Inner functions in Q_K type spaces¹

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Abstract. Inner functions in $Q_K(p, q)$ are studied, provided K satisfies certain regularity conditions. In particular, it is shown that the only inner functions in $Q_K(p, p-2)$, $p \geq 1$, are precisely the Blaschke products whose zeros $\{z_n\}$ satisfy $\sup_{a \in \mathbb{D}} \sum K(1 - |\varphi_a(z_n)|^2) < \infty$.

1. Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions in the unit disc $\mathbb{D} := \{z : |z| < 1\}$ of the complex plane. Let the Green's function of $\mathbb{D}$ be defined as $g(z, a) := -\log |\varphi_a(z)|$, where $\varphi_a(z) := (a - z)/(1 - \bar{a}z)$ is the automorphism of $\mathbb{D}$ which interchanges the origin and the point $a \in \mathbb{D}$. Let $0 < p < \infty$ and $-2 < q < \infty$, and let $K : [0, \infty) \rightarrow [0, \infty)$ be an arbitrary function. The space $Q_K(p, q)$ consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$(1.1) \quad \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \infty,$$

where dA is the element of the Lebesgue area measure on $\mathbb{D}$.

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The space $Q_K(p, q)$ was recently introduced in [13]. The definition is strongly motivated by the Q_p -theory and studies on Q_K and $F(p, q, s)$. While Q_K and $F(p, q, s)$ are different yet natural generalizations of Q_p , the definition of $Q_K(p, q)$ generalizes the concept of both of these spaces. Namely, $Q_K(2, 0) = Q_K$ and $Q_K(p, q) = F(p, q, s)$ when $K(t) = t^s$. In particular, if $K(t) = t$, then Q_K coincides with $BMOA$, the space of analytic functions in the Hardy space H^1 with boundary values of bounded mean oscillation [1]. The references [16, 17] and [10, 18] are mentioned as sources for the Q_p -theory and $F(p, q, s)$, while fundamental results on Q_K and $Q_K(p, q)$ can be found in [4, 5, 12, 15] and [13, 14], see also references therein.

The purpose of the present paper is to study inner functions in $Q_K(p, q)$ and $Q_{K,0}(p, q)$. The space $Q_{K,0}(p, q)$ is a subspace of $Q_K(p, q)$, and consists of those $f \in \mathcal{H}(\mathbb{D})$ for which the integral expression in (1.1) tends to zero as a approaches to the boundary $\mathbb{T} := \{z : |z| = 1\}$. The main results are stated in this section, but their proofs are postponed to forthcoming ones. In order to give the precise statements, the weight function K has to be specified. Throughout the paper, it is always assumed that $K : [0, \infty) \rightarrow [0, \infty)$ has the following properties:

- (a) K is nondecreasing, right-continuous, and vanishes at the origin and nowhere else;
- (b) $\int_0^1 (1 - r^2)^q K\left(\log \frac{1}{r}\right) r \, dr < \infty$;
- (c) $K(s) = K(1)$ for all $s \geq 1$;
- (d) $K(2s) \simeq K(s)$ for all $s \geq 0$.

The condition (a) ensures two things. First, the nondecreasing nature of K implies that $Q_K(p, q)$ is a subspace of $\mathcal{B}^{\frac{q+2}{p}}$ [13]. Second, if K does not vanish at the origin, then the weight function does not play any role in the definition, that is, the term $K(g(z, a))$ can be replaced by 1. As usual, for $\alpha > 0$, $\mathcal{B}^\alpha$ stands for the α -Bloch space which consists of those $f \in \mathcal{H}(\mathbb{D})$ for which $|f'(z)|(1 - |z|^2)^\alpha$ is uniformly bounded in $\mathbb{D}$. The condition (b) ensures that $Q_K(p, q)$ is not trivial [13], and (b) and (c) together imply that $Q_K(p, q)$ is determined by the behavior of K near the origin only. Note that (c) is not really a restriction since $Q_K(p, q) = Q_{K^*}(p, q)$ for K^* which is defined as $K^*(s) := K(s)$ if $0 \leq s \leq 1$ and $K^*(s) := K(1)$ if $s \geq 1$ [4]. The symbol $\simeq$ in (d) refers to the asymptotic equality which means that the quantities in other sides of the symbol are comparable, that is, their quotient is bounded and bounded away from zero.

In addition to the basic assumptions (a)-(d), the behavior of K has to be further restricted. In order to do so, denote $\varphi_K(s) := \sup_{0 < t \leq 1} K(st)/K(t)$.

In several occasions the following regularity properties appear to be essential:

- (A) $\int_1^\infty \varphi_K(s) s^{-2} ds < \infty$;
- (B) $\int_0^1 \varphi_K(s) s^{-1} ds < \infty$;
- (C) $\int_1^\infty (\varphi_K(s))^{\frac{1}{p}} s^{\frac{q+1}{p}-2} ds < \infty$.

The condition (A) is a restriction since it is satisfied for $K(s) = s^p$ if and only if $0 < p < 1$. Moreover, it is known that (A) implies (d) [15]. The conditions (B) and (C) play an essential role in the study of inner functions. The restriction (B) is of different nature than (A). For example, $K(s) = s^p$ satisfies (B) for all $p > 0$, but (B) fails for $K(s) = (1 - \log(1 - e^{-s}))^{-1}$. Note that in the latter case $K(s) \simeq (\log \frac{1}{s})^{-1}$ near the origin, and therefore $Q_K \subsetneq Q_p$ for all $p > 0$ [4]. The conditions (A) and (B) imply that K can be assumed to have the following properties:

- (i) K is concave;
- (ii) There is a $c > 0$ such that $\varphi_K(s) \leq s^{1-c}$ for all $s \geq 1$;
- (iii) $K(s)/s$ is decreasing and convex such that $\lim_{s \rightarrow 0^+} K(s)/s = \infty$.

Namely, if (A) is satisfied, then there is a concave weight K^* such that $Q_{K^*}(p, q) = Q_K(p, q)$ [5, 15]. Analogously, (A) and (B) together imply that $Q_{K^*}(p, q) = Q_K(p, q)$ for a suitably chosen weight K^* with the properties (ii) and (iii) [5].

Recall that $f \in \mathcal{H}(\mathbb{D})$ for which $|f(z)| \leq 1$ in $\mathbb{D}$ is an inner function if its modulus equals one almost everywhere on the boundary $\mathbb{T}$. It is well known that every inner function can be represented as a product of a Blaschke product and a singular inner function [2]. For a given sequence $\{z_n\}$ in $\mathbb{D}$ for which $\sum_{n=1}^\infty (1 - |z_n|^2)$ converges, the Blaschke product associated with a sequence $\{z_n\}$ is defined as

$$B(z) := \prod_{n=1}^\infty \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z} = \prod_{n=1}^\infty \frac{|z_n|}{z_n} \varphi_{z_n}(z).$$

Singular inner functions are of the form

$$S_\sigma(z) := \exp \left(\int_{\mathbb{T}} \frac{z+w}{z-w} d\sigma(w) \right),$$

where the measure σ on $\mathbb{T}$ is singular with respect to the Lebesgue measure. If the measure σ is atomic and consists of a point mass concentrated in a point $w \in \mathbb{T}$, then S is of the form

$$S_{\gamma, w}(z) := \exp \left(\gamma \frac{z+w}{z-w} \right),$$

where $0 < \gamma < \infty$.

The following auxiliary result deserves to be stated separately. It shows that applying the Schwarz-Pick lemma to the modulus of the derivative of an inner function inside of a radial integral does not cause any essential loss of information. Lemma 1 is proved by following the reasoning in the proof of [5, Lemma 5.1].

Lemma 1. *Let $1 \leq p < \infty$ and $-1 < q < \infty$, and assume that K satisfies (C). If S is an inner function, then*

$$(1.2) \quad \begin{aligned} & \int_{\delta}^1 (1 - |S(re^{i\theta})|^2)^p (1 - r^2)^{q-p} K\left(\frac{1-r}{1-\delta}\right) dr \\ & \simeq \int_{\delta}^1 |S'(re^{i\theta})|^p (1 - r^2)^q K\left(\frac{1-r}{1-\delta}\right) dr \end{aligned}$$

for all $0 \leq \delta < 1$ and almost all θ in $[0, 2\pi)$.

The proof of Theorem 2 relies strongly on Lemma 1 and a characterization of functions in $Q_K(p, q)$ in terms of so-called K -Carleson measures, introduced recently in [5]. A positive measure μ on $\mathbb{D}$ is a (bounded) K -Carleson measure if

$$\sup_I \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) d\mu(z) < \infty,$$

where $|I|$ denotes the arc length of a subarc I of $\mathbb{T}$, $S(I) := \{z \in \mathbb{D} : z/|z| \in I, 1 - |I| \leq |z|\}$ is the Carleson box based on I , and the supremum is taken over all subarcs I of $\mathbb{T}$ such that $|I| \leq 1$. Therefore the set $T(I)$ in Theorem 2 is the intersection of the Carleson box $S(I)$ and the annulus $\{z : 1 - |I|^2 \leq |z| < 1\}$.

Theorem 2. *Let $1 \leq p < \infty$ and $-1 < q < \infty$, and assume that K satisfies (B) and (C). Let S be a singular inner function, and denote $T(I) := \{z : z/|z| \in I, 1 - |I|^2 \leq |z| < 1\}$. Then $S \in Q_K(p, q)$ if and only if*

$$(1.3) \quad \int_{|I|}^1 K(s) s^{q-p} ds \lesssim |I|^{p-q-2}$$

for all subarc I of $\mathbb{T}$, and

$$(1.4) \quad \sup_I \int_{T(I)} (1 - |S(z)|)^p (1 - |z|)^{q-p} K\left(\frac{1-|z|}{|I|}\right) dA(z) < \infty.$$

As usual, the notation $A \lesssim B$ means that there is a positive constant C such that $A \leq CB$. The constant of comparison may depend on fixed parameters such as p , q and K , but is always independent of the essential variables of the quantities A and B . For example, in the case of (1.3), the C behind the symbol $\lesssim$ depends on the generating measure σ , but is independent of $|I|$. The notation $\gtrsim$ is understood in a similar fashion. In particular, if $A \lesssim B \lesssim A$, then $A \simeq B$.

If $K(t) = t^s$, then (1.3) reduces to the inequality $p \leq q + \frac{s+3}{2}$. It is known that this condition is necessary and sufficient for $S_{\gamma,w}$ to belong to $F(p, q, s)$ for all $0 < p < \infty$, $-2 < q < \infty$ and $0 < s \leq 1$ such that $q + s > -1$ [9]. It remains open whether (1.3) alone characterizes atomic singular inner functions in $Q_K(p, q)$ under the assumptions of Theorem 2.

The statement of Theorem 2 might look a bit impractical, but it appears to be useful since the condition (1.3) is easy to handle, as is seen next. The following lemma will be used also later on when studying Blaschke products.

Lemma 3. *Let $t \in \mathbb{R}$ and $0 < r_0 < 1$, and assume that K satisfies (d). Then*

$$(1.5) \quad K(r)r^{t+1} \lesssim \int_0^r K(s)s^t ds \leq K(r)r^{t+1} \int_0^1 \varphi_K(s)s^t ds, \quad 0 < r < 1,$$

and

$$(1.6) \quad K(r)r^{t+1} \lesssim \int_r^1 K(s)s^t ds \leq K(r)r^{t+1} \int_1^\infty \varphi_K(s)s^t ds, \quad 0 < r \leq r_0.$$

If K satisfies (A) and (B), then (ii) implies that (C) is satisfied for $q = p - 2$. Therefore Theorem 2 ensures that, for all $p \geq 1$, a necessary condition for a singular inner function S to belong to $Q_K(p, p - 2)$ is that the integral $\int_{|I|}^1 s^{-2} K(s) ds$ remains bounded as $|I| \rightarrow 0^+$. Further, since (A) implies (d), the (asymptotic) inequalities in (1.6) and (iii) yield $\int_{|I|}^1 s^{-2} K(s) ds \simeq K(|I|)/|I| \rightarrow \infty$, as $|I| \rightarrow 0^+$. It follows that $S \notin Q_K(p, p - 2)$ for $p \geq 1$. But now (a) implies that $Q_K(p, p - 2)$ is a subset of the Bloch space, and hence $Q_K(p, p - 2)$ gets larger with p . Therefore $S \notin Q_K(p, p - 2)$ for all $p > 0$. Since obviously $Q_K(p, q)$ gets larger with q , the following result is deduced.

Corollary 4. *Let $0 < p < \infty$ and $-1 < q < \infty$ such that $p \geq q + 2$, and assume that K satisfies (A) and (B). Then $Q_K(p, q)$ does not contain any singular inner functions.*

In the study of Blaschke products we need the following auxiliary result which is also of independent interest. It should be compared with the well-known Forelli-Rudin estimates, see, for example, [16, Lemma 1.4.1]

or [8, Proposition 2.1]. As a consequence we obtain alternative proofs for asymptotic inequalities (5.5) and (5.7) in [5]. Note that these inequalities were originally proved by passing to the upper half plane while the proof presented in Section 4 is done in the unit disc.

Lemma 5. *Let $c, t \in \mathbb{R}$ such that $c > -1$ and $c \geq t$, and assume that K satisfies (d) and*

$$(1.7) \quad \int_0^1 \varphi_K(s) s^t ds < \infty \quad \text{and} \quad \int_1^\infty \varphi_K(s) s^{t-c-1} ds < \infty.$$

Then

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^t K(1 - |z|^2)}{|1 - \bar{a}z|^{c+2}} dA(z) \simeq \frac{K(1 - |a|^2)}{(1 - |a|^2)^{c-t}}, \quad a \in \mathbb{D}.$$

The main result of the present paper on inner functions is Theorem 6. It gives a complete characterization of inner functions in $Q_K(p, p-2)$, provided $p \geq 1$ and K satisfies (A) and (B). It therefore generalizes the corresponding result in [5]. It is worth noticing that $Q_K(p, p-2)$ is a Möbius invariant subspace of the Bloch space for all p , and therefore of special interest among the spaces $Q_K(p, q)$.

Theorem 6. *Assume that K satisfies (A) and (B), and let S be an inner function. Then $S \in Q_K(p, p-2)$ for some (equivalently for all) $p \geq 1$ if and only if S is the Blaschke product associated with a sequence $\{z_n\}$ which satisfies*

$$(1.8) \quad \sup_{a \in \mathbb{D}} \sum_{n=1}^{\infty} K(1 - |\varphi_a(z_n)|^2) < \infty.$$

To see that the assumption (A) in Theorem 6 is necessary, consider $K(s) = s$. In this case, $Q_K(2, 0) = BMOA$ and therefore it contains all bounded analytic functions, while (1.8) now reads $\sup_{a \in \mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2) < \infty$. It is known that this condition is satisfied if and only if $\{z_n\}$ is a finite union of uniformly separated sequences [3].

The remainder of the paper is organized as follows. The results presented in this section are proved in Sections 2-5 in the sequential order. In addition, Section 5 contains Theorem 8 and Corollary 11 of which the first one gives a sufficient condition for the zeros of a Blaschke product B such that $B \in Q_K(p, q)$, and the second one characterizes Blaschke products in $Q_K(p, p-2)$ when $\frac{1}{2} < p < 1$.

2. Proof of Lemma 1

Let S be an inner function, and denote

$$I := \int_{\delta}^1 (1 - |S(re^{i\theta})|^2)^p (1 - r^2)^{q-p} K\left(\frac{1-r}{1-\delta}\right) dr.$$

By the Schwarz-Pick lemma, $|S'(z)|(1 - |z|^2) \leq 1 - |S(z)|^2$ for all $z \in \mathbb{D}$, and therefore

$$I \geq \int_{\delta}^1 |S'(re^{i\theta})|^p (1 - r^2)^q K\left(\frac{1-r}{1-\delta}\right) dr$$

for all p, q, K, δ and θ . To prove the other direction, the proof of [5, Lemma 5.1] is followed. Let $p \geq 1$. Since S is an inner function, the inequality

$$1 - |S(re^{i\theta})| \leq \int_r^1 |S'(se^{i\theta})| ds = \int_0^1 |S'((t + (1-t)r)e^{i\theta})|(1-r) dt$$

holds for almost all θ in $[0, 2\pi)$. Therefore Minkowski's integral inequality (Fubini's theorem in the case $p = 1$) yields

$$\begin{aligned} I &\lesssim \left(\int_0^1 \left(\int_{\delta}^1 |S'((t + (1-t)r)e^{i\theta})|^p (1-r)^q K\left(\frac{1-r}{1-\delta}\right) dr \right)^{\frac{1}{p}} dt \right)^p \\ &= \left(\int_0^1 \left(\int_{\delta+t(1-\delta)}^1 |S'(ue^{i\theta})|^p \left(\frac{1-u}{1-t}\right)^q K\left(\frac{1-u}{(1-t)(1-\delta)}\right) \frac{du}{1-t} \right)^{\frac{1}{p}} dt \right)^p. \end{aligned}$$

The definition of φ_K and the inequality $\delta \leq \delta + t(1-\delta)$ give

$$\begin{aligned} I &\lesssim \left(\int_{\delta}^1 |S'(ue^{i\theta})|^p (1-u)^q K\left(\frac{1-u}{1-\delta}\right) du \right) \left(\int_0^1 \left(\varphi_K\left(\frac{1}{1-t}\right) \right)^{\frac{1}{p}} \frac{dt}{(1-t)^{\frac{1}{p} + \frac{q}{p}}} \right)^p \\ &= \left(\int_{\delta}^1 |S'(ue^{i\theta})|^p (1-u)^q K\left(\frac{1-u}{1-\delta}\right) du \right) \left(\int_1^{\infty} (\varphi_K(s))^{\frac{1}{p}} s^{\frac{q+1}{p}-2} ds \right)^p, \end{aligned}$$

and the proof is complete.

3. Singular inner functions; proof of Theorem 2

The proof relies on Lemma 1 and the following characterization of functions in $Q_K(p, q)$ which follows readily by the proof of [5, Theorem 3.1].

Theorem 1. *Let $0 < p < \infty$ and $-2 < q < \infty$, assume that K satisfies (B). Let $f \in \mathcal{H}(\mathbb{D})$. Then the following statements are equivalent:*

- (1) $f \in Q_K(p, q)$;
- (2) $\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) < \infty$;
- (3) *The measure μ such that $d\mu(z) = |f'(z)|^p (1 - |z|^2)^q dA(z)$ is a K -Carleson measure.*

Let $1 \leq p < \infty$ and $-1 < q < \infty$, and assume that K satisfies (B) and (C). Let S be a singular inner function. By Theorem 1 and Lemma 1, $S \in Q_K(p, q)$ if and only if (1.4) with $T(I)$ being replaced by the Carleson box $S(I)$ is satisfied. Therefore it suffices to show that

$$(3.1) \quad \int_{S(I) \setminus T(I)} (1 - |S(z)|)^p (1 - |z|)^{q-p} K\left(\frac{1 - |z|}{|I|}\right) dA(z)$$

is uniformly bounded if and only if (1.3) is satisfied. To see this, the atomic and non-atomic cases are considered separately.

3.1. Atomic case. If S is of the form

$$S_{\gamma, w}(z) = \exp\left(\gamma \frac{z + w}{z - w}\right),$$

where $w \in \mathbb{T}$ and $0 < \gamma < \infty$, then

$$(3.2) \quad |S_{\gamma, w}(z)| = \exp\left(-\gamma \frac{1 - |z|^2}{|z - w|^2}\right) = \exp\left(-\gamma \frac{1 - |z|^2}{|1 - \bar{w}z|^2}\right).$$

Let $w = e^{i\theta}$ and $z = re^{it}$, and denote

$$J(t) := \int_{1-|I|}^{1-|I|^2} \left(1 - \exp\left(-\gamma \frac{1 - r^2}{|1 - re^{i(t-\theta)}|^2}\right)\right)^p (1 - r)^{q-p} K\left(\frac{1 - r}{|I|}\right) dr.$$

Then

$$J(t) \leq \int_{1-|I|}^{1-|I|^2} (1 - r)^{q-p} K\left(\frac{1 - r}{|I|}\right) dr = |I|^{q-p+1} \int_{|I|}^1 s^{q-p} K(s) ds,$$

and it follows that the integral in (3.1) is uniformly bounded if (1.3) is satisfied.

Conversely, consider an interval $I \subset \mathbb{T}$ centered at w , if $e^{it} \in I$, then

$$\begin{aligned}
J(t) &\geq \int_{1-|I|}^{1-|I|^2} \left(1 - \exp \left(-\gamma \frac{1-r}{|1-re^{i(t-\theta)}|^2} \right) \right)^p (1-r)^{q-p} K \left(\frac{1-r}{|I|} \right) dr \\
&\geq |I|^{q-p+1} \left(1 - e^{-c\frac{\gamma}{2}} \right)^p \int_{|I|}^1 s^{q-p} K(s) ds,
\end{aligned}$$

for some universal constant c , and it follows that (1.3) is satisfied if the integral in (3.1) is uniformly bounded.

3.2. Non-atomic case. If

$$S_\sigma(z) = \exp \left(\int_{\mathbb{T}} \frac{z+w}{z-w} d\sigma(w) \right),$$

where σ is a non-atomic singular measure on $\mathbb{T}$, then

$$(3.3) \quad |S_\sigma(z)| = \exp \left(- \int_{\mathbb{T}} \frac{1-|z|^2}{|w-z|^2} d\sigma(w) \right) = \exp \left(- \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{w}z|^2} d\sigma(w) \right).$$

Denote

$$J(t) := \int_{1-|I|}^{1-|I|^2} \left(1 - \exp \left(- \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{w}z|^2} d\sigma(w) \right) \right)^p (1-r)^{q-p} K \left(\frac{1-r}{|I|} \right) dr.$$

As in the atomic case we see that the integral in (3.1) is uniformly bounded if (1.3) is satisfied. Conversely,

$$\begin{aligned}
J(t) &\geq \int_{1-|I|}^{1-|I|^2} \left(1 - \exp \left(- \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{w}z|^2} d\sigma(w) \right) \right)^p (1-r)^{q-p} K \left(\frac{1-r}{|I|} \right) dr \\
&\geq |I|^{q-p+1} \left(1 - e^{-\frac{\sigma(\mathbb{T})}{2}} \right)^p \int_{|I|}^1 s^{q-p} K(s) ds,
\end{aligned}$$

and it follows that (1.3) is satisfied if the integral in (3.1) is uniformly bounded.

4. Proofs of Lemmas 3 and 5

The proof of Lemma 5 relies on Lemma 3 which is proved first. The author wishes to thank J. Rttý for offering the following proofs.

4.1. Proof of Lemma 3. Since K is nondecreasing,

$$\begin{aligned}
 \int_0^r K(s)s^t ds &\geq \int_{r/2}^r K(s)s^t ds \geq K(r/2) \int_{r/2}^r s^t ds \\
 (4.1) \quad &= K(r/2)r^{t+1} \cdot \begin{cases} \frac{1-2^{-(t+1)}}{t+1}, & t \neq -1 \\ \log 2, & t = -1, \end{cases} \\
 &\simeq K(r)r^{t+1},
 \end{aligned}$$

and the lower bound in (1.5) is proved. Moreover,

$$\begin{aligned}
 \int_0^r K(s)s^t ds &= \int_0^1 K(rs)(rs)^t r ds = K(r)r^{t+1} \int_0^1 \frac{K(rs)}{K(r)} s^t ds \\
 &\leq K(r)r^{t+1} \int_0^1 \varphi_K(s)s^t ds,
 \end{aligned}$$

which is the upper bound in (1.5).

Since $0 < r \leq r_0$,

$$\int_r^1 K(s)s^t ds \geq K(r) \int_r^{r/r_0} s^t ds,$$

and the lower bound follows as in (4.1). Moreover,

$$\begin{aligned}
 \int_r^1 K(s)s^t ds &= K(r) \int_r^1 \frac{K(r\frac{s}{r})}{K(r)} s^t ds \leq K(r) \int_r^1 \varphi_K(s/r)s^t ds \\
 &\leq K(r)r^{t+1} \int_1^\infty \varphi_K(s)s^t ds,
 \end{aligned}$$

which is the upper bound in (1.6).

4.2. Proof of Lemma 5. Let $c, t \in \mathbb{R}$ such that $c > -1$ and $c \geq t$, and assume that K satisfies (1.7). Denote

$$I_{t,c,K}(a) := \int_{\mathbb{D}} \frac{(1-|z|^2)^t K(1-|z|^2)}{|1-\bar{a}z|^{c+2}} dA(z).$$

For $a \in \mathbb{D}$, let $I_a := \{e^{i\zeta} : |\zeta - \arg a| \leq \frac{1}{2}(1-|a|)\}$, so that $|I_a| = 1-|a|$. Let $z \in S(I_a)$. Then

$$1 - \bar{a}z = 1 - \bar{a}(a + (z-a)) = 1 - |a|^2 - \bar{a}(z-a),$$

and therefore

$$|1 - \bar{a}z| \leq 1 - |a|^2 + |a||z-a| \leq 2(1-|a|) + |z-a|.$$

Now

$$\begin{aligned}
 |z - a|^2 &\leq \left| e^{i(\arg a + \frac{1-|a|}{2})} - |a|e^{i \arg a} \right|^2 = \left| 1 - |a|e^{-i\frac{1-|a|}{2}} \right|^2 \\
 &\leq 1 - 2|a| \left(1 - \frac{(1-|a|)^2}{8} \right) + |a|^2 = (1-|a|)^2 \left(1 + \frac{|a|}{4} \right) \\
 &\leq \frac{5}{4}(1-|a|)^2,
 \end{aligned}$$

and it follows that

$$|1 - \bar{a}z| \leq \left(2 + \frac{\sqrt{5}}{2} \right) (1 - |a|).$$

Therefore

$$\begin{aligned}
 I_{t,c,K}(a) &\geq \int_{S(I_a)} \frac{(1-|z|^2)^t K(1-|z|^2)}{|1 - \bar{a}z|^{c+2}} dA(z) \\
 &\gtrsim \frac{1}{(1-|a|)^{c+2}} \int_{S(I_a)} (1-|z|^2)^t K(1-|z|^2) dA(z) \\
 &\gtrsim \frac{1}{(1-|a|)^{c+1}} \int_0^{1-|a|} s^t K(s) ds
 \end{aligned}$$

from which Lemma 3 and (d) yield the desired lower estimate

$$I_{t,c,K}(a) \gtrsim \frac{K(1-|a|^2)}{(1-|a|^2)^{c-t}}.$$

To see the other direction, let $0 < r_0 < 1$ be fixed. If $1 - |a| > r_0$, then clearly

$$(4.2) \quad I_{t,c,K}(a) \leq \frac{\pi}{r_0^{c+2} K(r_0)} \int_0^1 s^t K(s) ds \frac{K(1-|a|^2)}{(1-|a|^2)^{c-t}} \lesssim \frac{K(1-|a|^2)}{(1-|a|^2)^{c-t}}.$$

If $1 - |a| \leq r_0$, then [2, p. 65 Lemma] gives

$$(4.3) \quad I_{t,c,K}(a) \lesssim \int_0^1 \frac{(1-r)^t K(1-r)}{(1-|a|r)^{c+1}} dr$$

since $c > -1$ by the assumption. By Lemma 3 and (d),

$$(4.4) \quad \int_0^{|a|} \frac{(1-r)^t K(1-r)}{(1-|a|r)^{c+1}} dr \leq \int_{1-|a|}^1 s^{t-c-1} K(s) ds \simeq \frac{K(1-|a|^2)}{(1-|a|^2)^{c-t}}$$

and

$$(4.5) \quad \int_{|a|}^1 \frac{(1-r)^t K(1-r)}{(1-|a|r)^{c+1}} dr \leq \frac{1}{(1-|a|)^{c+1}} \int_0^{1-|a|} s^t K(s) ds \simeq \frac{K(1-|a|^2)}{(1-|a|^2)^{c-t}}.$$

The upper estimate follows by combining (4.2)-(4.5).

5. Blaschke products; proof of Theorem 6

The proof of Theorem 6 is done in several steps. Some of these steps are essentially known, but for convenience of the reader the proofs are given in detail. The final reasoning which yields the assertion of Theorem 6 is given in Section 5.

We begin with a consequence of Lemma 1 which shows that a finite product of inner functions belongs to $Q_K(p, q)$ if and only if it's every member does. This is a natural analogue of [5, Corollary 5.1].

Proposition 7. *Let $1 \leq p < \infty$ and $-1 < q < \infty$, and assume that K satisfies (B) and (C). Let $S := \prod_{j=1}^n S_j$, where S_j is an inner function for all $j = 1, 2, \dots, n$. Then $S \in Q_K(p, q)$ if and only if $S_j \in Q_K(p, q)$ for all $j = 1, 2, \dots, n$. Similarly, $S \in Q_{K,0}(p, q)$ if and only if $S_j \in Q_{K,0}(p, q)$ for all $j = 1, 2, \dots, n$.*

Proof. Let first $S_j \in Q_K(p, q)$ for all $j = 1, 2, \dots, n$, and denote $S := \prod_{j=1}^n S_j$. Then $S' = \sum_{j=1}^n S'_j \prod_{k \neq j} S_k$ from which Hölder's inequality yields $|S'(z)|^p \leq n^{p-1} \sum_{j=1}^n |S'_j(z)|^p$ for all $z \in \mathbb{D}$, and it follows that $S \in Q_K(p, q)$.

Let now $S \in Q_K(p, q)$, and denote $d\mu(z) := K\left(\frac{1-|z|}{|I|}\right) dA(z)$. Since $|S_j(z)| \geq |S(z)|$, for all $z \in \mathbb{D}$ and $j = 1, 2, \dots, n$, Lemma 1 implies

$$\begin{aligned} \int_{S(I)} |S'_j(z)|^p (1-|z|^2)^q d\mu(z) &\leq \int_{S(I)} (1-|S_j(z)|^2)^p (1-|z|^2)^{q-p} d\mu(z) \\ &\leq \int_{S(I)} (1-|S(z)|^2)^p (1-|z|^2)^{q-p} d\mu(z) \\ &\lesssim \int_{S(I)} |S'(z)|^p (1-|z|^2)^q d\mu(z) \end{aligned}$$

from which Theorem 1 yields $S_j \in Q_K(p, q)$ for $j = 1, \dots, n$. The assertion for the little spaces follows readily by the proof above. $\square$

The following result gives a sufficient condition for the zeros of a Blaschke product B such that $B \in Q_K(p, q)$, provided K satisfies (B).

Proposition 8. *Let $0 < p < \infty$ and $-2 < q < \infty$, and assume that K satisfies (B). Let B be the Blaschke product associated with a sequence $\{z_n\}$. If either $0 < p \leq 1$ and*

$$\sup_{a \in \mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2)^p (1 - |a|^2)^{q+2-p} \int_{\mathbb{D}} \frac{(1 - |z|^2)^q K(1 - |z|^2)}{|1 - \overline{\varphi_a(z_n)}z|^{2p} |1 - \overline{a}z|^{2(q+2-p)}} dA(z) < \infty,$$

or $1 \leq p < \infty$ and

$$\sup_{a \in \mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2) (1 - |a|^2)^{q+2-p} \int_{\mathbb{D}} \frac{(1 - |z|^2)^{q+1-p} K(1 - |z|^2)}{|1 - \overline{\varphi_a(z_n)}z|^{2p} |1 - \overline{a}z|^{2(q+2-p)}} dA(z) < \infty,$$

then $B \in Q_K(p, q)$.

Proof. Since

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \frac{|z_n|^2 - 1}{(1 - \overline{z_n}z)(z_n - z)},$$

it follows that

$$\begin{aligned} |B'(z)| &= \left| \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{(1 - \overline{z_n}z)(z_n - z)} \right| \left| \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k}z} \right| \\ (5.1) \quad &\leq \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|z_n - z| |1 - \overline{z_n}z|} \frac{|z_n - z|}{|1 - \overline{z_n}z|} |B_n(z)| \leq \sum_{n=1}^{\infty} |\varphi'_{z_n}(z)|, \end{aligned}$$

where $B_n(z) = \prod_{k \neq n} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k}z}$. The inequality (5.1) yields

$$|(B \circ \varphi_a)'(z)| \leq \sum_{n=1}^{\infty} |\varphi'_{z_n}(\varphi_a(z))| |\varphi'_a(z)| = \sum_{n=1}^{\infty} |\varphi'_{\varphi_a(z_n)}(z)|,$$

and it follows that

$$|(B \circ \varphi_a)'(z)|^p \leq \sum_{n=1}^{\infty} |\varphi'_{\varphi_a(z_n)}(z)|^p$$

for $0 < p \leq 1$. The change of variable $w = \varphi_a(z)$ gives

$$\begin{aligned} &\int_{\mathbb{D}} |B'(w)|^p (1 - |w|^2)^q K(1 - |\varphi_a(w)|^2) dA(w) \\ &= \int_{\mathbb{D}} |(B \circ \varphi_a)'(z)|^p |\varphi'_a(z)|^{q+2-p} (1 - |z|^2)^q K(1 - |z|^2) dA(z) \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{D}} \sum_{n=1}^{\infty} \frac{(1 - |\varphi_a(z_n)|^2)^p}{|1 - \overline{\varphi_a(z_n)}z|^{2p}} |\varphi'_a(z)|^{q+2-p} (1 - |z|^2)^q K(1 - |z|^2) dA(z) \\
&= \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2)^p (1 - |a|^2)^{q+2-p} \int_{\mathbb{D}} \frac{(1 - |z|^2)^q K(1 - |z|^2)}{|1 - \overline{\varphi_a(z_n)}z|^{2p} |1 - \overline{a}z|^{2(q+2-p)}} dA(z),
\end{aligned}$$

from which Theorem 1 yields the desired result for $0 < p \leq 1$. If $1 \leq p < \infty$, then the Schwarz-Pick lemma and the change of variable $w = \varphi_a(z)$ give

$$\begin{aligned}
&\int_{\mathbb{D}} |B'(w)|^p (1 - |w|^2)^q K(1 - |\varphi_a(w)|^2) dA(w) \\
&= \int_{\mathbb{D}} |(B \circ \varphi_a)'(z)|^p |\varphi'_a(z)|^{q+2-p} (1 - |z|^2)^q K(1 - |z|^2) dA(z) \\
&\leq \int_{\mathbb{D}} \sum_{n=1}^{\infty} \frac{1 - |\varphi_a(z_n)|^2}{|1 - \overline{\varphi_a(z_n)}z|^2} |\varphi'_a(z)|^{q+2-p} (1 - |z|^2)^{q-p+1} K(1 - |z|^2) dA(z) \\
&= \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2) (1 - |a|^2)^{q+2-p} \int_{\mathbb{D}} \frac{(1 - |z|^2)^{q-p+1} K(1 - |z|^2)}{|1 - \overline{\varphi_a(z_n)}z|^2 |1 - \overline{a}z|^{2(q+2-p)}} dA(z),
\end{aligned}$$

and again Theorem 1 yields the assertion. $\square$

Undoubtedly the assertion in Proposition 8 looks nasty, but in the case $q = p - 2$ it combined with Lemma 5 yields the following result.

Corollary 9. *Let $\frac{1}{2} < p \leq 1$ and assume that K satisfies*

$$(5.2) \quad \int_0^1 \varphi_K(s) s^{p-2} ds < \infty \quad \text{and} \quad \int_1^\infty \varphi_K(s) s^{-p-1} ds < \infty.$$

Let B be the Blaschke product associated with a sequence $\{z_n\}$. If

$$(5.3) \quad \sup_{a \in \mathbb{D}} \sum_{n=1}^{\infty} K(1 - |\varphi_a(z_n)|^2) < \infty,$$

then $B \in Q_K(p, p - 2)$. In particular, if K satisfies (A) and (B), and the zeros of B satisfy (5.3), then $B \in Q_K(p, p - 2)$ for all $p \geq 1$.

As to the converse of Corollary 9, we prove the following result. The reasoning used in the proof can be found in the existing literature, see, for example, [5, 6, 9, 11].

Theorem 10. *Let $0 < p < \infty$ and assume that K satisfies (A) and (B). Let B be the Blaschke product associated with a sequence $\{z_n\}$. If*

$B \in Q_K(p, p-2)$, then

$$\sup_{a \in \mathbb{D}} \sum_{n=1}^{\infty} K(1 - |\varphi_a(z_n)|^2) < \infty.$$

Proof. Let $B \in Q_K(p, p-2)$. Since $Q_K(p, p-2)$ gets larger with p , we may assume that $p \geq 1$. Let $I \subset \mathbb{T}$ such that $|I| \leq \frac{1}{2}$. Since

$$2 \log |B(z)| = \sum_{n=1}^{\infty} \log (|\varphi_{z_n}(z)|^2) \leq - \sum_{n=1}^{\infty} (1 - |\varphi_{z_n}(z)|^2),$$

it follows that $|B(z)|^2 \leq \exp(-\sum_{n=1}^{\infty} (1 - |\varphi_{z_n}(z)|^2))$, and therefore

$$(5.4) \quad 1 - |B(z)|^2 \geq 1 - \exp\left(-\sum_{n=1}^{\infty} (1 - |\varphi_{z_n}(z)|^2)\right).$$

For shortness, denote $d\mu(z) := K\left(\frac{1-|z|}{|I|}\right) dA(z)$. Recall that (A) and (B) imply (C) via (ii). Therefore Lemma 1 and the inequality (5.4) yield

$$\begin{aligned} J_I &:= \int_{S(I)} |B'(z)|^p (1 - |z|^2)^{p-2} d\mu(z) \\ &\gtrsim \int_{S(I)} \left(1 - \exp\left(-\sum_{n=1}^{\infty} (1 - |\varphi_{z_n}(z)|^2)\right)\right)^p (1 - |z|^2)^{-2} d\mu(z) \\ &\geq \int_{S(I)} \left(1 - \exp\left(-(1 - |z|^2) \sum_{z_n \in S(I)} \frac{1 - |z_n|^2}{|1 - z\bar{z}_n|^2}\right)\right)^p (1 - |z|^2)^{-2} d\mu(z). \end{aligned}$$

Denoting $R(I) := \sum_{z_n \in S(I)} (1 - |z_n|^2)$, we obtain

$$\begin{aligned} J_I &\gtrsim \int_{S(I)} \left(1 - \exp\left(-\frac{1 - |z|}{2|I|^2} R(I)\right)\right)^p (1 - |z|^2)^{-2} d\mu(z) \\ &\simeq |I| \int_{1-|I|}^1 \left(1 - \exp\left(-\frac{1-r}{2|I|^2} R(I)\right)\right)^p (1 - r^2)^{-2} K\left(\frac{1-r}{|I|}\right) dr \\ &\simeq \left(\frac{R(I)}{|I|}\right) \int_0^{\frac{R(I)}{2|I|}} (1 - e^{-s})^p s^{-2} K\left(\frac{2|I|s}{R(I)}\right) ds \\ &\gtrsim \left(\frac{R(I)}{|I|}\right) \int_1^{\frac{R(I)}{2|I|}} s^{-2} K\left(\frac{2|I|s}{R(I)}\right) ds \simeq \int_{\frac{2|I|}{R(I)}}^1 K(t) \frac{dt}{t^2}. \end{aligned}$$

As in the reasoning before Corollary 4 we deduce that $\int_r^1 s^{-2} K(s) ds \simeq K(r)/r \rightarrow \infty$, as $r \rightarrow 0^+$. So, since $B \in Q_K(p, p-2)$, Theorem 1 implies that J_I is uniformly bounded, and it follows that $R(I) \lesssim |I|$, which yields

$$(5.5) \quad \sup_{z \in \mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_{z_n}(z)|^2) < \infty$$

by [7, VI Lemma 3.3]. From (5.4) and (5.5) we obtain

$$(5.6) \quad \begin{aligned} 1 - |B(z)|^2 &\geq \sum_{n=1}^{\infty} (1 - |\varphi_{z_n}(z)|^2) \frac{1 - \exp(-\sup_{z \in \mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_{z_n}(z)|^2))}{\sup_{z \in \mathbb{D}} \sum_{n=1}^{\infty} (1 - |\varphi_{z_n}(z)|^2)} \\ &\gtrsim \sum_{n=1}^{\infty} (1 - |\varphi_{z_n}(z)|^2). \end{aligned}$$

If B_{φ_a} denotes the Blaschke product associated with a sequence $\{\varphi_a(z_n)\}$, then $|B(\varphi_a(z))| = |B_{\varphi_a}(z)|$, and (5.6) yields

$$(5.7) \quad 1 - |B(\varphi_a(z))|^2 = 1 - |B_{\varphi_a}(z)|^2 \gtrsim \sum_{n=1}^{\infty} (1 - |\varphi_{\varphi_a(z_n)}(z)|^2).$$

By the change of variable $w = \varphi_a(z)$, Lemma 1 and (5.7),

$$\begin{aligned} J(a) &:= \int_{\mathbb{D}} |B'(w)|^p (1 - |w|^2)^{p-2} K(1 - |\varphi_a(w)|^2) dA(w) \\ &= \int_{\mathbb{D}} |(B \circ \varphi_a)'(z)|^p (1 - |z|^2)^{p-2} K(1 - |z|^2) dA(z) \\ &\simeq \int_{\mathbb{D}} (1 - |(B \circ \varphi_a)(z)|^2)^p (1 - |z|^2)^{-2} K(1 - |z|^2) dA(z) \\ &\gtrsim \int_{\mathbb{D}} \left(\sum_{n=1}^{\infty} (1 - |\varphi_{\varphi_a(z_n)}(z)|^2) \right)^p (1 - |z|^2)^{-2} K(1 - |z|^2) dA(z) \end{aligned}$$

from which the inequality $(\sum |b_k|)^p \geq \sum |b_k|^p$ finally yields

$$J(a) \gtrsim \sum_{n=1}^{\infty} (1 - |\varphi_a(z_n)|^2)^p \int_{\mathbb{D}} \frac{(1 - |z|^2)^{p-2} K(1 - |z|^2)}{|1 - \overline{\varphi_a(z_n)} z|^{2p}} dA(z).$$

Since Theorem 1 ensures that $J(a)$ is uniformly bounded, the assertion follows by Lemma 5. $\square$

5.1. Proof of Theorem 6. Let $1 \leq p < \infty$ and $-1 < q < \infty$, and assume that K satisfies (A) and (B). By Corollary 4 and Proposition 7, the only possible inner functions in $Q_K(p, p-2)$ are Blaschke products.

Therefore it suffices to show that the Blaschke product B associated with a sequence $\{z_n\}$ belongs to $Q_K(p, p-2)$ for some (equivalently for all) $p \geq 1$ if and only if (1.8) is satisfied. If (1.8) is satisfied, then Corollary 9 shows that $B \in Q_K(p, p-2)$ for all $p \geq 1$. On the other hand, if $B \in Q_K(p, p-2)$ for some $p \geq 1$, then Theorem 10 implies that (1.8) is satisfied. Thus Theorem 6 is proved.

It is worth noticing that Corollary 9 and Theorem 10 yield the following result which finishes the paper.

Corollary 11. *Let $\frac{1}{2} < p < 1$, and assume that K satisfies (5.2). Let B be a Blaschke product associated with a sequence $\{z_n\}$. Then $B \in Q_K(p, p-2)$ if and only if the sequence $\{z_n\}$ satisfies (1.8).*

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