) = b(\infty) = \infty$,

and a continuous kernel $k(x, y) > 0$ on $\mathcal{R} := \{(x, y) : x > 0, a(x) < y < b(x)\}$ satisfying generalized *Oinarov's conditions* $\mathcal{O}_b$ or/and $\mathcal{O}_a$.

Definition 1.1. We write $k(x, y) \in \mathcal{O}_b$ if there exists a constant $D \geq 1$ independent on x, y, z such that

(1.3)

$$D^{-1}k(x, y) \leq k(x, b(z)) + k(z, y) \leq Dk(x, y), \quad z \leq x, \quad a(x) \leq y \leq b(z).$$

Similarly, $k(x, y) \in \mathcal{O}_a$ if

(1.4)

$$D^{-1}k(x, y) \leq k(x, a(z)) + k(z, y) \leq Dk(x, y), \quad x \leq z, \quad a(z) \leq y \leq b(x).$$

Operators of the type (1.1) have been studied in [1] – [4], [6], [9] and [11]. In particular, $L_p - L_q$ boundedness and compactness criteria for $\mathcal{K}$ were obtained in [9].

Theorem 1.1. [9, Theorems 2, 3] *Let $1 < p \leq q < \infty$ and the operator $\mathcal{K}$ be defined by (1.1) with the border functions $a(x)$, $b(x)$ satisfying (1.2). If the kernel $k(x, y) \in \mathcal{O}_b$, then*

$$(1.5) \quad \alpha_1(p, q)\mathcal{A}_b \leq \|\mathcal{K}\|_{L_p \rightarrow L_q} \leq \alpha_2(p, q)\mathcal{A}_b,$$

where positive constants α_1, α_2 depend only on the parameters p, q and $\mathcal{A}_b := \mathcal{A}_{b,0} + \mathcal{A}_{b,1}$ with

(1.6)

$$\mathcal{A}_{b,0} := \sup_{t>0} \mathcal{A}_{b,0}(t) = \sup_{t>0} \sup_{b^{-1}(a(t)) \leq s \leq t} \left(\int_s^t k^q(x, b(s)) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} v^{p'}(y) dy \right)^{\frac{1}{p'}},$$

(1.7)

$$\mathcal{A}_{b,1} := \sup_{t>0} \mathcal{A}_{b,1}(t) = \sup_{t>0} \sup_{b^{-1}(a(t)) \leq s \leq t} \left(\int_s^t w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} k^{p'}(s, y) v^{p'}(y) dy \right)^{\frac{1}{p'}}.$$

Moreover, $\mathcal{K}$ is compact if and only if $\mathcal{A}_b < \infty$ and $\lim_{s \rightarrow 0} \mathcal{A}_{b,i}(s) = \lim_{s \rightarrow \infty} \mathcal{A}_{b,i}(s) = 0, i = 0, 1$.

If the kernel $k(x, y) \in \mathcal{O}_a$, then

$$(1.8) \quad \beta_1(p, q)\mathcal{A}_a \leq \|\mathcal{K}\|_{L_p \rightarrow L_q} \leq \beta_2(p, q)\mathcal{A}_a,$$

where positive constants β_1, β_2 depend only on the parameters p, q and $\mathcal{A}_a := \mathcal{A}_{a,0} + \mathcal{A}_{a,1}$ with

$$\mathcal{A}_{a,0} := \sup_{s>0} \mathcal{A}_{a,0}(s) = \sup_{s>0} \sup_{s \leq t \leq a^{-1}(b(s))} \left(\int_s^t k^q(x, a(t)) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} v^{p'}(y) dy \right)^{\frac{1}{p'}},$$

$$\mathcal{A}_{a,1} := \sup_{s>0} \mathcal{A}_{a,1}(s) = \sup_{s>0} \sup_{s \leq t \leq a^{-1}(b(s))} \left(\int_s^t w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{1}{p'}}.$$

Besides, $\mathcal{K}$ is compact if and only if $\mathcal{A}_a < \infty$ and $\lim_{s \rightarrow 0} \mathcal{A}_{a,i}(s) = \lim_{s \rightarrow \infty} \mathcal{A}_{a,i}(s) = 0$, $i = 0, 1$.

Similar to (1.5) and (1.8) two-sided estimates for the case $1 < q < p < \infty$ were obtained in [9] in a discrete form. This fact together with double supremum in the definitions of $\mathcal{A}_a$ and $\mathcal{A}_b$ may be rather inconvenient for further development.

In this work we obtain new necessary and new sufficient conditions for $L_p - L_q$ boundedness and compactness of $\mathcal{K}$ (see Sections 3 and 4) which become a criterion either under some additional requirements on weight functions or when a kernel satisfies (1.3) and (1.4) simultaneously (see Section 5). Unlike to the above results the new criteria have one supremum if $1 < p \leq q < \infty$ and a continuous form for $1 < q < p < \infty$. We start the paper with Preliminaries (Section 2) and conclude by Examples (Section 6).

Throughout the paper products of the form $0 \cdot \infty$ are taken to be equal to 0. Relations $A \ll B$ mean $A \leq cB$ with some constants c depending only on parameters of summations and, possibly, on the constants of equivalence in the inequalities of the type (1.3). We write $A \approx B$ instead of $A \ll B \ll A$ or $A = cB$. $\mathbb{Z}$ denotes the set of all integers and χ_E stands for a characteristic function (indicator) of a subset $E \subset \mathbb{R}^+$. Also we make use of marks $: =$ and $=:$ for introducing new quantities and suppose $p' := p/(p-1)$ for $1 < p < \infty$ and $r := pq/(p-q)$ for $1 < q < p < \infty$.

2. Preliminaries

Here we collect some statements, which are necessary to prove our results. We start with Lemma 2.1 on a block-diagonal operator.

Lemma 2.1 ([10, Lemma 1]). *Let $0 < q < p < \infty$, let $U = \bigsqcup_k U_k$ and $V = \bigsqcup_k V_k$ be unions of non-overlapping measurable sets and let $T = \sum_k T_k$, where $T_k : L_p(U_k) \rightarrow L_q(V_k)$. Then*

$$\|T\|_{L_p(U) \rightarrow L_q(V)} \approx \left(\sum_k \|T_k\|_{L_p(U_k) \rightarrow L_q(V_k)}^r \right)^{1/r}, \quad 1/r = 1/q - 1/p.$$

The next Theorem 2.1 is a known result for the Hardy type operator

$$(2.1) \quad Kf(x) := w(x) \int_c^x k(x, y) f(y) v(y) dy, \quad 0 \leq c \leq x \leq d \leq \infty,$$

with the kernel $k(x, y) \geq 0$ satisfying the Oinarov's condition of the form: there exists a constant $D \geq 1$ independent on x, y, z such that

$$(2.2) \quad D^{-1}k(x, y) \leq k(x, z) + k(z, y) \leq Dk(x, y), \quad 0 \leq c \leq y \leq z \leq x \leq d \leq \infty.$$

Theorem 2.1. ([8, Theorem 2]) *Let $1 < q < p < \infty$. Then $\|K\|_{L_p(c, d) \rightarrow L_q(c, d)} \approx B_0 + B_1$, where*

$$B_0 := \left(\int_c^d \left[\int_t^d k^q(x, t) w^q(x) dx \right]^{\frac{r}{q}} \left[\int_c^t v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}},$$

$$B_1 := \left(\int_c^d \left[\int_t^d w^q(x) dx \right]^{\frac{r}{p}} \left[\int_c^t k^{p'}(t, y) v^{p'}(y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}}.$$

The result of Theorem 2.1 can be extended to a more general then (2.1) operator $K_b : L_p(b(c), b(d)) \rightarrow L_q(c, d)$ of the form

$$(2.3) \quad K_b f(x) := w(x) \int_{b(c)}^{b(x)} k(x, y) f(y) v(y) dy, \quad 0 \leq c \leq x \leq d \leq \infty,$$

with a differentiable and strictly increasing on $[c, d)$ function $b(x) \geq 0$ and a non-negative kernel $k(x, y)$ satisfying the condition: there exists a constant $D \geq 1$ such that

$$(2.4) \quad D^{-1}k(x, y) \leq k(x, b(z)) + k(z, y) \leq Dk(x, y), \quad \begin{cases} 0 \leq c \leq z \leq x \leq d \leq \infty, \\ 0 \leq b(c) \leq y \leq b(z). \end{cases}$$

Corollary 2.1. *Let $1 < q < p < \infty$, $r = pq/(p - q)$ and the operator K_b be given by (2.3) with a differentiable and strictly increasing on $[c, d)$ function $b(x) \geq 0$ and a non-negative kernel $k(x, y) \in (2.4)$. Then*

$$(2.5) \quad \|K_b\|_{L_p(b(c), b(d)) \rightarrow L_q(c, d)} \approx B_{b,0} + B_{b,1},$$

where

(2.6)

$$B_{b,0} := \left(\int_{b(c)}^{b(d)} \left[\int_{b^{-1}(t)}^d k^q(x,t) w^q(x) dx \right]^{\frac{r}{q}} \left[\int_{b(c)}^t v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}},$$

(2.7)

$$B_{b,1} := \left(\int_c^d \left[\int_t^d w^q(x) dx \right]^{\frac{r}{p}} \left[\int_{b(c)}^{b(t)} k^{p'}(t,y) v^{p'}(y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}}.$$

By duality and Theorem 2.1 we can obtain norm estimates for the operator

$$(2.8) \quad K_a f(x) := w(x) \int_{a(x)}^{a(d)} k(x,y) f(y) v(y) dy, \quad 0 \leq c \leq x \leq d \leq \infty,$$

with a differentiable and strictly increasing on $[c, d]$ function $a(x) \geq 0$ and a non-negative kernel $k(x, y)$ satisfying the condition: there exists a constant $D \geq 1$ such that

(2.9)

$$D^{-1}k(x, y) \leq k(x, a(z)) + k(z, y) \leq Dk(x, y), \quad \begin{cases} 0 \leq c \leq x \leq z \leq d \leq \infty, \\ 0 \leq a(z) \leq y \leq a(d). \end{cases}$$

Corollary 2.2. *Let $1 < q < p < \infty$, $r = pq/(p - q)$ and the operator K_a be defined by (2.8) with a differentiable and strictly increasing on $[c, d]$ function $a(x) \geq 0$ and a non-negative kernel $k(x, y) \in (2.9)$. Then $\|K_a\|_{L_p(a(c), a(d)) \rightarrow L_q(c, d)} \approx B_{a,0} + B_{a,1}$, where*

$$B_{a,0} := \left(\int_{a(c)}^{a(d)} \left[\int_c^{a^{-1}(t)} k^q(x,t) w^q(x) dx \right]^{\frac{r}{q}} \left[\int_t^{a(d)} v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}},$$

$$B_{a,1} := \left(\int_c^d \left[\int_c^t w^q(x) dx \right]^{\frac{r}{p}} \left[\int_{a(t)}^{a(d)} k^{p'}(t,y) v^{p'}(y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}}.$$

Now we can state the next auxiliary result of the paper.

Lemma 2.2. *Let $1 < q < p < \infty$. Suppose $b(x) \geq 0$ is differentiable and strictly increasing on $[c, d]$, and a satisfying (2.4) non-negative function $k(x, y)$ is continuous on $\mathcal{R}_b := \{(x, y) : x > 0, b(c) < y < b(x)\}$. Then we*

have

$$\begin{aligned}
 B_{b,0} \ll \mathcal{B}_{b,0} &:= \left(\int_{b(c)}^{b(d)} \left[\int_t^{b(d)} \left\{ \int_{b^{-1}(y)}^d k^q(x, y) w^q(x) dx \right\}^{p'} v^{p'}(y) dy \right]^{\frac{r}{q'}} \times \right. \\
 (2.10) \quad &\quad \times \left. \left[\int_{b^{-1}(t)}^d k^q(x, t) w^q(x) dx \right]^{p' - \frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}},
 \end{aligned}$$

$$\begin{aligned}
 B_{b,1} \ll \mathbb{B}_{b,1} &:= \left(\int_c^d \left[\int_c^t \left\{ \int_{b(c)}^{b(x)} k^{p'}(x, y) v^{p'}(y) dy \right\}^q w^q(x) dx \right]^{\frac{r}{p}} \times \right. \\
 (2.11) \quad &\quad \times \left. \left[\int_{b(c)}^{b(t)} k^{p'}(t, y) v^{p'}(y) dy \right]^{q - \frac{r}{p}} w^q(t) dt \right)^{\frac{1}{r}},
 \end{aligned}$$

where $B_{b,0}$ and $B_{b,1}$ are defined by (2.6) and (2.7) respectively.

For the proof of Lemma 2.2, we refer first to the paper [5].

Definition 2.1. ([5, Definition 2.2(a)]) A nonnegative sequence $\{a_k\}_{k \in \mathbb{Z}}$ is said to be *strongly increasing* (*strongly decreasing*), denoted by $a_k \uparrow\uparrow$ ($a_k \downarrow\downarrow$), if

$$\inf_{k \in \mathbb{Z}} \frac{a_{k+1}}{a_k} > 1 \quad \left(\sup_{k \in \mathbb{Z}} \frac{a_{k+1}}{a_k} < 1 \right).$$

Proposition 2.1. ([5, Proposition 2.1]) Let $\{a_k\}_{k \in \mathbb{Z}}$, $\{\sigma_k\}_{k \in \mathbb{Z}}$ and $\{\tau_k\}_{k \in \mathbb{Z}}$ be nonnegative sequences and $0 < p < \infty$.

- (a) If $\sigma_k \uparrow\uparrow$, then $\left(\sum_{k \in \mathbb{Z}} \left[\sum_{m \geq k} a_m \right]^p \sigma_k^p \right)^{\frac{1}{p}} \ll \left(\sum_{m \in \mathbb{Z}} [a_m \sigma_m]^p \right)^{\frac{1}{p}}.$
- (b) If $\tau_k \downarrow\downarrow$, then $\left(\sum_{k \in \mathbb{Z}} \left[\sum_{m \leq k} a_m \right]^p \tau_k^p \right)^{\frac{1}{p}} \ll \left(\sum_{m \in \mathbb{Z}} [a_m \tau_m]^p \right)^{\frac{1}{p}}.$

Proof. [Proof of Lemma 2.2.] By the substitution $\tau = b^{-1}(t)$ we reduce $B_{b,0}$, $B_{b,1}$ to the constants B_0 and B_1 respectively. Analogously we arrive to $\mathcal{B}_0 := \mathcal{B}_{b(x)=x,0}$ and $\mathbb{B}_1 := \mathbb{B}_{b(x)=x,1}$. Now it is sufficient to prove (2.10), (2.11) for the case $b(x) = x$ only. Note that under this condition the kernel $k(x, t) = k(x, b(\tau)) =: k_b(x, \tau)$ becomes satisfying (2.2). Moreover, without loss of generality we can assume that $k_b(x, \tau)$ is non-decreasing with respect to the variable x and non-increasing with respect to y . Otherwise we can consider the kernel $\bar{k}_b(x, \tau) = \sup_{\tau \leq s \leq x} \tilde{k}_b(s, \tau)$, where $\tilde{k}_b(s, \tau) = \sup_{\tau \leq t \leq s} k_b(s, t)$, which satisfies the pointed monotonicity properties and $k_b(x, \tau) \leq \bar{k}_b(x, \tau) \leq D^2 k_b(x, \tau)$ (for details see [7, Lemma 3]).

For the proof of (2.10) we put

$$t_k = \inf \left\{ t: \int_t^d k^q(x, t) w^q(x) dx \leq 2^{-k} \right\}, \quad k \in \mathbb{Z},$$

and denote $\mathbb{Z}_1 := \{k \in \mathbb{Z}: t_k < t_{k+1}\}$. Observe that the function $W_q(t) := \int_t^d k^q(x, t) w^q(x) dx$ is continuous, non-increasing and such that

$$(2.12) \quad 2^{-k-1} \leq W_q(t) \leq 2^{-k}, \quad t_k \leq t \leq t_{k+1}.$$

Put $V(t) := \int_c^t v^{p'}(y) dy$. We have

$$\begin{aligned} B_0^r &= \frac{p'}{r} \sum_{k \in \mathbb{Z}_1} \int_{t_k}^{t_{k+1}} [W_q(t)]^{\frac{r}{q}} d[V(t)]^{\frac{r}{p'}} \ll \sum_{k \in \mathbb{Z}_1} 2^{-k \frac{r}{q}} \int_{t_k}^{t_{k+1}} d[V(t)]^{\frac{r}{p'}} \\ &\leq \sum_{k \in \mathbb{Z}_1} 2^{-k \frac{r}{q}} [V(t_{k+1})]^{\frac{r}{p'}} = \sum_{k \in \mathbb{Z}_1} 2^{-k \frac{r}{q}} \left(\sum_{m \leq k} \int_{t_m}^{t_{m+1}} v^{p'}(y) dy \right)^{\frac{r}{p'}}. \end{aligned}$$

Applying Proposition 2.1(b) we find that

$$\begin{aligned} B_0^r &\ll \sum_{k \in \mathbb{Z}_1} 2^{-k \frac{r}{q}} \left(\int_{t_k}^{t_{k+1}} v^{p'}(y) dy \right)^{\frac{r}{p'}} \\ (2.12) \quad &\stackrel{\approx}{\sim} \sum_{k \in \mathbb{Z}_1} 2^{k \frac{r}{q'}} \left(\int_{t_k}^{t_{k+1}} v^{p'}(y) dy \right)^{\frac{r}{p'}} [W_q(t_{k+1})]^r \\ &\leq \sum_{k \in \mathbb{Z}_1} 2^{k \frac{r}{q'}} \left(\int_{t_k}^{t_{k+1}} [W_q(y)]^{p'} v^{p'}(y) dy \right)^{\frac{r}{p'}} \\ &\approx \sum_{k \in \mathbb{Z}_1} [W_q(t_k)]^{-\frac{r}{q'}} \int_{t_k}^{t_{k+1}} \left(\int_t^{t_{k+1}} [W_q(y)]^{p'} v^{p'}(y) dy \right)^{\frac{r}{q'}} [W_q(t)]^{p'} v^{p'}(t) dt \\ &\leq \sum_{k \in \mathbb{Z}_1} \int_{t_k}^{t_{k+1}} \left(\int_t^d [W_q(y)]^{p'} v^{p'}(y) dy \right)^{\frac{r}{q'}} [W_q(t)]^{p' - \frac{r}{q'}} v^{p'}(t) dt = B_0^r. \end{aligned}$$

Analogously, to prove (2.11) we put

$$t_l = \sup \left\{ t: \int_c^t k^{p'}(t, y) v^{p'}(y) dy \leq 2^l \right\}, \quad l \in \mathbb{Z},$$

and $\mathbb{Z}_2 := \{l \in \mathbb{Z}: t_l < t_{l+1}\}$. Note that $V_{p'}(t) := \int_c^t k^{p'}(t, y) v^{p'}(y) dy$ is continuous, non-decreasing and such that

$$(2.13) \quad 2^l \leq V_{p'}(t) \leq 2^{l+1}, \quad t_l \leq t \leq t_{l+1}.$$

Put $W(t) := \int_t^d w^q(x)dx$. Then

$$\begin{aligned} B_1^r &= \frac{q}{r} \sum_{l \in \mathbb{Z}_2} \int_{t_l}^{t_{l+1}} [V_{p'}(t)]^{\frac{r}{p'}} d \left(-[W(t)]^{\frac{r}{q}} \right) \ll \sum_{l \in \mathbb{Z}_2} 2^{l \frac{r}{p'}} \int_{t_l}^{t_{l+1}} d \left(-[W(t)]^{\frac{r}{q}} \right) \\ &\leq \sum_{l \in \mathbb{Z}_2} 2^{l \frac{r}{p'}} [W(t_l)]^{\frac{r}{q}} = \sum_{k \in \mathbb{Z}_2} 2^{l \frac{r}{p'}} \left(\sum_{m \geq l} \int_{t_m}^{t_{m+1}} w^q(x)dx \right)^{\frac{r}{q}}. \end{aligned}$$

Applying Proposition 2.1(a) we obtain

$$\begin{aligned} B_1^r &\ll \sum_{l \in \mathbb{Z}_2} 2^{l \frac{r}{p'}} \left(\int_{t_l}^{t_{l+1}} w^q(x)dx \right)^{\frac{r}{q}} \stackrel{(2.13)}{\simeq} \sum_{l \in \mathbb{Z}_2} 2^{-l \frac{r}{p}} \left(\int_{t_l}^{t_{l+1}} w^q(x)dx \right)^{\frac{r}{q}} [V_{p'}(t_l)]^r \\ &\leq \sum_{l \in \mathbb{Z}_2} 2^{-l \frac{r}{p}} \left(\int_{t_l}^{t_{l+1}} [V_{p'}(x)]^q w^q(x)dx \right)^{\frac{r}{q}} \\ &\approx \sum_{l \in \mathbb{Z}_2} V_{p'}(t_{l+1})^{-\frac{r}{p}} \int_{t_l}^{t_{l+1}} \left(\int_{t_l}^t [V_{p'}(x)]^q w^q(x)dx \right)^{\frac{r}{p}} [V_{p'}(t)]^q w^q(t)dt \\ &\leq \sum_{l \in \mathbb{Z}_2} \int_{t_l}^{t_{l+1}} \left(\int_c^t [V_{p'}(x)]^q w^q(x)dx \right)^{\frac{r}{p}} [V_{p'}(t)]^{q-\frac{r}{p}} w^q(t)dt = \mathbb{B}_1^r. \end{aligned}$$

□

Analogous result is true for the constants $B_{a,0}$ and $B_{a,1}$.

Lemma 2.3. *Let $1 < q < p < \infty$. Suppose that a function $a(x) \geq 0$ is differentiable and strictly increasing on $[c, d]$, and a continuous function $k(x, y) \geq 0$ on $\mathcal{R}_a := \{(x, y) : x > 0, a(x) < y < a(d)\}$ is from the class (2.9). Then*

$$\begin{aligned} B_{a,0} &\ll \mathcal{B}_{a,0} := \left(\int_{a(c)}^{a(d)} \left[\int_{a(c)}^t \left\{ \int_c^{a^{-1}(y)} k^q(x, y) w^q(x) dx \right\}^{p'} v^{p'}(y) dy \right]^{\frac{r}{q'}} \times \right. \\ &\quad \left. \times \left[\int_c^{a^{-1}(t)} k^q(x, t) w^q(x) dx \right]^{p' - \frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}}, \end{aligned}$$

$$B_{a,1} \ll \mathbb{B}_{a,1} := \left(\int_c^d \left[\int_t^d \left\{ \int_{a(x)}^{a(d)} k^{p'}(x,y) v^{p'}(y) dy \right\}^q w^q(x) dx \right]^{\frac{r}{p}} \times \right. \\ \left. \times \left[\int_{a(t)}^{a(d)} k^{p'}(t,y) v^{p'}(y) dy \right]^{q-\frac{r}{p}} w(t) dt \right)^{\frac{1}{r}},$$

where $B_{a,0}$, $B_{a,1}$ are defined by (2.10) and (2.10) respectively.

We conclude this section by the following two statements.

Lemma 2.4. [11, Lemma 1] *Let $0 < q < p < \infty$, $p > 1$, $1/r = 1/q - 1/p$ and $0 \leq c < d \leq \infty$, $0 \leq a < \infty$. Suppose that the function $b(x)$ is differentiable, strictly increasing and such that $a \leq b(x) < \infty$, $x \in [c, d]$. Let the operator S be defined by $Sf(x) := w(x) \int_a^{b(x)} f(y) v(y) dy$. Then*

(2.14)

$$\|S\|_{L_p(a,b(d)) \rightarrow L_q(c,d)}^r \approx \int_c^d \left(\int_t^d w^q(x) dx \right)^{\frac{r}{p}} \left(\int_a^{b(t)} v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt,$$

(2.15)

$$\approx \int_c^d \left(\int_c^t \left[\int_a^{b(x)} v^{p'}(y) dy \right]^q w^q(x) dx \right)^{\frac{r}{p}} \left(\int_a^{b(t)} v^{p'}(y) dy \right)^{q-\frac{r}{p}} w^q(t) dt.$$

Lemma 2.5. [11, Lemma 2] *Let $0 < q < p < \infty$, $p > 1$, $1/r = 1/q - 1/p$ and $0 \leq c < d \leq \infty$, $0 < b \leq \infty$. Suppose that the function $a(x)$ is differentiable, strictly increasing and such that $0 < a(x) \leq b$, $x \in (c, d]$. Let the operator T be defined by $Tf(x) := w(x) \int_{a(x)}^b f(y) v(y) dy$. Then*

(2.16)

$$\|T\|_{L_p(a(c),b) \rightarrow L_q(c,d)}^r \approx \int_c^d \left(\int_c^t w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{a(t)}^b v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt,$$

(2.17)

$$\approx \int_c^d \left(\int_t^d \left[\int_{a(x)}^b v^{p'}(y) dy \right]^q w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{a(t)}^b v^{p'}(y) dy \right)^{q-\frac{r}{p}} w^q(t) dt.$$

3. The main result

Let functions $\sigma(x)$ and $\rho(y)$ on $\mathbb{R}^+ \cup \{+\infty\}$ be such that $a(x) \leq \sigma(x) \leq b(x)$, $b^{-1}(y) \leq \rho(y) \leq a^{-1}(y)$ and be *fairway*-functions satisfying the following

Definition 3.1. Given boundary functions $a(x)$ and $b(x)$ satisfying the conditions (1.2), numbers $p, q \in (1, \infty)$, a continuous kernel $0 < k(x, y) < \infty$ a.e on $\mathcal{R} = \{(x, y) : x > 0, a(x) < y < b(x)\}$ and weight functions $0 < v, w < \infty$ a.e. on $\mathbb{R}^+$ such that for any fixed $x > 0$ the function $k^{p'}(x, y)v^{p'}(y)$ is locally integrable on $\mathbb{R}^+$ with respect to the variable y as well as for any $y > 0$ the function $k^q(x, y)w^q(x)$ is locally integrable on $\mathbb{R}^+$ with respect to x , we define two *fairways* – the functions $\sigma(x)$ and $\rho(y)$ such that $a(x) < \sigma(x) < b(x)$, $b^{-1}(y) < \rho(y) < a^{-1}(y)$ and

$$(3.1) \quad \int_{a(x)}^{\sigma(x)} k^{p'}(x, y)v^{p'}(y)dy = \int_{\sigma(x)}^{b(x)} k^{p'}(x, y)v^{p'}(y)dy \quad \text{for all } x > 0,$$

$$(3.2) \quad \int_{b^{-1}(y)}^{\rho(y)} k^q(x, y)w^q(x)dx = \int_{\rho(y)}^{a^{-1}(y)} k^q(x, y)w^q(x)dx \quad \text{for all } y > 0.$$

By assumptions of the definition the fairways $\sigma(x)$ and $\rho(y)$ are continuous functions. Put

$$\Theta(t) := \Theta^-(t) \cup \Theta^+(t), \quad \Theta^-(t) := [b^{-1}(t), \rho(t)), \quad \Theta^+(t) := [\rho(t), a^{-1}(t)),$$

$$\vartheta(t) := \vartheta^-(t) \cup \vartheta^+(t), \quad \vartheta^-(t) := [a(\rho(t)), t), \quad \vartheta^+(t) := [t, b(\rho(t))),$$

$$\delta(t) := \delta^-(t) \cup \delta^+(t), \quad \delta^-(t) := [b^{-1}(\sigma(t)), t), \quad \delta^+(t) := [t, a^{-1}(\sigma(t))),$$

$$\Delta(t) := \Delta^-(t) \cup \Delta^+(t), \quad \Delta^-(t) := [a(t), \sigma(t)), \quad \Delta^+(t) := [\sigma(t), b(t))$$

and denote

$$\begin{aligned} \mathcal{A}_\rho^\pm &:= \sup_{t>0} \mathcal{A}_\rho^\pm(t) = \sup_{t>0} \left(\int_{\Theta(t)} k^q(x, t)w^q(x)dx \right)^{\frac{1}{q}} \left(\int_{\vartheta^\pm(t)} v^{p'}(y)dy \right)^{\frac{1}{p'}}, \\ \mathcal{A}_\sigma^\pm &:= \sup_{t>0} \mathcal{A}_\sigma^\pm(t) = \sup_{t>0} \left(\int_{\delta^\pm(t)} w^q(x)dx \right)^{\frac{1}{q}} \left(\int_{\Delta(t)} k^{p'}(t, y)v^{p'}(y)dy \right)^{\frac{1}{p'}}, \end{aligned}$$

$$\begin{aligned}
\mathcal{B}_\rho^\pm &:= \left(\int_0^\infty \mathcal{B}_\rho^\pm(t) dt \right)^{\frac{1}{r}} = \left(\int_0^\infty \left[\int_{\Theta(t)} k^q(x, t) w^q(x) dx \right]^{\frac{r}{q}} \times \right. \\
&\quad \left. \times \left[\int_{\vartheta^\pm(t)} v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}}, \\
\mathcal{B}_\sigma^\pm &:= \left(\int_0^\infty \mathcal{B}_\sigma^\pm(t) dt \right)^{\frac{1}{r}} = \left(\int_0^\infty \left[\int_{\delta^\pm(t)} w^q(x) dx \right]^{\frac{r}{p}} \times \right. \\
&\quad \left. \times \left[\int_{\Delta(t)} k^{p'}(t, y) v^{p'}(y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}}, \\
\mathcal{A}_\rho &:= \sup_{t>0} \mathcal{A}_\rho(t) = \sup_{t>0} \left(\int_{\Theta(t)} k^q(x, t) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{\vartheta(t)} v^{p'}(y) dy \right)^{\frac{1}{p'}}, \\
\mathcal{A}_\sigma &:= \sup_{t>0} \mathcal{A}_\sigma(t) = \sup_{t>0} \left(\int_{\delta(t)} w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{\Delta(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{1}{p'}}, \\
\mathcal{B}_\rho &:= \left(\int_0^\infty \mathcal{B}_\rho(t) dt \right)^{\frac{1}{r}} = \left(\int_0^\infty \left[\int_{\Theta(t)} k^q(x, t) w^q(x) dx \right]^{\frac{r}{q}} \times \right. \\
&\quad \left. \times \left[\int_{\vartheta(t)} v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}}, \\
\mathcal{B}_\sigma &:= \left(\int_0^\infty \mathcal{B}_\sigma(t) dt \right)^{\frac{1}{r}} = \left(\int_0^\infty \left[\int_{\delta(t)} w^q(x) dx \right]^{\frac{r}{p}} \times \right. \\
&\quad \left. \times \left[\int_{\Delta(t)} k^{p'}(t, y) v^{p'}(y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}},
\end{aligned}$$

where $r = pq/(p - q)$. The main result of the paper is proved in Section 4 and reads

Theorem 3.1. *Let the operator $\mathcal{K}$ be defined by (1.1) with the border functions $a(x)$, $b(x)$ satisfying (1.2) and a continuous positive kernel $k(x, y)$ on $\mathcal{R}$ from the Oinarov's type class $\mathcal{O}_b$. Suppose the functions $\rho(y)$, $\sigma(x)$ on $\mathbb{R}^+$ are strictly increasing fairways from Definition 3.1.*

(a) If $1 < p \leq q < \infty$, then

$$(3.3) \quad \alpha_3(p, q) [\mathcal{A}_\rho^- + \mathcal{A}_\sigma^+] \leq \|\mathcal{K}\|_{L_p \rightarrow L_q} \leq \alpha_4(p, q) [\mathcal{A}_\rho + \mathcal{A}_\sigma].$$

Moreover, if $\mathcal{K}$ is compact then $\mathcal{A}_\rho^-, \mathcal{A}_\sigma^+ < \infty$, $\lim_{t \rightarrow 0} \mathcal{A}_\rho^-(t) = \lim_{t \rightarrow \infty} \mathcal{A}_\sigma^+(t) = 0$ and $\mathcal{K}$ is compact if $\mathcal{A}_\rho, \mathcal{A}_\sigma < \infty$, $\lim_{t \rightarrow 0} \mathcal{A}_\rho(t) = \lim_{t \rightarrow \infty} \mathcal{A}_\sigma(t) = 0$.

(b) If $1 < q < p < \infty$, then

$$(3.4) \quad \beta_3(p, q) [\mathcal{B}_\rho^- + \mathcal{B}_\sigma^+] \leq \|\mathcal{K}\|_{L_p \rightarrow L_q} \leq \beta_4(p, q) [\mathcal{B}_\rho + \mathcal{B}_\sigma].$$

Moreover, the operator $\mathcal{K}$ is compact if $\mathcal{B}_\rho, \mathcal{B}_\sigma < \infty$ and if $\mathcal{K}$ is compact then $\mathcal{B}_\rho^-, \mathcal{B}_\sigma^+ < \infty$.

Analogously we obtain a similar result for $\mathcal{K}$ with $k(x, y)$ satisfying the condition (1.4):

Theorem 3.2. Let the operator $\mathcal{K}$ be defined by (1.1) with the border functions $a(x)$, $b(x)$ satisfying (1.2) and a continuous kernel $k(x, y) > 0$ on $\mathcal{R}$ from the Oinarov's type class $\mathcal{O}_a$. Suppose that the functions $\rho(y)$, $\sigma(x)$ on $\mathbb{R}^+$ are strictly increasing fairways from Definition 3.1.

(a) If $1 < p \leq q < \infty$, then

$$\alpha_5(p, q) [\mathcal{A}_\rho^+ + \mathcal{A}_\sigma^-] \leq \|\mathcal{K}\|_{L_p \rightarrow L_q} \leq \alpha_6(p, q) [\mathcal{A}_\rho + \mathcal{A}_\sigma].$$

Moreover, if $\mathcal{K}$ is compact then $\mathcal{A}_\rho^+, \mathcal{A}_\sigma^- < \infty$, $\lim_{t \rightarrow 0} \mathcal{A}_\rho^+(t) = \lim_{t \rightarrow \infty} \mathcal{A}_\sigma^-(t) = 0$ and $\mathcal{K}$ is compact if $\mathcal{A}_\rho, \mathcal{A}_\sigma < \infty$, $\lim_{t \rightarrow 0} \mathcal{A}_\rho(t) = \lim_{t \rightarrow \infty} \mathcal{A}_\sigma(t) = 0$.

(b) If $1 < q < p < \infty$, then

$$\beta_5(p, q) [\mathcal{B}_\rho^+ + \mathcal{B}_\sigma^-] \leq \|\mathcal{K}\|_{L_p \rightarrow L_q} \leq \beta_6(p, q) [\mathcal{B}_\rho + \mathcal{B}_\sigma].$$

Moreover, the operator $\mathcal{K}$ is compact if $\mathcal{B}_\rho, \mathcal{B}_\sigma < \infty$ and if $\mathcal{K}$ is compact then $\mathcal{B}_\rho^+, \mathcal{B}_\sigma^- < \infty$.

4. Proof of Theorem 3.1

(a) *The lower estimate.* Let $1 < p \leq q < \infty$. It follows from (1.5) of Theorem 1.1 that

$$(4.1) \quad \|\mathcal{K}\|_{L_p \rightarrow L_q} \approx \sup_{t > 0} \sup_{b^{-1}(a(t)) \leq s \leq t} [\mathcal{A}_0(s, t) + \mathcal{A}_1(s, t)],$$

where

$$\begin{aligned}\mathcal{A}_0(s, t) &:= \left(\int_s^t k^q(x, b(s)) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} v^{p'}(y) dy \right)^{\frac{1}{p'}}, \\ \mathcal{A}_1(s, t) &:= \left(\int_s^t w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} k^{p'}(s, y) v^{p'}(y) dy \right)^{\frac{1}{p'}}.\end{aligned}$$

Using (3.2) we find that

$$\begin{aligned}\mathcal{A}_\rho^-(t) &= 2^{1/q} \left(\int_{\Theta^-(t)} k^q(x, t) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{\vartheta^-(t)} v^{p'}(y) dy \right)^{\frac{1}{p'}} \\ &= 2^{1/q} \left(\int_{b^{-1}(\rho^{-1}(s))}^s k^q(x, \rho^{-1}(s)) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(s)}^{\rho^{-1}(s)} v^{p'}(y) dy \right)^{\frac{1}{p'}} \\ &= 2^{1/q} \mathcal{A}_0(b^{-1}(\rho^{-1}(s)), s) \leq \sup_{t>0} \sup_{b^{-1}(a(t)) \leq s \leq t} \mathcal{A}_0(s, t).\end{aligned}$$

On the strength of (4.1) it implies $\mathcal{A}_\rho^- \ll \|\mathcal{K}\|_{L_p \rightarrow L_q}$. Analogously,

$$\begin{aligned}\mathcal{A}_\sigma^+(t) &\stackrel{(3.1)}{=} 2^{1/p'} \left(\int_{\delta^+(t)} w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{\Delta^+(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{1}{p'}} \\ &= 2^{1/p'} \mathcal{A}_1(t, a^{-1}(\sigma(t))) \leq \sup_{t>0} \sup_{b^{-1}(a(t)) \leq s \leq t} \mathcal{A}_1(s, t)\end{aligned}$$

implies $\mathcal{A}_\sigma^+ \ll \|\mathcal{K}\|_{L_p \rightarrow L_q}$, and the lower estimate in (3.3) is proved.

The upper bound. For the opposite estimate we put $\tau_0 := \rho(a(t))$ and write

$$\begin{aligned}\sup_{b^{-1}(a(t)) \leq s \leq t} \mathcal{A}_0(s, t) &\leq \sup_{b^{-1}(a(t)) \leq s \leq \rho(a(t)) < t} \mathcal{A}_0(s, t) + \sup_{\rho(a(t)) \leq s \leq t} \mathcal{A}_0(s, t) \\ &\leq \sup_{b^{-1}(\rho^{-1}(\tau_0)) \leq s \leq \tau_0} \left(\int_s^{a^{-1}(\rho^{-1}(\tau_0))} k^q(x, b(s)) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{\rho^{-1}(\tau_0)}^{b(s)} v^{p'}(y) dy \right)^{\frac{1}{p'}} \\ &\quad + \sup_{\rho(a(t)) \leq s \leq t} \left(\int_s^{a^{-1}(\rho^{-1}(s))} k^q(x, b(s)) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(s)}^{b(s)} v^{p'}(y) dy \right)^{\frac{1}{p'}} \\ &=: H_1(\tau_0) + H_2(t).\end{aligned}$$

Indeed, if $b^{-1}(a(t)) \leq s \leq \rho(a(t)) < t$ that is $b^{-1}(\rho^{-1}(\tau_0)) \leq s \leq \tau_0$ then $(s, t) = (s, a^{-1}(\rho^{-1}(\tau_0)))$ and $(a(t), b(s)) = (\rho^{-1}(\tau_0), b(s))$. If $\rho(a(t)) \leq s \leq t$, then $(s, t) \subset (s, a^{-1}(\rho^{-1}(s)))$ and $(a(t), b(s)) \subset (a(s), b(s))$.

To estimate $H_1(\tau_0)$ we use (1.3) with $y = \rho^{-1}(\tau_0)$ and $z = s$, that is since $s \leq x$ and $a(x) \leq \rho^{-1}(\tau_0) \leq b(s)$ then $k(x, b(s)) \ll k(x, \rho^{-1}(\tau_0))$. Therefore,

$$\begin{aligned} H_1(\tau_0) &\ll \left(\int_{b^{-1}(\rho^{-1}(\tau_0))}^{a^{-1}(\rho^{-1}(\tau_0))} k^q(x, \rho^{-1}(\tau_0)) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{\rho^{-1}(\tau_0)}^{b(\tau_0)} v^{p'}(y) dy \right)^{\frac{1}{p'}} \\ &= \left(\int_{b^{-1}(z)}^{a^{-1}(z)} k^q(x, z) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_z^{b(\rho(z))} v^{p'}(y) dy \right)^{\frac{1}{p'}} = \mathcal{A}_\rho^+(z) \leq \mathcal{A}_\rho^+. \end{aligned}$$

Since $s \leq x$ and $a(x) \leq \rho^{-1}(s) \leq b(s)$ in $H_2(t)$ we obtain by using (1.3) with $z = s$ and $y = \rho^{-1}(s)$

$$\begin{aligned} H_2(t) &\ll \sup_{\rho(a(t)) \leq s \leq t} \left(\int_s^{a^{-1}(\rho^{-1}(s))} k^q(x, \rho^{-1}(s)) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(s)}^{b(s)} v^{p'}(y) dy \right)^{\frac{1}{p'}} \\ &= \sup_{a(t) \leq z \leq \rho^{-1}(t)} \left(\int_{\rho(z)}^{a^{-1}(z)} k^q(x, z) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(\rho(z))}^{b(\rho(z))} v^{p'}(y) dy \right)^{\frac{1}{p'}} \leq \mathcal{A}_\rho. \end{aligned}$$

Thus,

$$(4.2) \quad \sup_{t>0} \sup_{b^{-1}(a(t)) \leq s \leq t} \mathcal{A}_0(s, t) \ll \mathcal{A}_\rho.$$

Analogously, we put $\tau_1 := \sigma^{-1}(a(t))$ and write

$$\begin{aligned} \sup_{b^{-1}(a(t)) \leq s \leq t} \mathcal{A}_1(s, t) &\leq \sup_{b^{-1}(a(t)) \leq s \leq \sigma^{-1}(a(t)) < t} \mathcal{A}_1(s, t) + \sup_{\sigma^{-1}(a(t)) \leq s \leq t} \mathcal{A}_1(s, t) \\ &\leq \sup_{b^{-1}(\sigma(\tau_1)) \leq s \leq \tau_1} \left(\int_s^{a^{-1}(\sigma(\tau_1))} w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{\sigma(\tau_1)}^{b(s)} k^{p'}(s, y) v^{p'}(y) dy \right)^{\frac{1}{p'}} \\ &\quad + \sup_{\sigma^{-1}(a(t)) \leq s \leq t} \left(\int_s^{a^{-1}(\sigma(s))} w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{a(s)}^{b(s)} k^{p'}(s, y) v^{p'}(y) dy \right)^{\frac{1}{p'}} \\ &=: H_3(\tau_1) + H_4(t). \end{aligned}$$

Obviously, $H_4(t) \leq \sup_{\sigma^{-1}(a(t)) \leq s \leq t} \mathcal{A}_\sigma^+(s) \leq \mathcal{A}_\sigma^+$. For $H_3(\tau_1)$ we apply (1.3) with $z = s \leq \tau_1 = x$ and $a(\tau_1) < \sigma(\tau_1) \leq y \leq b(s)$:

$$\begin{aligned} H_3(\tau_1) &\ll \sup_{b^{-1}(\sigma(\tau_1)) \leq s \leq \tau_1} \left(\int_s^{a^{-1}(\sigma(\tau_1))} w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{\sigma(\tau_1)}^{b(s)} k^{p'}(\tau_1, y) v^{p'}(y) dy \right)^{\frac{1}{p'}} \\ &\leq \mathcal{A}_\sigma(\tau_1) \leq \mathcal{A}_\sigma. \end{aligned}$$

Thus,

$$(4.3) \quad \sup_{t>0} \sup_{b^{-1}(a(t)) \leq s \leq t} \mathcal{A}_1(s, t) \ll \mathcal{A}_\sigma.$$

Combining (4.1), (4.2) and (4.3), we obtain the upper estimate in (3.3). Necessary and sufficient compactness conditions for $1 < p \leq q < \infty$ follow from Theorem 1.1.

(b) Now we consider the case $1 < q < p < \infty$. We prove first *the upper estimate* in (3.4). To this end we take a point sequence $\{\xi_k\}_{k \in \mathbb{Z}} \subset (0, \infty)$ such that $\xi_0 = 1, \xi_k = (a^{-1} \circ b)^k(1), k \in \mathbb{Z}$, and put

$$\eta_k = a(\xi_k) = b(\xi_{k-1}), \quad \Delta_k = [\xi_k, \xi_{k+1}), \quad \delta_k = [\eta_k, \eta_{k+1}), \quad k \in \mathbb{Z}.$$

Breaking the semiaxis $(0, \infty)$ by points $\{\xi_k\}_{k \in \mathbb{Z}}$ we decompose the operator $\mathcal{K}$ into the sum

$$(4.4) \quad \mathcal{K} = \mathcal{T} + \mathcal{S}$$

of block-diagonal operators $\mathcal{T}$ and $\mathcal{S}$ such that

$$(4.5) \quad \mathcal{T} = \sum_{k \in \mathbb{Z}} T_k, \quad \mathcal{S} = \sum_{k \in \mathbb{Z}} S_k,$$

where

$$T_k f(x) = w(x) \int_{a(x)}^{a(\xi_{k+1})} k(x, y) f(y) v(y) dy, \quad T_k : L_p(\delta_k) \rightarrow L_q(\Delta_k),$$

$$S_k f(x) = w(x) \int_{b(\xi_k)}^{b(x)} k(x, y) f(y) v(y) dy, \quad S_k : L_p(\delta_{k+1}) \rightarrow L_q(\Delta_k).$$

Kernels $k(x, y)$ of the operators T_k and S_k satisfy the condition (1.3) for $z \leq x, x \in [\xi_k, \xi_{k+1}]$ and

$$(4.6) \quad a(x) \leq y \leq b(\xi_k), \quad b(\xi_k) \leq y \leq b(z),$$

respectively. Since $\sqcup_{k \in \mathbb{Z}} [\xi_k, \xi_{k+1}) = (0, \infty)$ it follows from (4.4) – (4.5) that

$$(4.7) \quad \|\mathcal{K}f\|_q^q = \sum_{k \in \mathbb{Z}} \|\mathcal{K}f\|_{L_q(\Delta_k)}^q \approx \sum_{k \in \mathbb{Z}} \|T_k f\|_{L_q(\Delta_k)}^q + \sum_{k \in \mathbb{Z}} \|S_k f\|_{L_q(\Delta_k)}^q.$$

To estimate a norm of the operator S_k we take into account two key points $s_\rho := b^{-1}(\rho^{-1}(\xi_{k+1}))$, $s_\sigma := \sigma^{-1}(b(\xi_k)) = \sigma^{-1}(a(\xi_{k+1}))$ and consider three

only possible variants:

$$(i) \quad s_\rho < s_\sigma, \quad (ii) \quad s_\rho = s_\sigma, \quad (iii) \quad s_\rho > s_\sigma.$$

FIGURE 1. Case (i)

In the case (i) we have

$$(4.8) \quad S_k f(x) = \sum_{i=1}^3 S_{k,i} f(x) + \sum_{i=1}^3 H_{k,i} f(x),$$

where

$$\begin{aligned} S_{k,1} f(x) &= \chi_{[\xi_k, s_\rho]}(x) S_k f(x), \quad S_{k,1} : L_p(b(\xi_k), b(s_\rho)) \rightarrow L_q(\xi_k, s_\rho), \\ H_{k,1} f(x) &= \chi_{[s_\rho, s_\sigma]}(x) S_k (f \chi_{[b(\xi_k), b(s_\rho)]}) (x), \quad H_{k,1} : L_p(b(\xi_k), b(s_\rho)) \rightarrow L_q(s_\rho, s_\sigma), \\ S_{k,2} f(x) &= \chi_{[s_\rho, s_\sigma]}(x) S_k (f \chi_{[b(s_\rho), b(s_\sigma)]}) (x), \quad S_{k,2} : L_p(b(s_\rho), b(s_\sigma)) \rightarrow L_q(s_\rho, s_\sigma), \\ H_{k,2} f(x) &= \chi_{[s_\sigma, \xi_{k+1}]}(x) S_k (f \chi_{[b(\xi_k), b(s_\rho)]}) (x), \quad H_{k,2} : L_p(b(\xi_k), b(s_\rho)) \rightarrow L_q(s_\sigma, \xi_{k+1}), \\ H_{k,3} f(x) &= \chi_{[s_\sigma, \xi_{k+1}]}(x) S_k (f \chi_{[b(s_\rho), b(s_\sigma)]}) (x), \quad H_{k,3} : L_p(b(s_\rho), b(s_\sigma)) \rightarrow L_q(s_\sigma, \xi_{k+1}), \\ S_{k,3} f(x) &= \chi_{[s_\sigma, \xi_{k+1}]}(x) S_k (f \chi_{[b(s_\sigma), b(\xi_{k+1})]}) (x), \quad S_{k,3} : L_p(b(s_\sigma), b(\xi_{k+1})) \rightarrow L_q(s_\sigma, \xi_{k+1}). \end{aligned}$$

Applying Corollary 2.1 with $K_b = S_{k,1}$, $c = \xi_k$, $d = s_\rho$ and Lemma 2.2 we obtain

(4.9)

$$\begin{aligned}
\|S_{k,1}\|^r &:= \|S_{k,1}\|_{L_p(b(\xi_k), b(s_\rho)) \rightarrow L_q(\xi_k, s_\rho)}^r \\
&\approx B_{b,0}^r + B_{b,1}^r \stackrel{(2.11)}{\ll} B_{b,0}^r + \mathbb{B}_{b,1}^r \\
&= \int_{b(\xi_k)}^{b(s_\rho)} \left(\int_{b^{-1}(t)}^{s_\rho} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_{b(\xi_k)}^t v^{p'}(y) dy \right)^{\frac{r}{q'}} v^{p'}(t) dt \\
&\quad + \int_{\xi_k}^{s_\rho} \left(\int_{\xi_k}^t \left[\int_{b(\xi_k)}^{b(x)} k^{p'}(x, y) v^{p'}(y) dy \right]^q w^q(x) dx \right)^{\frac{r}{p}} \times \\
&\quad \times \left[\int_{b(\xi_k)}^{b(t)} k^{p'}(t, y) v^{p'}(y) dy \right]^{q - \frac{r}{p}} w^q(t) dt.
\end{aligned}$$

Since $t \geq b(\xi_k) > a(s_\rho)$ in $B_{b,0}$ then $[b^{-1}(t), s_\rho] \subset [b^{-1}(t), a^{-1}(t)]$, and $[b(\xi_k), t] \subseteq [a(\rho(t)), t]$ because of $t \leq b(s_\rho) = \rho^{-1}(\xi_{k+1}) \implies \rho(t) \leq \xi_{k+1} \implies a(\rho(t)) \leq a(\xi_{k+1}) = b(\xi_k)$. Therefore

$$\begin{aligned}
B_{b,0}^r &\leq \int_{b(\xi_k)}^{b(s_\rho)} \left(\int_{\Theta(t)} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_{\vartheta^-(t)} v^{p'}(y) dy \right)^{\frac{r}{q'}} v^{p'}(t) dt \\
(4.10) \quad &\leq \int_{\delta_{k+1}} \mathcal{B}_\rho^-(t) dt.
\end{aligned}$$

To estimate $\mathbb{B}_{b,1}$ note that in view of (1.3) we have $k(x, y) \ll k(t, y)$, where $x \leq t$, $a(t) < b(\xi_k) \leq y \leq b(x)$ and

$$\mathbb{B}_{b,1}^r \ll \int_{\xi_k}^{s_\rho} \left(\int_{\xi_k}^t w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{b(\xi_k)}^{b(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt.$$

Here $[\xi_k, t] \subset [b^{-1}(\sigma(t)), t]$ because $t \leq s_\rho < s_\sigma = \sigma^{-1}(b(\xi_k)) \implies \sigma(t) < b(\xi_k) \implies b^{-1}(\sigma(t)) < \xi_k$. Obviously, $[b(\xi_k), b(t)] \subset [a(t), b(t)]$. Therefore

$$\begin{aligned}
\mathbb{B}_{b,1}^r &\ll \int_{\xi_k}^{s_\rho} \left(\int_{\delta^-(t)} w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{\Delta(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt \\
(4.11) \quad &\leq \int_{\Delta_k} \mathcal{B}_\sigma^-(t) dt.
\end{aligned}$$

Next, we decompose the operator $H_{k,1}$ into a sum by using (1.3) with $z = s_\rho \leq x \leq s_\sigma$, $a(x) < b(\xi_k) \leq y \leq b(s_\rho) = b(z)$:

$$\begin{aligned} H_{k,1}f(x) &= w(x)k(x, b(s_\rho)) \int_{b(\xi_k)}^{b(s_\rho)} f(y)v(y)dy + w(x) \int_{b(\xi_k)}^{b(s_\rho)} k(s_\rho, y)f(y)v(y)dy \\ &=: H_{k,1}^w f(x) + H_{k,1}^v f(x). \end{aligned}$$

By Hölder's inequality and (1.3)

$$\begin{aligned} \|H_{k,1}^w\|^r &= \left(\int_{s_\rho}^{s_\sigma} k^q(x, b(s_\rho))w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{b(\xi_k)}^{b(s_\rho)} v^{p'}(y)dy \right)^{\frac{r}{p'}} \\ &\stackrel{(1.3)}{\ll} \int_{b(\xi_k)}^{b(s_\rho)} \left(\int_{b(\xi_k)}^t v^{p'}(y)dy \right)^{\frac{r}{q'}} \left(\int_{b^{-1}(t)}^{s_\sigma} k^q(x, t)w^q(x)dx \right)^{\frac{r}{q}} v^{p'}(t)dt \\ &< \int_{b(\xi_k)}^{b(s_\rho)} \left(\int_{\Theta(t)} k^q(x, t)w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{\vartheta^-(t)} v^{p'}(y)dy \right)^{\frac{r}{q'}} v^{p'}(t)dt \\ (4.12) \quad &\leq \int_{\delta_{k+1}} \mathcal{B}_\rho^-(t)dt \end{aligned}$$

because $a(s_\sigma) < t$ and $t \leq b(s_\rho) = \rho^{-1}(\xi_{k+1}) \implies \rho(t) \leq \xi_{k+1} \implies a(\rho(t)) \leq a(\xi_{k+1}) = b(\xi_k)$. Analogously,

$$\begin{aligned} \|H_{k,1}^v\|^r &= \left(\int_{s_\rho}^{s_\sigma} w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{b(\xi_k)}^{b(s_\rho)} k^{p'}(s_\rho, y)v^{p'}(y)dy \right)^{\frac{r}{p'}} \\ &\ll \int_{s_\rho}^{s_\sigma} \left(\int_{s_\rho}^t w^q(x)dx \right)^{\frac{r}{p}} \left(\int_{b(\xi_k)}^{b(t)} k^{p'}(t, y)v^{p'}(y)dy \right)^{\frac{r}{p'}} w^q(t)dt \\ &< \int_{s_\rho}^{s_\sigma} \left(\int_{\delta^-(t)} w^q(x)dx \right)^{\frac{r}{p}} \left(\int_{\Delta(t)} k^{p'}(t, y)v^{p'}(y)dy \right)^{\frac{r}{p'}} w^q(t)dt \\ (4.13) \quad &\leq \int_{\Delta_k} \mathcal{B}_\sigma^-(t)dt \end{aligned}$$

on the strength of $a(t) < a(\xi_{k+1}) = b(\xi_k)$ and $t \leq s_\sigma \implies \sigma(t) \leq b(\xi_k) \implies b^{-1}(\sigma(t)) \leq \xi_k < s_\rho$. For $S_{k,2}$ we use the result of Corollary 2.1 with

$c = s_\rho$, $d = s_\sigma$ and Lemma 2.2:

$$\begin{aligned}
\|S_{k,2}\|^r &:= \|S_{k,2}\|_{L_p(b(s_\rho), b(s_\sigma)) \rightarrow L_q(s_\rho, s_\sigma)}^r \\
(4.14) \quad &\approx B_{b,0}^r + B_{b,1}^r \stackrel{(2.10), (2.11)}{\ll} \mathcal{B}_{b,0}^r + \mathbb{B}_{b,1}^r \\
&= \int_{b(s_\rho)}^{b(s_\sigma)} \left(\int_t^{b(s_\sigma)} \left[\int_{b^{-1}(y)}^{s_\sigma} k^q(x, y) w^q(x) dx \right]^{p'} v^{p'}(y) dy \right)^{\frac{r}{q'}} \times \\
&\quad \times \left[\int_{b^{-1}(t)}^{s_\sigma} k^q(x, t) w^q(x) dx \right]^{p' - \frac{r}{q'}} v^{p'}(t) dt \\
&\quad + \int_{s_\rho}^{s_\sigma} \left(\int_{s_\rho}^t \left[\int_{b(s_\rho)}^{b(x)} k^{p'}(x, y) v^{p'}(y) dy \right]^q w^q(x) dx \right)^{\frac{r}{p}} \times \\
&\quad \times \left[\int_{b(s_\rho)}^{b(t)} k^{p'}(t, y) v^{p'}(y) dy \right]^{q - \frac{r}{p}} w^q(t) dt.
\end{aligned}$$

Since $a(x) < b(\xi_k) < b(s_\rho) \leq t \leq y = b(z)$ with $z = b^{-1}(y) \leq x$ in $\mathcal{B}_{b,0}$ then $k(x, y) = k(x, b(z)) \ll k(x, t)$. Therefore,

$$\mathcal{B}_{b,0}^r \ll \int_{b(s_\rho)}^{b(s_\sigma)} \left(\int_t^{b(s_\sigma)} v^{p'}(y) dy \right)^{\frac{r}{q'}} \left(\int_{b^{-1}(t)}^{s_\sigma} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q'}} v^{p'}(t) dt.$$

Since $s_\sigma < \xi_{k+1}$ then $a(s_\sigma) < \rho^{-1}(s_\sigma) < \rho^{-1}(\xi_{k+1}) = b(s_\rho) \leq t \implies s_\sigma < a^{-1}(t)$ and $s_\sigma < \rho(t) \implies b(s_\sigma) < b(\rho(t))$. Thus, $[t, b(s_\sigma)] \subset [t, b(\rho(t))]$, $[b^{-1}(t), s_\sigma] \subset [b^{-1}(t), a^{-1}(t)]$ and

$$\begin{aligned}
\mathcal{B}_{b,0}^r &\ll \int_{b(s_\rho)}^{b(s_\sigma)} \left(\int_{\Theta(t)} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_{\vartheta^+(t)} v^{p'}(y) dy \right)^{\frac{r}{q'}} v^{p'}(t) dt \\
(4.15) \quad &\leq \int_{\delta_{k+1}} \mathcal{B}_\rho^+(t) dt.
\end{aligned}$$

In $\mathbb{B}_{b,1}$ since $x \leq t$, $a(t) < b(\xi_k) < b(s_\rho) \leq y \leq b(x)$ we have $k(x, y) \ll k(t, y)$. Therefore, in view of $\xi_k < s_\rho \leq t \leq s_\sigma < \xi_{k+1} \implies \sigma(t) \leq b(\xi_k) \implies b^{-1}(\sigma(t)) \leq \xi_k < s_\rho$ and $a(t) < b(\xi_k)$

$$\mathbb{B}_{b,1}^r \ll \int_{s_\rho}^{s_\sigma} \left(\int_{s_\rho}^t w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{b(s_\rho)}^{b(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt$$

$$\begin{aligned}
& < \int_{s_\rho}^{s_\sigma} \left(\int_{\delta^-(t)} w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{\Delta(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt \\
(4.16) \quad & \leq \int_{\Delta_k} \mathcal{B}_\sigma^-(t) dt.
\end{aligned}$$

By using (1.3) with $z = s_\rho < x \leq \xi_{k+1}$, $a(x) < b(\xi_k) \leq y \leq b(s_\rho) = b(z)$ we decompose $H_{k,2}$ into the sum

$$\begin{aligned}
H_{k,2}f(x) &= w(x)k(x, b(s_\rho)) \int_{b(\xi_k)}^{b(s_\rho)} f(y)v(y)dy + w(x) \int_{b(\xi_k)}^{b(s_\rho)} k(s_\rho, y)f(y)v(y)dy \\
&=: H_{k,2}^w f(x) + H_{k,2}^v f(x).
\end{aligned}$$

Denote $\|H_{k,2}\| := \|H_{k,2}\|_{L_p(b(\xi_k), b(s_\rho)) \rightarrow L_q(s_\sigma, \xi_{k+1})}$. By Hölder's inequality and (1.3)

$$\begin{aligned}
\|H_{k,2}^w\|^r &< \left(\int_{s_\rho}^{\xi_{k+1}} k^q(x, b(s_\rho)) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_{b(\xi_k)}^{b(s_\rho)} v^{p'}(y) dy \right)^{\frac{r}{p'}} \\
&\ll \int_{b(\xi_k)}^{b(s_\rho)} \left(\int_{b(\xi_k)}^t v^{p'}(y) dy \right)^{\frac{r}{q'}} \left(\int_{b^{-1}(t)}^{\xi_{k+1}} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q}} v^{p'}(t) dt \\
&< \int_{b(\xi_k)}^{b(s_\rho)} \left(\int_{\Theta(t)} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_{\vartheta^-(t)} v^{p'}(y) dy \right)^{\frac{r}{q'}} v^{p'}(t) dt \\
(4.17) \quad &\leq \int_{\delta_{k+1}} \mathcal{B}_\rho^-(t) dt
\end{aligned}$$

since t is still not greater then $\rho^{-1}(\xi_{k+1}) = b(s_\rho)$. Again by Hölder's inequality and (1.3)

$$\begin{aligned}
\|H_{k,2}^v\|^r &= \left(\int_{s_\sigma}^{\xi_{k+1}} w^q(x) dx \right)^{\frac{r}{q}} \left(\int_{b(\xi_k)}^{b(s_\rho)} k^{p'}(s_\rho, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} \\
&\ll \int_{s_\sigma}^{\xi_{k+1}} \left(\int_t^{\xi_{k+1}} w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{b(\xi_k)}^{b(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt \\
&< \int_{s_\sigma}^{\xi_{k+1}} \left(\int_{\delta^+(t)} w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{\Delta(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt \\
(4.18) \quad &\leq \int_{\Delta_k} \mathcal{B}_\sigma^+(t) dt
\end{aligned}$$

on the strength of $a(t) \leq b(\xi_k)$ and $s_\sigma \leq t \implies a(\xi_{k+1}) \leq \sigma(t) \implies \xi_{k+1} \leq a^{-1}(\sigma(t))$. To estimate $S_{k,3}$ we use again Corollary 2.1 with $c = s_\sigma$, $d = \xi_{k+1}$ and Lemma 2.2:

$$\begin{aligned}
\|S_{k,3}\|^r &:= \|S_{k,3}\|_{L_p(b(s_\sigma), b(\xi_{k+1})) \rightarrow L_q(s_\sigma, \xi_{k+1})}^r \\
(4.19) \quad &\approx B_{b,0}^r + B_{b,1}^r \\
&\stackrel{(2.10)}{\ll} \mathcal{B}_{b,0}^r + \mathcal{B}_{b,1}^r \\
&= \int_{b(s_\sigma)}^{b(\xi_{k+1})} \left(\int_t^{b(\xi_{k+1})} \left[\int_{b^{-1}(y)}^{\xi_{k+1}} k^q(x, y) w^q(x) dx \right]^{p'} v^{p'}(y) dy \right)^{\frac{r}{q'}} \times \\
&\quad \times \left[\int_{b^{-1}(t)}^{\xi_{k+1}} k^q(x, t) w^q(x) dx \right]^{p' - \frac{r}{q'}} v^{p'}(t) dt \\
&\quad + \int_{s_\sigma}^{\xi_{k+1}} \left(\int_t^{\xi_{k+1}} w^q(x) dx \right)^{\frac{r}{p'}} \left(\int_{b(s_\sigma)}^{b(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p}} w^q(t) dt.
\end{aligned}$$

As before $k(x, y) \ll k(x, t)$ for $a(x) \leq b(\xi_k) < b(s_\sigma) \leq t \leq y = b(z)$ with $z = b^{-1}(y) \leq x$ in $\mathcal{B}_{b,0}$, and also $b(\xi_{k+1}) < b(\rho(t))$ since $\rho^{-1}(\xi_{k+1}) < t$. Therefore,

$$\begin{aligned}
\mathcal{B}_{b,0}^r &\ll \int_{b(s_\sigma)}^{b(\xi_{k+1})} \left(\int_t^{b(\xi_{k+1})} v^{p'}(y) dy \right)^{\frac{r}{q'}} \left(\int_{b^{-1}(t)}^{\xi_{k+1}} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q}} v^{p'}(t) dt \\
&< \int_{b(s_\sigma)}^{b(\xi_{k+1})} \left(\int_{\Theta(t)} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_{\vartheta^+(t)} v^{p'}(y) dy \right)^{\frac{r}{q'}} v^{p'}(t) dt \\
(4.20) \quad &\leq \int_{\delta_{k+1}} \mathcal{B}_\rho^+(t) dt.
\end{aligned}$$

The estimate

$$\begin{aligned}
B_{b,1}^r &\ll \int_{s_\sigma}^{\xi_{k+1}} \left(\int_{\delta^+(t)} w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{\Delta(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt \\
(4.21) \quad &\leq \int_{\Delta_k} \mathcal{B}_\sigma^+(t) dt
\end{aligned}$$

follows from $a(t) \leq a(\xi_{k+1}) = b(\xi_k) < b(s_\sigma)$ and $s_\sigma \leq t \implies a(\xi_{k+1}) \leq \sigma(t) \implies \xi_{k+1} \leq a^{-1}(\sigma(t))$. To estimate the last operator norm $\|H_{k,3}\| :=$

$\|H_{k,3}\|_{L_p(b(s_\rho), b(s_\sigma)) \rightarrow L_q(s_\sigma, \xi_{k+1})}$ we make a decomposition

$$\begin{aligned} H_{k,3}f(x) &= w(x)k(x, b(s_\sigma)) \int_{b(s_\rho)}^{b(s_\sigma)} f(y)v(y)dy + w(x) \int_{b(s_\rho)}^{b(s_\sigma)} k(s_\sigma, y)f(y)v(y)dy \\ &=: H_{k,3}^w f(x) + H_{k,3}^v f(x) \end{aligned}$$

because $k(x, y) \approx k(x, b(s_\sigma)) + k(s_\sigma, y)$ for $s_\sigma \leq x$ and $a(x) \leq y \leq b(s_\sigma)$.

By Hölder's inequality

$$\begin{aligned} \|H_{k,3}^w\|^r &= \left(\int_{s_\sigma}^{\xi_{k+1}} k^q(x, b(s_\sigma))w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{b(s_\rho)}^{b(s_\sigma)} v^{p'}(y)dy \right)^{\frac{r}{p'}} \\ &\ll \int_{b(s_\rho)}^{b(s_\sigma)} \left(\int_t^{b(s_\sigma)} v^{p'}(y)dy \right)^{\frac{r}{q'}} \left(\int_{b^{-1}(t)}^{\xi_{k+1}} k^q(x, t)w^q(x)dx \right)^{\frac{r}{q}} v^{p'}(t)dt \\ &< \int_{b(s_\rho)}^{b(s_\sigma)} \left(\int_{\Theta(t)} k^q(x, t)w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{\vartheta^+(t)} v^{p'}(y)dy \right)^{\frac{r}{q'}} v^{p'}(t)dt \\ (4.22) &\leq \int_{\delta_{k+1}} \mathcal{B}_\rho^+(t)dt \end{aligned}$$

since $\rho^{-1}(s_\sigma) < \rho^{-1}(\xi_{k+1}) = b(s_\rho) < b(s_\sigma) \leq t \implies s_\sigma < \rho(t) \implies b(s_\sigma) < b(\rho(t))$. We have also

$$\begin{aligned} \|H_{k,3}^v\|^r &= \left(\int_{s_\sigma}^{\xi_{k+1}} w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{b(s_\rho)}^{b(s_\sigma)} k^{p'}(s_\sigma, y)v^{p'}(y)dy \right)^{\frac{r}{p'}} \\ &\ll \int_{s_\sigma}^{\xi_{k+1}} \left(\int_t^{\xi_{k+1}} w^q(x)dx \right)^{\frac{r}{p}} \left(\int_{b(s_\rho)}^{b(t)} k^{p'}(t, y)v^{p'}(y)dy \right)^{\frac{r}{p'}} w^q(t)dt \\ &< \int_{s_\sigma}^{\xi_{k+1}} \left(\int_{\delta^+(t)} w^q(x)dx \right)^{\frac{r}{p}} \left(\int_{\Delta(t)} k^{p'}(t, y)v^{p'}(y)dy \right)^{\frac{r}{p'}} w^q(t)dt \\ (4.23) &\leq \int_{\Delta_k} \mathcal{B}_\sigma^+(t)dt \end{aligned}$$

since $s_\sigma \leq t \implies a(\xi_{k+1}) \leq \sigma(t) \implies \xi_{k+1} \leq a^{-1}(\sigma(t))$ and $a(t) \leq b(\xi_k) < b(s_\sigma)$. Thus, by (4.8) – (4.23) it holds for the case (i) that

$$(4.24) \quad \|S_k\|_{L_p(\delta_{k+1}) \rightarrow L_q(\Delta_k)}^r \ll \int_{\delta_{k+1}} \mathcal{B}_\rho(t)dt + \int_{\Delta_k} \mathcal{B}_\sigma(t)dt.$$

FIGURE 2. Case (iii)

In the case **(iii)** we have

$$(4.25) \quad S_k f(x) = \sum_{i=1}^3 S_{k,i} f(x) + \sum_{i=1}^2 H_{k,i} f(x),$$

where

$$\begin{aligned} S_{k,1} f(x) &= \chi_{[\xi_k, s_\sigma]}(x) S_k f(x), \quad S_{k,1} : L_p(b(\xi_k), b(s_\sigma)) \rightarrow L_q(\xi_k, s_\sigma), \\ H_{k,1} f(x) &= \chi_{[s_\sigma, s_\rho]}(x) S_k (f \chi_{[b(\xi_k), b(s_\sigma)]}) (x), \quad H_{k,1} : L_p(b(\xi_k), b(s_\sigma)) \rightarrow L_q(s_\sigma, s_\rho), \\ S_{k,2} f(x) &= \chi_{[s_\sigma, s_\rho]}(x) S_k (f \chi_{[b(s_\sigma), b(s_\rho)]}) (x), \quad S_{k,2} : L_p(b(s_\sigma), b(s_\rho)) \rightarrow L_q(s_\sigma, s_\rho), \\ H_{k,2} f(x) &= \chi_{[s_\rho, \xi_{k+1}]}(x) S_k (f \chi_{[b(\xi_k), b(s_\rho)]}) (x), \quad H_{k,2} : L_p(b(\xi_k), b(s_\rho)) \rightarrow L_q(s_\rho, \xi_{k+1}), \\ S_{k,3} f(x) &= \chi_{[s_\rho, \xi_{k+1}]}(x) S_k (f \chi_{[b(s_\rho), b(\xi_{k+1})]}) (x), \quad S_{k,3} : L_p(b(s_\rho), b(\xi_{k+1})) \rightarrow L_q(s_\rho, \xi_{k+1}). \end{aligned}$$

The estimate

$$(4.26) \quad \|S_{k,1}\|^r := \|S_{k,1}\|_{L_p(b(\xi_k), b(s_\sigma)) \rightarrow L_q(\xi_k, s_\sigma)}^r \ll \int_{\delta_{k+1}} \mathcal{B}_\rho^-(t) dt + \int_{\Delta_k} \mathcal{B}_\sigma^-(t) dt$$

can be obtained analogously to the case (i). The next operator $H_{k,1}$ should be decomposed by (1.3) with $z = s_\sigma \leq x \leq s_\rho$, $a(x) < b(\xi_k) \leq y \leq b(s_\sigma) = b(z)$ into the sum

$$\begin{aligned} H_{k,1}f(x) &= w(x)k(x, b(s_\sigma)) \int_{b(\xi_k)}^{b(s_\sigma)} f(y)v(y)dy + w(x) \int_{b(\xi_k)}^{b(s_\sigma)} k(s_\sigma, y)f(y)v(y)dy \\ &=: H_{k,1}^w f(x) + H_{k,1}^v f(x). \end{aligned}$$

Putting $\|H_{k,1}\| := \|H_{k,1}\|_{L_p(b(\xi_k), b(s_\sigma)) \rightarrow L_q(s_\sigma, s_\rho)}$, by Hölder's inequality and (1.3) we obtain

$$\begin{aligned} \|H_{k,1}^w\|^r &= \left(\int_{s_\sigma}^{s_\rho} k^q(x, b(s_\sigma))w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{b(\xi_k)}^{b(s_\sigma)} v^{p'}(y)dy \right)^{\frac{r}{p'}} \\ &\ll \int_{b(\xi_k)}^{b(s_\sigma)} \left(\int_{b(\xi_k)}^t v^{p'}(y)dy \right)^{\frac{r}{q'}} \left(\int_{b^{-1}(t)}^{s_\rho} k^q(x, t)w^q(x)dx \right)^{\frac{r}{q}} v^{p'}(t)dt \\ &< \int_{b(\xi_k)}^{b(s_\sigma)} \left(\int_{\Theta(t)} k^q(x, t)w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{\vartheta^-(t)} v^{p'}(y)dy \right)^{\frac{r}{q'}} v^{p'}(t)dt \\ (4.27) \quad &\leq \int_{\delta_{k+1}} \mathcal{B}_\rho^-(t)dt \end{aligned}$$

since $a(s_\rho) < t$ and $t < b(s_\rho) = \rho^{-1}(\xi_{k+1}) \implies \rho(t) < \xi_{k+1} \implies a(\rho(t)) < a(\xi_{k+1}) = b(\xi_k)$, and

$$\begin{aligned} \|H_{k,1}^v\|^r &= \left(\int_{s_\sigma}^{s_\rho} w^q(x)dx \right)^{\frac{r}{q}} \left(\int_{b(\xi_k)}^{b(s_\sigma)} k^{p'}(s_\sigma, y)v^{p'}(y)dy \right)^{\frac{r}{p'}} \\ &\ll \int_{s_\sigma}^{s_\rho} \left(\int_t^{s_\rho} w^q(x)dx \right)^{\frac{r}{p}} \left(\int_{b(\xi_k)}^{b(t)} k^{p'}(t, y)v^{p'}(y)dy \right)^{\frac{r}{p'}} w^q(t)dt \\ &< \int_{s_\sigma}^{s_\rho} \left(\int_{\delta^+(t)} w^q(x)dx \right)^{\frac{r}{p}} \left(\int_{\Delta(t)} k^{p'}(t, y)v^{p'}(y)dy \right)^{\frac{r}{p'}} w^q(t)dt \\ (4.28) \quad &\leq \int_{\Delta_k} \mathcal{B}_\sigma^+(t)dt \end{aligned}$$

on the strength of $a(t) < a(\xi_{k+1}) = b(\xi_k)$ and $a(s_\rho) < b(\xi_k) = \sigma(s_\sigma) \leq \sigma(t) \implies s_\rho < a^{-1}(\sigma(t))$. For $S_{k,2}$ we use Corollary 2.1 with $c = s_\sigma$ and $d = s_\rho$:

$$\begin{aligned}
\|S_{k,2}\|^r &:= \|S_{k,2}\|_{L_p(b(s_\sigma), b(s_\rho)) \rightarrow L_q(s_\sigma, s_\rho)}^r \\
(4.29) \quad &\approx B_{b,0}^r + B_{b,1}^r \\
&= \int_{b(s_\sigma)}^{b(s_\rho)} \left(\int_{b^{-1}(t)}^{s_\rho} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_{b(s_\sigma)}^t v^{p'}(y) dy \right)^{\frac{r}{q'}} v^{p'}(t) dt \\
&\quad + \int_{s_\sigma}^{s_\rho} \left(\int_t^{s_\rho} w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{b(s_\sigma)}^{b(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt.
\end{aligned}$$

Since $a(s_\rho) < b(s_\sigma) \leq t \leq b(s_\rho) = \rho^{-1}(\xi_{k+1})$ we have in $B_{b,0}$ that $s_\rho < a^{-1}(t)$ and $\rho(t) \leq \xi_{k+1} \implies a(\rho(t)) \leq a(\xi_{k+1}) < b(s_\sigma)$. Therefore,

$$\begin{aligned}
B_{b,0}^r &\ll \int_{b(s_\sigma)}^{b(s_\rho)} \left(\int_{\Theta(t)} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_{\vartheta^-(t)} v^{p'}(y) dy \right)^{\frac{r}{q'}} v^{p'}(t) dt \\
(4.30) \quad &\leq \int_{\delta_{k+1}} \mathcal{B}_\rho^-(t) dt.
\end{aligned}$$

In $B_{b,1}$ in view of $a(s_\rho) < b(\xi_k) = \sigma(s_\sigma) < \sigma(t) \implies s_\rho < a^{-1}(\sigma(t))$ and $a(t) < b(s_\sigma)$ we have

$$\begin{aligned}
B_{b,1}^r &\ll \int_{s_\sigma}^{s_\rho} \left(\int_{\delta^+(t)} w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{\Delta(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt \\
(4.31) \quad &\leq \int_{\Delta_k} \mathcal{B}_\sigma^+(t) dt.
\end{aligned}$$

Analogously to the case (i) it holds that

$$\begin{aligned}
(4.32) \quad \|H_{k,2}\| &:= \|H_{k,2}\|_{L_p(b(\xi_k), b(s_\rho)) \rightarrow L_q(s_\rho, \xi_{k+1})} \ll \int_{\delta_{k+1}} \mathcal{B}_\rho^-(t) dt + \int_{\Delta_k} \mathcal{B}_\sigma^+(t) dt
\end{aligned}$$

and

$$\begin{aligned}
(4.33) \quad \|S_{k,3}\|^r &:= \|S_{k,3}\|_{L_p(b(s_\rho), b(\xi_{k+1})) \rightarrow L_q(s_\rho, \xi_{k+1})}^r \ll \int_{\delta_{k+1}} \mathcal{B}_\rho^+(t) dt + \int_{\Delta_k} \mathcal{B}_\sigma^+(t) dt.
\end{aligned}$$

Now from (4.25) – (4.33) we have the estimate (4.24) for the case (iii) too. The case **(ii)** is clear from either (i) or (iii). Now we obtain from (4.5) by Lemma 2.1

$$(4.34) \quad \|\mathcal{S}\| \approx \left(\sum_k \|S_k\|_{L_p(\delta_{k+1}) \rightarrow L_q(\Delta_k)}^r \right)^{\frac{1}{r}} \ll \mathcal{B}_\rho + \mathcal{B}_\sigma.$$

To estimate the norm of the operator T_k we decompose it by (1.3), (4.6) into the sum

$$(4.35) \quad \begin{aligned} T_k f(x) &\approx w(x)k(x, b(\xi_k)) \int_{a(x)}^{b(\xi_k)} f(y)v(y)dy \\ &+ w(x) \int_{a(x)}^{b(\xi_k)} k(\xi_k, y)f(y)v(y)dy =: T_k^w f(x) + T_k^v f(x). \end{aligned}$$

Then we find two points $t_\rho := a^{-1}(\rho^{-1}(\xi_k))$, $t_\sigma := \sigma^{-1}(b(\xi_k))$ and make two more decompositions

$$(4.36) \quad T_k^w f(x) = T_k^w (f\chi_{[a(\xi_k), \rho^{-1}(\xi_k)]})(x) + T_k^w (f\chi_{[\rho^{-1}(\xi_k), b(\xi_k)]})(x) =: T_{k,1}^w f(x) + T_{k,2}^w f(x),$$

$$(4.37) \quad T_k^v f(x) = T_k^v f(x)\chi_{[\xi_k, t_\sigma]}(x) + T_k^v f(x)\chi_{[t_\sigma, \xi_{k+1}]}(x) =: T_{k,1}^v f(x) + T_{k,2}^v f(x)$$

(see Figure 1 and Figure 2 respectively). By duality and (2.15) it follows from Lemma 2.4 with $c = a(\xi_k)$, $d = \rho^{-1}(\xi_k)$, $a = \xi_k$, $b(x) \rightarrow a^{-1}(y)$, $v(y) \rightarrow w(x)k(x, b(\xi_k))$, $w(x) \rightarrow v(y)$ and $q = p'$, $p = q'$ that

$$\begin{aligned} \|T_{k,1}^w\|^r &:= \|T_{k,1}^w\|_{L_p(a(\xi_k), \rho^{-1}(\xi_k)) \rightarrow L_q(\xi_k, t_\rho)}^r \approx \\ &\approx \int_{a(\xi_k)}^{\rho^{-1}(\xi_k)} \left(\int_{a(\xi_k)}^t \left[\int_{\xi_k}^{a^{-1}(y)} w^q(x)k^q(x, b(\xi_k))dx \right]^{p'} v^{p'}(y)dy \right)^{\frac{r}{q'}} \\ &\quad \times \left(\int_{\xi_k}^{a^{-1}(t)} w^q(x)k^q(x, b(\xi_k))dx \right)^{p' - \frac{r}{q'}} v^{p'}(t)dt. \end{aligned}$$

Since $\xi_k \leq x$ and $a(x) \leq y \leq t \leq \rho^{-1}(\xi_k) < b(\xi_k)$ then $k(x, b(\xi_k)) \ll k(x, t)$ and, therefore,

$$\|T_{k,1}^w\|^r \ll \int_{a(\xi_k)}^{\rho^{-1}(\xi_k)} \left(\int_{a(\xi_k)}^t v^{p'}(y)dy \right)^{\frac{r}{q'}} \left(\int_{\xi_k}^{a^{-1}(t)} k^q(x, t)w^q(x)dx \right)^{\frac{r}{q'}} v^{p'}(t)dt$$

$$\begin{aligned}
& < \int_{a(\xi_k)}^{\rho^{-1}(\xi_k)} \left(\int_{\Theta(t)} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_{\vartheta^-(t)} v^{p'}(y) dy \right)^{\frac{r}{q'}} v^{p'}(t) dt \\
(4.38) \quad & \leq \int_{\delta_k} \mathcal{B}_\rho^-(t) dt
\end{aligned}$$

on the strength of $t < b(\xi_k) \implies b^{-1}(t) < \xi_k$ and $t \leq \rho^{-1}(\xi_k) \implies a(\rho(t)) \leq a(\xi_k)$. Further, again by duality and (2.14), we obtain from Lemma 2.4 with $c = \rho^{-1}(\xi_k)$, $d = b(\xi_k)$, $a = \xi_k$, $b(x) \rightarrow a^{-1}(y)$, $v(y) \rightarrow w(x)k(x, b(\xi_k))$, $w(x) \rightarrow v(y)$, $q = p'$, $p = q'$ provided $a^{-1}(\rho^{-1}(\xi_k)) = t_\rho$ that

$$\begin{aligned}
\|T_{k,2}^w\|^r &:= \|T_{k,2}^w\|_{L_p(\rho^{-1}(\xi_k), b(\xi_k)) \rightarrow L_q(\xi_k, \xi_{k+1})}^r \approx \\
&\approx \int_{\rho^{-1}(\xi_k)}^{b(\xi_k)} \left(\int_{\xi_k}^{a^{-1}(t)} w^q(x) k^q(x, b(\xi_k)) dx \right)^{\frac{r}{q}} \left(\int_t^{b(\xi_k)} v^{p'}(y) dy \right)^{\frac{r}{q'}} v^{p'}(t) dt.
\end{aligned}$$

Since $\xi_k \leq x$ and $a(x) \leq t \leq b(\xi_k)$ then $k(x, b(\xi_k)) \ll k(x, t)$ and, therefore,

$$\begin{aligned}
\|T_{k,2}^w\|^r &\ll \int_{\rho^{-1}(\xi_k)}^{b(\xi_k)} \left(\int_{\xi_k}^{a^{-1}(t)} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_t^{b(\xi_k)} v^{p'}(y) dy \right)^{\frac{r}{q'}} v^{p'}(t) dt \\
(4.39) \quad &< \int_{\rho^{-1}(\xi_k)}^{b(\xi_k)} \left(\int_{\Theta(t)} k^q(x, t) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_{\vartheta^+(t)} v^{p'}(y) dy \right)^{\frac{r}{q'}} v^{p'}(t) dt \leq \int_{\delta_k} \mathcal{B}_\rho^+(t) dt
\end{aligned}$$

on the strength of $t \leq b(\xi_k) \implies b^{-1}(t) \leq \xi_k$ and $t \geq \rho^{-1}(\xi_k) \implies b(\xi_k) \leq b(\rho(t))$. Now we have by (2.16) from Lemma 2.5 and (1.3) with $\xi_k \leq t$, $a(t) \leq y \leq b(\xi_k)$

$$\begin{aligned}
\|T_{k,1}^v\|^r &:= \|T_{k,1}^v\|_{L_p(a(\xi_k), b(\xi_k)) \rightarrow L_q(\xi_k, t_\sigma)}^r \approx \\
&\approx \int_{\xi_k}^{t_\sigma} \left(\int_{\xi_k}^t w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{a(t)}^{b(\xi_k)} k^{p'}(\xi_k, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt \\
(1.3) \quad &\ll \int_{\xi_k}^{t_\sigma} \left(\int_{\xi_k}^t w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{a(t)}^{b(\xi_k)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt
\end{aligned}$$

$$\begin{aligned}
&< \int_{\xi_k}^{t_\sigma} \left(\int_{\delta^-(t)} w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{\Delta(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt \\
(4.40) &\leq \int_{\Delta_k} \mathcal{B}_\sigma^-(t) dt
\end{aligned}$$

since $\sigma(t) \leq b(\xi_k) \implies b^{-1}(\sigma(t)) \leq \xi_k$ and $b(\xi_k) \leq b(t)$. Finally, by (2.17), from Lemma 2.5 and (1.3) with $\xi_k < t_\sigma \leq t$, $a(t) \leq a(x) \leq y \leq b(\xi_k)$ we obtain

$$\begin{aligned}
\|T_{k,2}^v\|^r &:= \|T_{k,2}^v\|_{L_p(a(t_\sigma), b(\xi_k)) \rightarrow L_q(t_\sigma, \xi_{k+1})}^r \approx \\
&\approx \int_{t_\sigma}^{\xi_{k+1}} \left(\int_t^{\xi_{k+1}} \left[\int_{a(x)}^{b(\xi_k)} k^{p'}(\xi_k, y) v^{p'}(y) dy \right]^q w^q(x) dx \right)^{\frac{r}{p}} \\
&\quad \times \left(\int_{a(t)}^{b(\xi_k)} k^{p'}(\xi_k, y) v^{p'}(y) dy \right)^{q - \frac{r}{p}} w^q(t) dt \\
&\stackrel{(1.3)}{\ll} \int_{t_\sigma}^{\xi_{k+1}} \left(\int_t^{\xi_{k+1}} w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{a(t)}^{b(\xi_k)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt \\
&< \int_{t_\sigma}^{\xi_{k+1}} \left(\int_{\delta^+(t)} w^q(x) dx \right)^{\frac{r}{p}} \left(\int_{\Delta(t)} k^{p'}(t, y) v^{p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt \\
(4.41) &\leq \int_{\Delta_k} \mathcal{B}_\sigma^+(t) dt
\end{aligned}$$

since $a(\xi_{k+1}) = b(\xi_k) \leq \sigma(t) \implies \xi_{k+1} \leq a^{-1}(\sigma(t))$ and $b(\xi_k) < b(t)$. Now from (4.35) – (4.41) we have

$$(4.42) \quad \|T_k\|_{L_p(\delta_k) \rightarrow L_q(\Delta_k)}^r \ll \int_{\delta_k} \mathcal{B}_\rho(t) dt + \int_{\Delta_k} \mathcal{B}_\sigma(t) dt.$$

Therefore, from (4.5) by Lemma 2.1

$$\|\mathcal{T}\| \approx \left(\sum_k \|T_k\|_{L_p(\delta_k) \rightarrow L_q(\Delta_k)}^r \right)^{\frac{1}{r}} \ll \mathcal{B}_\rho + \mathcal{B}_\sigma.$$

Thus and from (4.34) the upper estimate in (3.4) follows in view of (4.4).

The lower estimate. Suppose that the inequality

$$(4.43) \quad \|\mathcal{K}f\|_q \leq \|\mathcal{K}\| \|f\|_p$$

holds. To prove

$$(4.44) \quad \|\mathcal{K}\|_{L_p \rightarrow L_q} \gg \mathcal{B}_\rho^-$$

we take a point sequence $\{\xi_k\}_{k \in \mathbb{Z}} \subset (0, \infty)$ such that

$$(4.45) \quad \xi_0 = 1, \quad \xi_k = (a^{-1} \circ b)^k(1), \quad k \in \mathbb{Z},$$

and denote

$$W_\rho(t) = \int_{b^{-1}(t)}^{a^{-1}(t)} k^q(x, t) w^q(x) dx, \quad V_\rho^-(t) = \int_{a(\rho(t))}^t v^{p'}(y) dy.$$

Note that $[W_\rho(t)]^{r/pq} [V_\rho^-(t)]^{r/pq'} [v(t)]^{p'-1} = \mathcal{B}_\rho^-(t)^{1/p}$. If we put

$$f_\rho(t) := [W_\rho(t)]^{r/pq} [V_\rho^-(t)]^{r/pq'} [v(t)]^{p'-1},$$

then $\|f_\rho\|_p = (\mathcal{B}_\rho^-)^{r/p}$. Thus, since $\sqcup_k [\xi_k, \xi_{k+1}) = (0, \infty)$ and (4.43) holds we have

$$(4.46) \quad \begin{aligned} \|\mathcal{K}\| (\mathcal{B}_\rho^-)^{r/p} &\stackrel{(4.43)}{\geq} \|\mathcal{K}f_\rho\|_q = \left(\sum_k \int_{\xi_k}^{\xi_{k+1}} (\mathcal{K}f_\rho)^q(x) dx \right)^{\frac{1}{q}} \\ &\geq 2^{-1/q} \left(\sum_k \int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_{k+1}} (\mathcal{K}f_\rho)^q(x) dx \right)^{\frac{1}{q}}. \end{aligned}$$

FIGURE 3. Necessity of $\mathcal{B}_\rho^-$

Using the explicit form of the operator $\mathcal{K}$ we find that

$$\begin{aligned}
& \int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_{k+1}} (\mathcal{K}f_\rho)^q(x) dx \\
&= \int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_{k+1}} w^q(x) \left(\int_{a(x)}^{b(x)} k(x, y) f_\rho(y) v(y) dy \right)^q dx \\
&= \int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_{k+1}} w^q(x) \left(\int_{a(x)}^{b(x)} k(x, t) f_\rho(t) v(t) dt \right) \times \\
&\quad \times \left(\int_{a(x)}^{b(x)} k(x, y) f_\rho(y) v(y) dy \right)^{q-1} dx \\
&\geq \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} f_\rho(t) v(t) \left(\int_{b^{-1}(t)}^{\min\{a^{-1}(t), \xi_{k+1}\}} k(x, t) w^q(x) \times \right. \\
&\quad \times \left. \left[\int_{a(x)}^{b(x)} k(x, y) f_\rho(y) v(y) dy \right]^{q-1} dx \right) dt \\
&\geq \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} f_\rho(t) v(t) \left(\int_{b^{-1}(t)}^{\rho(t)} k(x, t) w^q(x) \times \right. \\
&\quad \times \left. \left[\int_{a(x)}^t k(x, y) f_\rho(y) v(y) dy \right]^{q-1} dx \right) dt \\
&\quad [\text{applying (1.3) with } z = b^{-1}(t)] \\
&\gg \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} f_\rho(t) v(t) \left(\int_{b^{-1}(t)}^{\rho(t)} k^q(x, t) w^q(x) \times \right. \\
&\quad \times \left. \left[\int_{a(x)}^t f_\rho(y) v(y) dy \right]^{q-1} dx \right) dt \\
&\geq \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} f_\rho(t) v(t) \left(\int_{b^{-1}(t)}^{\rho(t)} k^q(x, t) w^q(x) dx \right) \times \\
&\quad \times \left(\int_{a(\rho(t))}^t f_\rho(y) v(y) dy \right)^{q-1} dt \\
&= \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} [W_\rho(t)]^{r/pq} [V_\rho^-(t)]^{r/pq'} \left(\int_{b^{-1}(t)}^{\rho(t)} k^q(x, t) w^q(x) dx \right) \\
&\quad \times \left(\int_{a(\rho(t))}^t [W_\rho(y)]^{r/pq} [V_\rho^-(y)]^{r/pq'} v^{p'}(y) dy \right)^{q-1} v^{p'}(t) dt
\end{aligned}$$

$$\begin{aligned}
& \stackrel{(3.1)}{\gg} \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} [V_\rho^-(t)]^{r/pq'} [W_\rho(t)]^{r/pq+1} \\
& \quad \times \left(\int_{a(\rho(t))}^t \left[\int_{b^{-1}(t)}^{\rho(t)} k^q(z, y) w^q(z) dz \right]^{\frac{r}{pq}} [V_\rho^-(y)]^{r/pq'} v^{p'}(y) dy \right)^{q-1} v^{p'}(t) dt.
\end{aligned}$$

It follows from (1.3) that $k(z, y) \gg k(z, t)$ provided $a(z) \leq a(\rho(t)) \leq y \leq t = b(\tau)$, $\tau = b^{-1}(t) \leq z$. Therefore, in view of (3.1)

$$\begin{aligned}
\int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_{k+1}} (\mathcal{K}f_\rho)^q(x) dx & \gg \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} [V_\rho^-(t)]^{r/pq'} [W_\rho(t)]^{r/q} \\
& \quad \times \left(\int_{a(\rho(t))}^t \left[\int_{a(\rho(t))}^y v^{p'}(z) dz \right]^{r/pq'} v^{p'}(y) dy \right)^{q-1} v^{p'}(t) dt \\
& \approx \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} \mathcal{B}_\rho^-(t) dt.
\end{aligned}$$

Since $\sqcup_k[\rho^{-1}(\xi_k), \rho^{-1}(\xi_{k+1})) = (0, \infty)$, it yields

$$\left(\sum_k \int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_{k+1}} (\mathcal{K}f_\rho)^q(x) dx \right)^{\frac{1}{q}} \gg (\mathcal{B}_\rho^-)^{r/q}.$$

Thus, by (4.46) we obtain (4.44).

To prove

$$(4.47) \quad \|\mathcal{K}\|_{L_p \rightarrow L_q} \gg \mathcal{B}_\sigma^+$$

we use the dual to (4.43) inequality

$$(4.48) \quad \left(\int_0^\infty \left[\int_{b^{-1}(y)}^{a^{-1}(y)} k(x, y) g(x) w(x) dx \right]^{p'} v^{p'}(y) dy \right)^{\frac{1}{p'}} \leq \|\bar{\mathcal{K}}\| \left(\int_0^\infty g^{q'}(x) dx \right)^{\frac{1}{q'}}.$$

Note that the kernel $k(x, y)$ is satisfying a condition following from (1.3) :

$$(4.49) \quad D^{-1}k(x, y) \leq k(x, z) + k(b^{-1}(z), y) \leq Dk(x, y), \quad y \leq z, \quad b^{-1}(z) \leq x \leq a^{-1}(y).$$

Break the semiaxis $(0, \infty)$ by the point sequence (4.45) and put

$$W_\sigma^+(t) = \int_t^{a^{-1}(\sigma(t))} w^q(x) dx, \quad V_\sigma(t) = \int_{a(t)}^{b(t)} k^{p'}(t, y) v^{p'}(y) dy.$$

Note that $[W_\sigma^+(t)]^{r/pq'} [V_\sigma(t)]^{r/p'q'} [w(t)]^{q-1} = \mathcal{B}_\sigma^+(t)^{1/q'}$. If we take

$$g_\sigma(t) = [W_\sigma^+(t)]^{r/pq'} [V_\sigma(t)]^{r/p'q'} [w(t)]^{q-1},$$

then $\left(\int_0^\infty g_\sigma^{q'}(x) dx\right)^{\frac{1}{q'}} = (\mathcal{B}_\sigma^+)^{r/q'}$. Thus, by $\sqcup_k[\sigma(\xi_k), \sigma(\xi_{k+1})) = (0, \infty)$ and (4.48), we have

$$\begin{aligned} \|\bar{\mathcal{K}}\| (\mathcal{B}_\sigma^+)^{r/q'} &\stackrel{(4.48)}{\geq} \|\bar{\mathcal{K}}g_\sigma\|_{p'} = \left(\sum_k \int_{\sigma(\xi_k)}^{\sigma(\xi_{k+1})} (\bar{\mathcal{K}}g_\sigma)^{p'}(y) dy \right)^{\frac{1}{p'}} \\ (4.50) \quad &\geq 2^{-1/p'} \left(\sum_k \int_{\sigma(\xi_k)}^{b(\xi_{k+1})} (\bar{\mathcal{K}}g_\sigma)^{p'}(y) dy \right)^{\frac{1}{p'}}, \end{aligned}$$

where $\bar{\mathcal{K}}g(y) := v(y) \int_{b^{-1}(y)}^{a^{-1}(y)} k(x, y) g_\sigma(x) w(x) dx$. We find that

$$\begin{aligned} \int_{\sigma(\xi_k)}^{b(\xi_{k+1})} (\bar{\mathcal{K}}g_\sigma)^{p'}(y) dy &= \int_{\sigma(\xi_k)}^{b(\xi_{k+1})} v^{p'}(y) \left(\int_{b^{-1}(y)}^{a^{-1}(y)} k(x, y) g_\sigma(x) w(x) dx \right)^{p'} dy \\ &= \int_{\sigma(\xi_k)}^{b(\xi_{k+1})} v^{p'}(y) \left(\int_{b^{-1}(y)}^{a^{-1}(y)} k(t, y) g_\sigma(t) w(t) dt \right) \times \\ &\quad \times \left(\int_{b^{-1}(y)}^{a^{-1}(y)} k(x, y) g_\sigma(x) w(x) dx \right)^{p'-1} dx \\ &\geq \int_{\xi_k}^{\xi_{k+1}} g_\sigma(t) w(t) \left(\int_{\max\{a(t), \sigma(\xi_k)\}}^{b(t)} k(t, y) v^{p'}(y) \times \right. \\ &\quad \times \left. \left[\int_{b^{-1}(y)}^{a^{-1}(y)} k(x, y) g_\sigma(x) w(x) dx \right]^{p'-1} dy \right) dt \\ &\geq \int_{\xi_k}^{\xi_{k+1}} g_\sigma(t) w(t) \left(\int_{\sigma(t)}^{b(t)} k(t, y) v^{p'}(y) \times \right. \\ &\quad \times \left. \left[\int_t^{a^{-1}(y)} k(x, y) g_\sigma(x) w(x) dx \right]^{p'-1} dy \right) dt \end{aligned}$$

[applying (4.49) with $z = b(t)$]

$$\begin{aligned}
& \gg \int_{\xi_k}^{\xi_{k+1}} g_\sigma(t) w(t) \left(\int_{\sigma(t)}^{b(t)} k^{p'}(t, y) v^{p'}(y) \left[\int_t^{a^{-1}(y)} g_\sigma(x) w(x) dx \right]^{p'-1} dy \right) dt \\
& \geq \int_{\xi_k}^{\xi_{k+1}} g_\sigma(t) w(t) \left(\int_{\sigma(t)}^{b(t)} k^{p'}(t, y) v^{p'}(y) dy \right) \left(\int_t^{a^{-1}(\sigma(t))} g_\sigma(x) w(x) dx \right)^{p'-1} dt \\
& = \int_{\xi_k}^{\xi_{k+1}} [W_\sigma^+(t)]^{r/pq'} [V_\sigma(t)]^{r/p'q'} \left(\int_{\sigma(t)}^{b(t)} k^{p'}(t, y) v^{p'}(y) dy \right) \\
& \quad \times \left(\int_t^{a^{-1}(\sigma(t))} [W_\sigma^+(x)]^{r/pq'} [V_\sigma(x)]^{r/p'q'} w^q(x) dx \right)^{p'-1} w^q(t) dt \\
& \stackrel{(3.2)}{\gg} \int_{\xi_k}^{\xi_{k+1}} [W_\sigma^+(t)]^{r/pq'} [V_\sigma(t)]^{r/p'q'+1} \\
& \quad \times \left(\int_t^{a^{-1}(\sigma(t))} \left[\int_{\sigma(t)}^{b(t)} k^{p'}(x, z) v^{p'}(z) dz \right]^{\frac{r}{p'q'}} [W_\sigma^+(x)]^{r/pq'} w^q(x) dx \right)^{p'-1} w^q(t) dt.
\end{aligned}$$

It follows from (4.49) that $k(x, z) \gg k(t, z)$ because $b^{-1}(\tau) = t \leq x \leq a^{-1}(\sigma(t)) \leq a^{-1}(z)$ and $z \leq \tau = b(t)$. Therefore, by (3.2)

$$\begin{aligned}
& \int_{\sigma(\xi_k)}^{b(\xi_{k+1})} (\bar{\mathcal{K}}_\sigma)^{p'}(y) dy \\
& \gg \int_{\xi_k}^{\xi_{k+1}} [W_\sigma^+(t)]^{r/pq'} [V_\sigma(t)]^{r/p'} \\
& \quad \times \left(\int_t^{a^{-1}(\sigma(t))} \left[\int_x^{a^{-1}(\sigma(t))} w^q(z) dz \right]^{r/pq'} w^q(x) dx \right)^{p'-1} w^q(t) dt \\
& \approx \int_{\xi_k}^{\xi_{k+1}} \mathcal{B}_\sigma^+(t) dt.
\end{aligned}$$

Thus and from (4.50)

$$(4.51) \quad \|\mathcal{K}\|_{L_p \rightarrow L_q} = \|\bar{\mathcal{K}}\|_{L_{q'} \rightarrow L_{p'}} \gg \mathcal{B}_\sigma^+.$$

Combining (4.44) and (4.51) we obtain the lower estimate in (3.4). The assertion about compactness for $q < p$ is a direct corollary of the obtained boundedness criterion and Ando's theorem.

FIGURE 4. Necessity of $\mathcal{B}_\sigma^+$

The assertions of Theorem 3.2 can be proved analogously by using (1.8), Corollary 2.2, Lemma 2.3 and Lemmas 2.4, 2.5.

5. Criterion cases

Here we consider two cases when the results of Theorems 3.1 and 3.2 give criteria.

Theorem 5.1. *Let the operator $\mathcal{K}$ be defined by (1.1) with $a(x)$, $b(x)$ satisfying (1.2) and a continuous kernel $k(x, y) > 0$. Suppose that the functions $\rho(x)$, $\sigma(x)$ on $\mathbb{R}^+$ are strictly increasing fairways from Definition 3.1.*

(a) *If $k(x, y) \in \mathcal{O}_b$ and*

$$(5.1) \quad \int_{\vartheta^-(t)} v^{p'}(y) dy \approx \int_{\vartheta(t)} v^{p'}(y) dy, \quad t > 0,$$

$$(5.2) \quad \int_{\delta^+(t)} w^q(x) dx \approx \int_{\delta(t)} w^q(x) dx, \quad t > 0,$$

then

$$(5.3) \quad \|\mathcal{K}\|_{L_p \rightarrow L_q} \approx \begin{cases} \mathcal{A}_\rho + \mathcal{A}_\sigma, & 1 < p \leq q < \infty, \\ \mathcal{B}_\rho + \mathcal{B}_\sigma, & 1 < q < p < \infty, \end{cases}$$

and $\mathcal{K}$ is compact if and only if $\mathcal{A}_\rho, \mathcal{A}_\sigma < \infty$, $\lim_{t \rightarrow 0} \mathcal{A}_\rho(t) = \lim_{t \rightarrow \infty} \mathcal{A}_\sigma(t) = 0$ in the case $1 < p \leq q < \infty$ and if and only if $\mathcal{B}_\rho, \mathcal{B}_\sigma < \infty$ for $1 < q < p < \infty$.

(b) If $k(x, y) \in \mathcal{O}_a$ and

$$(5.4) \quad \int_{\vartheta^+(t)} v^{p'}(y) dy \approx \int_{\vartheta(t)} v^{p'}(y) dy, \quad t > 0,$$

$$(5.5) \quad \int_{\delta^-(t)} w^q(x) dx \approx \int_{\delta(t)} w^q(x) dx, \quad t > 0,$$

then the estimate (5.3) holds and $\mathcal{K}$ is compact if and only if $\mathcal{A}_\rho, \mathcal{A}_\sigma < \infty$, $\lim_{t \rightarrow 0} \mathcal{A}_\rho(t) = \lim_{t \rightarrow \infty} \mathcal{A}_\sigma(t) = 0$ in the case $1 < p \leq q < \infty$ and if and only if $\mathcal{B}_\rho, \mathcal{B}_\sigma < \infty$ for $1 < q < p < \infty$.

Theorem 5.2. Let the operator $\mathcal{K}$ be given by (1.1) with $a(x)$, $b(x) \in (1.2)$ and a continuous kernel $0 < k(x, y) \in \mathcal{O}_a \cap \mathcal{O}_b$. Suppose $\rho(x)$, $\sigma(x)$ on $\mathbb{R}^+$ are strictly increasing fairways satisfying Definition 3.1

(a) If $1 < p \leq q < \infty$, then

$$(5.6) \quad \alpha_7(p, q) [\mathcal{A}_\rho + \mathcal{A}_\sigma] \leq \|\mathcal{K}\|_{L_p \rightarrow L_q} \leq \alpha_8(p, q) [\mathcal{A}_\rho + \mathcal{A}_\sigma],$$

and $\mathcal{K}$ is compact if and only if $\mathcal{A}_\rho, \mathcal{A}_\sigma < \infty$, $\lim_{t \rightarrow 0} \mathcal{A}_\rho(t) = \lim_{t \rightarrow \infty} \mathcal{A}_\sigma(t) = 0$.

(b) If $1 < q < p < \infty$ then

$$(5.7) \quad \beta_7(p, q) [\mathcal{B}_\rho + \mathcal{B}_\sigma] \leq \|\mathcal{K}\|_{L_p \rightarrow L_q} \leq \beta_8(p, q) [\mathcal{B}_\rho + \mathcal{B}_\sigma],$$

and $\mathcal{K}$ is compact if and only if $\mathcal{B}_\rho, \mathcal{B}_\sigma < \infty$.

Proofs of Theorems 5.1 and 5.2 easy follow from Theorems 3.1 and 3.2.

6. Examples

We conclude the paper by illustrating some of our results by examples. The first of them is about a criterion case from Theorem 5.1.

Example 6.1. Let $p = q = 2$, $v(y) = y^{-3/2}$, $w(x) = 1$, $k(x, y) = (y - a(x))^{1/2}$, $a(x) = x/2$, $b(x) = 2x$, that is

$$K_1 f(x) = \int_{x/2}^{2x} (y - x/2)^{1/2} f(y) y^{-3/2} dy.$$

According to (3.1)

$$L_v(x) := \int_{x/2}^{\sigma(x)} (y - x/2) y^{-3} dy = \int_{\sigma(x)}^{2x} (y - x/2) y^{-3} dy = R_v(x).$$

Integrating by parts we find

$$\begin{aligned} L_v(x) &= -\frac{1}{2} \int_{x/2}^{\sigma(x)} (y - x/2) dy^{-2} \\ &= -\frac{1}{2} (\sigma(x) - x/2) \sigma^{-2}(x) - \frac{1}{2} \int_{x/2}^{\sigma(x)} dy^{-1} \\ &= \frac{1}{x} - \sigma^{-1}(x) + \frac{x}{4} \sigma^{-2}(x) \end{aligned}$$

and

$$R_v(x) = -\frac{1}{2} \int_{\sigma(x)}^{2x} (y - x/2) dy^{-2} = -\frac{5}{8x} + \sigma^{-1}(x) - \frac{x}{4} \sigma^{-2}(x).$$

Thus, since $x/2 \leq \sigma(x) \leq 2x$

$$\sigma(x) = \frac{2(4 + \sqrt{3})}{13} x.$$

Analogously, on the strength of (3.2)

$$L_w(y) := \int_{y/2}^{\rho(y)} (y - x/2) dx = \int_{\rho(y)}^{2y} (y - x/2) dx =: R_w(y).$$

We have

$$L_w(y) = \int_{y-\rho(y)/2}^{y-y/4} dz^2 = \left(\frac{3}{4}y\right)^2 - \left(y - \frac{\rho(y)}{2}\right)^2$$

and

$$R_w(y) = \int_0^{y-\rho(y)/2} dz^2 = \left(y - \frac{\rho(y)}{2}\right)^2.$$

Therefore,

$$\rho(y) = \frac{4\sqrt{2}-3}{2\sqrt{2}} y.$$

The operator K_1 with $k(x, y) = (y - a(x))^{1/2} \in \mathcal{O}_a$ is bounded from L_2 to L_2 . Indeed, since

$$\begin{aligned} A_\rho &= \sup_{t>0} \left(\int_{t/2}^{2t} (t - x/2) dx \right)^{\frac{1}{2}} \left(\int_{\frac{4\sqrt{2}-3}{4\sqrt{2}}t}^{\frac{4\sqrt{2}-3}{\sqrt{2}}t} y^{-3} dy \right)^{\frac{1}{2}} \\ &= \sup_{t>0} \frac{1}{\sqrt{2}} \left(\int_0^{\frac{3}{4}t} dz^2 \right)^{\frac{1}{2}} \left(\int_{\frac{4\sqrt{2}-3}{4\sqrt{2}}t}^{\frac{4\sqrt{2}-3}{\sqrt{2}}t} dy^{-2} \right)^{\frac{1}{2}} = \frac{3\sqrt{15}}{4(4\sqrt{2}-3)} < \infty \end{aligned}$$

and

$$\begin{aligned} A_\sigma &= \sup_{t>0} \left(\int_{\frac{4+\sqrt{3}}{13}t}^{\frac{4(4+\sqrt{3})}{13}t} dx \right)^{\frac{1}{2}} \left(\int_{t/2}^{2t} (y - t/2)y^{-3} dy \right)^{\frac{1}{2}} \\ &= \sup_{t>0} \left(\frac{3(4+\sqrt{3})}{13}t \right)^{\frac{1}{2}} \left(\frac{15}{16t} \right)^{\frac{1}{2}} = \frac{3}{4} \sqrt{\frac{5(4+\sqrt{3})}{13}} < \infty \end{aligned}$$

by (3.3) we have $\|K_1\|_{L_2 \rightarrow L_2} < \infty$. Moreover, since

$$\int_{\vartheta^+(t)} v^{p'}(y) dy = \int_t^{\frac{4\sqrt{2}-3}{\sqrt{2}}t} y^{-3} dy = \frac{39 - 24\sqrt{2}}{2(41 - 24\sqrt{2})} t^{-2},$$

$$\int_{\vartheta(t)} v^{p'}(y) dy = \int_{\frac{4\sqrt{2}-3}{4\sqrt{2}}t}^{\frac{4\sqrt{2}-3}{\sqrt{2}}t} y^{-3} dy = \frac{15}{41 - 24\sqrt{2}} t^{-2}$$

and

$$\int_{\delta^-(t)} w^q(x) dx = \int_{\frac{4+\sqrt{3}}{13}t}^t dx = \frac{9 - \sqrt{3}}{13} t,$$

$$\int_{\delta(t)} w^q(x) dx = \int_{\frac{4+\sqrt{3}}{13}t}^{\frac{4(4+\sqrt{3})}{13}t} dx = \frac{3(4+\sqrt{3})}{13} t,$$

the requirements (5.4), (5.5) are satisfied and on the strength of Theorem 5.1 (b) the conditions $A_\rho < \infty, A_\sigma < \infty$ are also necessary for the boundedness of the operator $K_1 : L_2 \rightarrow L_2$.

Note that proofs of upper norm estimates in Theorems 3.1, 3.2 do not use fairway's integral properties (3.1), (3.2). Therefore, if we need to state boundedness of $\mathcal{K}$ only, it is correct and easier to take a strongly increasing fairway function $\phi(x)$ with the only property $a(x) < \phi(x) < b(x)$, $x > 0$, instead of $\sigma(x)$ and $\rho^{-1}(x)$ equipped by (3.1), (3.2).

Example 6.2. Let $p = 3$, $q = 2$, $k(x, y) = \left| \int_y^x h(z) dz \right|^\gamma$ with $\gamma > 0$, $v(y) = h(y)^{2/3}$, $w(x) = h(x)^{1/2}$, where $0 \leq h \in L_{loc}(\mathbb{R}^+)$, $a(x) = \ln \sqrt{x+1}$ and $b(x) = \exp 2x - 1$. That is

$$\begin{aligned} K_2 f(x) &= h(x)^{1/2} \int_{\ln \sqrt{x+1}}^{\exp 2x-1} \left| \int_y^x h(z) dz \right|^\gamma f(y) h(y)^{2/3} dy \\ &= h(x)^{1/2} \int_{\ln \sqrt{x+1}}^x \left(\int_y^x h(z) dz \right)^\gamma f(y) h(y)^{2/3} dy \\ &\quad + h(x)^{1/2} \int_x^{\exp 2x-1} \left(\int_x^y h(z) dz \right)^\gamma f(y) h(y)^{2/3} dy \\ &=: K_{2,1} f(x) + K_{2,2} f(x), \end{aligned}$$

$p' = 3/2$, $q' = 2$ and $r = 6$. Note that the kernel $k_{2,1}(x, y) := \left(\int_y^x h(z) dz \right)^\gamma$ of the operator $K_{2,1}$ is from the class $\mathcal{O}_b$, while the kernel $k_{2,2}(x, y) := \left(\int_x^y h(z) dz \right)^\gamma$ of $K_{2,2}$ is of the type $\mathcal{O}_a$.

To state boundedness of $K_{2,1}$ with $h(z) = \exp(-z)$ we shall use the line $\phi_1(x) = x/2$, which is $a(x) = \ln \sqrt{x+1} < \phi_1(x) < x =: b_1(x)$, instead of $\rho^{-1}(x)$ and $\sigma(x)$. Since $b_1^{-1}(t) = t$, $a^{-1}(t) = \exp 2t - 1$, $\phi_1^{-1}(t) = 2t$ then

$$\begin{aligned} B_{1,\rho=\phi^{-1}}^6 &:= \\ &:= \int_0^\infty \left(\int_t^{\exp 2t-1} \left[\int_t^x h(z) dz \right]^{2\gamma} h(x) dx \right)^3 \left(\int_{\ln \sqrt{2t+1}}^{2t} h(y) dy \right)^3 h(t) dt \\ &= \int_0^\infty \left(\int_t^{\exp 2t-1} \left[\int_t^x h(z) dz \right]^{2\gamma} d \int_t^x h(z) dz \right)^3 \left(\int_{\ln \sqrt{2t+1}}^{2t} h(y) dy \right)^3 h(t) dt \\ &= \frac{1}{(2\gamma+1)^3} \int_0^\infty \left(\int_t^{\exp 2t-1} h(z) dz \right)^{3(2\gamma+1)} \left(\int_{\ln \sqrt{2t+1}}^{2t} h(y) dy \right)^3 h(t) dt \end{aligned}$$

and

$$\begin{aligned} B_{1,\sigma=\phi}^6 &:= \\ &:= \int_0^\infty \left(\int_{t/2}^{\exp t-1} h(x) dx \right)^2 \left(\int_{\ln \sqrt{t+1}}^t \left[\int_y^t h(z) dz \right]^{3\gamma/2} h(y) dy \right)^4 h(t) dt \\ &= \int_0^\infty \left(\int_{t/2}^{\exp t-1} h(x) dx \right)^2 \left(\int_{\ln \sqrt{t+1}}^t \left[\int_y^t h(z) dz \right]^{3\gamma/2} d \left[- \int_y^t h(z) dz \right] \right)^4 h(t) dt \\ &= \left(\frac{2}{3\gamma+2} \right)^4 \int_0^\infty \left(\int_{t/2}^{\exp t-1} h(x) dx \right)^2 \left(\int_{\ln \sqrt{t+1}}^t h(z) dz \right)^{6\gamma+4} h(t) dt. \end{aligned}$$

Let $h(z) = \exp(-z)$, then

$$\begin{aligned}
B_{1,\rho=\phi^{-1}}^6 &= \frac{1}{(2\gamma+1)^3} \int_0^\infty \left(\int_t^{\exp 2t-1} \exp(-z) dz \right)^{3(2\gamma+1)} \\
&\quad \times \left(\int_{\ln \sqrt{2t+1}}^{2t} \exp(-y) dy \right)^3 \exp(-t) dt \\
&\leq \frac{1}{(2\gamma+1)^3} \int_0^\infty \exp(-3(2\gamma+1)t) \frac{1}{(2t+1)^{3/2}} \exp(-t) dt \\
&= \frac{1}{(2\gamma+1)^3} \int_0^\infty \frac{\exp(-2(3\gamma+2)t)}{(2t+1)^{3/2}} dt \\
&\leq \frac{1}{(2\gamma+1)^3} \int_0^\infty \exp(-2(3\gamma+2)t) dt \\
&= \frac{1}{2(3\gamma+2)(2\gamma+1)^3} < \infty
\end{aligned}$$

and

$$\begin{aligned}
B_{1,\sigma=\phi}^6 &= \left(\frac{2}{3\gamma+2} \right)^4 \int_0^\infty \left(\int_{t/2}^{\exp t-1} \exp(-x) dx \right)^2 \\
&\quad \times \left(\int_{\ln \sqrt{t+1}}^t \exp(-z) dz \right)^{6\gamma+4} \exp(-t) dt \\
&\leq \left(\frac{2}{3\gamma+2} \right)^4 \int_0^\infty \frac{\exp(-2t)}{(t+1)^{3\gamma+2}} dt \leq \frac{8}{(3\gamma+2)^4} < \infty.
\end{aligned}$$

Thus, by Theorem 3.1 the operator $K_{2,1}$ is bounded and compact from L_3 to L_2 . Analogously by Theorem 3.2 with $\phi_2(x) = 2x$ we can prove the boundedness and compactness of $K_{2,2}$. Therefore, if $h(z) = \exp(-z)$, then the initial operator K_2 is bounded and compact from L_3 to L_2 .

The last example is about a criterion case from Theorem 5.2.

Example 6.3. Let $q = 3/2$, $p = 3$, $v = w = 1$, $k(x, y) = \frac{1}{(xy+1)^{4/3}}$, $a(x) = x/2$, $b(x) = 2x$:

$$K_3 f(x) = \int_{x/2}^{2x} \frac{f(y) dy}{(xy+1)^{4/3}}.$$

It holds that $k(x, y) \in \mathcal{O}_a \cap \mathcal{O}_b$ because for $x/2 \leq z/2 = a(z) \leq y \leq b(x) = 2x \leq 2z$ we have $k(x, a(z)) = \frac{1}{x \cdot z/2+1} \leq \frac{1}{x \cdot y/4+1} < 4k(x, y)$, $k(z, y) = \frac{1}{zy+1} \leq k(x, y)$, $k(x, a(z)) + k(z, y) > \frac{1}{x \cdot z/2+1} \geq k(x, y)$ as well as for $z/2 \leq x/2 = a(x) \leq y \leq b(z) = 2z \leq 2x$ it holds that

$k(x, b(z)) = \frac{1}{2xz+1} \leq k(x, y)$, $k(z, y) = \frac{1}{zy+1} \leq \frac{1}{y \cdot x/4+1} < 4k(x, y)$,
 $k(x, b(z)) + k(z, y) > \frac{1}{zy+1} > k(x, y)$. Since $a^{-1}(y) = 2y$ and $b^{-1}(y) = y/2$
we write by (3.2)

$$\int_{y/2}^{\rho(y)} \frac{dx}{(xy+1)^2} = \int_{\rho(y)}^{2y} \frac{dx}{(xy+1)^2} \implies \int_{y \cdot \rho(y)+1}^{y^2/2+1} dz^{-1} = \int_{2y^2+1}^{y \cdot \rho(y)+1} dz^{-1},$$

which yields

$$\rho(y) = \frac{y(4y^2+5)}{5y^2+4}.$$

Since $a = b^{-1}$ we have by (3.1) that

$$\sigma(x) = \frac{x(4x^2+5)}{5x^2+4}$$

too. Important that $\rho = \sigma$ are strictly increasing functions. Thus, since $q = p' = 3/2$, $p = q' = 3$, $r = 3$, $a = b^{-1}$ and $(4t^2+5)/(5t^2+4) \leq 5/4$ we obtain the condition

$$\begin{aligned} B_{\rho=\sigma}^3 &= \frac{27}{2} \int_0^\infty \frac{t^3(4t^2+5)dt}{(t^2+2)^2(2t^2+1)^2(5t^2+4)} \leq \frac{27 \cdot 5}{2 \cdot 4} \int_0^\infty \frac{t^3 dt}{(t^2+2)^2(2t^2+1)^2} \\ &< \frac{135}{8} \int_0^\infty \frac{tdt}{(t^2+2)(2t^2+1)^2} \leq \frac{135}{32} \int_1^\infty \frac{dz}{z^3} = \frac{135}{64} < \infty, \end{aligned}$$

which is necessary and sufficient for the boundedness and compactness of K_3 by Theorem 5.2 (b). This fact can be also proved by using Theorem 5.1.

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References

- [1] A. L. Bernardis, F. J. Martin-Reyes and P. Ortega Salvador, *Weighted inequalities for Hardy-Steklov operators*, Canad. J. Math., 59 (2007), 276–295.

- [2] A. L. Bernardis, F. J. Martin-Reyes and P. Ortega Salvador, *Weighted weak type inequalities for generalized Hardy operators*, J. Inequal. Appl., 2006, Art. ID 62426, 10 pp.
- [3] T. Chen and G. Sinnamon, *Generalized Hardy operators and normalizing measures*, J. Ineq. Appl., 7 (2002), 829–866.
- [4] A. Gogatishvili and J. Lang, *The generalized Hardy operators with kernel and variable integral limits in Banach function spaces*, J. Inequal. Appl., 4 (1999), 1–16.
- [5] M. L. Goldman, H. P. Heinig, and V. D. Stepanov, *On the principle of duality in Lorentz spaces*, Canad. J Math., 48 (1996), 959–979.
- [6] H. P. Heinig and G. Sinnamon, *Mapping properties of integral averaging operators*, Studia Math., 129 (1998), 157–177.
- [7] E. Lomakina and V. D. Stepanov, *On the Hardy-type integral operators in Banach function spaces*, Publ. Mat., 42 (1998), 165–194.
- [8] V. D. Stepanov, *Weighted norm inequalities of Hardy type for the class of integral operators*, J. London Math. Soc., 50 (1994), 105–120.
- [9] V. D. Stepanov and E. P. Ushakova, *On integral operators with variable limits of integration*, Proc. Steklov Inst. Math., 232 (2001), 290–309.
- [10] V. D. Stepanov and E. P. Ushakova, *Hardy operator with variable limits on monotone functions*, J. Funct. Spaces Appl., 1 (2003), 1–15.
- [11] V. D. Stepanov and E. P. Ushakova, *On the geometric mean operator with variable limits of integration*, Proc. Steklov Inst. Math., 260 (2008), 254–278.

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