Fuzzy approximation of an additive functional equation

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Abstract. In this paper, we investigate the generalized Hyers–Ulam–Rassias stability of the functional equation

$$\sum_{i=1}^m f(mx_i + \sum_{j=1, j \neq i}^m x_j) + f\left(\sum_{i=1}^m x_i\right) = 2f\left(\sum_{i=1}^m mx_i\right)$$

in fuzzy Banach spaces and some applications of our results in the stability of above mapping from a normed space to a Banach space will be exhibited.

1. Introduction and preliminaries

In 1940, S.M. Ulam [27] asked the fundamental question for the stability of approximate homomorphisms. In 1941, D. H. Hyers [7] provided a partial solution to Ulam’s problem for mappings between Banach spaces. This result was generalized by Aoki [2] for additive mappings and by Th. M. Rassias [26] for linear mappings by considering an *unbounded Cauchy difference*. The paper of Th. M. Rassias [26] has provided a lot of influence in the

development of what we now call generalized Hyers-Ulam-Rassias stability of functional equations. In 1994, a generalization of Th. M. Rassias' Theorem was obtained by Găvruta [5]. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [6], [9], [15]- [25]). We also refer the readers to the books [1], [3], [8] and [10].

We recall some basic facts concerning fuzzy normed space.

Let X be a real linear space. A function $N : X \times \mathbb{R} \rightarrow [0, 1]$ (so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

- (N1) $N(x, t) = 0$ for $t \leq 0$;
- (N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
- (N3) $N(sx, t) = N(x, t/|s|)$ if $s \neq 0$;
- (N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
- (N5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and

$$\lim_{t \rightarrow \infty} N(x, t) = 1$$

The pair (X, N) is called a fuzzy normed linear space.

Example 1.1. Let $(X, \|\cdot\|)$ be a normed linear space. Then

$$N(x, t) = \begin{cases} \frac{t}{t + \|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X .

Example 1.2. Let $(X, \|\cdot\|)$ be a normed linear space. Then

$$N(x, t) = \begin{cases} 0 & t \leq \|x\|, x \in X \\ 1 & t > \|x\|, x \in X \end{cases}$$

is a fuzzy norm on X .

Let (X, N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{n \rightarrow \infty} N(x_n - x, t) = 1$$

for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$, and we write $N - \lim_{n \rightarrow \infty} x_n = x$.

A sequence $\{x_n\}$ in a fuzzy normed space (X, N) is called Cauchy if for each $\epsilon > 0$ and each $\delta > 0$, one can find some n_0 such that

$$N(x_m - x_n, \delta) > 1 - \epsilon$$

for all $n, m \geq n_0$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy. If in a fuzzy normed space, each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Stability of Cauchy, Jensen, quadratic and cubic function equation in fuzzy normed spaces have been investigated in [11]-[14]. C. Park [20] has proved the generalized Hyers-Ulam stability of a functional equation associated with inner product spaces in fuzzy Banach spaces.

In this paper, we investigate the generalized Hyers-Ulam-Rassias stability of the functional equation

$$(1) \quad \sum_{i=1}^m f(mx_i + \sum_{j=1, j \neq i}^m x_j) + f(\sum_{i=1}^m x_i) = 2f(\sum_{i=1}^m mx_i)$$

in fuzzy Banach spaces, where m is a fixed positive integer with $m \geq 2$.

2. Fuzzy Stability of functional equation (1)

Hereafter, unless otherwise stated, we will assume that X is real vectors space, (Y, N) is a complete fuzzy normed space and m is a fixed integer greater than 1. For convenience, we use the following abbreviation for a given mapping f

$$Df(x_1, \dots, x_m) := \sum_{i=1}^m f(mx_i + \sum_{j=1, j \neq i}^m x_j) + f(\sum_{i=1}^m x_i) - 2f(\sum_{i=1}^m mx_i).$$

We will use the following lemma [4]:

Lemma 2.1. *A mapping $f : X \rightarrow Y$ satisfies (1) if and only if it is additive.*

Theorem 2.2. *Let (Z, N') be a fuzzy normed space and $\varphi : \underbrace{X \times X \times \dots \times X}_{m\text{-times}} \rightarrow Z$ be a mapping such that $\varphi(0, \dots, \underbrace{mx}_{j\text{th}}, \dots, 0) =$*

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$\alpha\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0)$ for some α with $0 < \alpha < m$. Assume that $f : X \rightarrow Y$ be mapping such that

$$(2) \quad N(Df(x_1, \dots, x_m), t) \geq N'(\varphi(x_1, \dots, x_m), t)$$

for all $x_i \in X$ ($1 \leq i \leq m$) and all positive real number t . Then there is a unique additive mapping $T : X \rightarrow Y$ such that $T(x) = N - \lim_{n \rightarrow \infty} f(m^n x)/m^n$ and

$$N(T(x) - f(x), t) \geq N'(\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0), \frac{m - \alpha}{2\alpha}t)$$

for all $x \in X$.

Proof. Letting $x_j = x$ and $x_i = 0$ for all ($1 \leq i \leq m$) with $i \neq j$ in (2), we get

$$(3) \quad N(f(mx) - mf(x), t) \geq N'(\varphi(0, \dots, \underbrace{mx}_{jth}, \dots, 0), t)$$

for all $x \in X$ and all positive real number t . Replacing x by $m^n x$ in (3), we get

$$(4) \quad N(f(m^{n+1}x) - mf(m^n x), t) \geq N'(\varphi(0, \dots, \underbrace{m^{n+1}x}_{jth}, \dots, 0), t).$$

Thus

$$(5) \quad N\left(\frac{f(m^{n+1}x)}{m^{n+1}} - \frac{f(m^n x)}{m^n}, \frac{t}{m^{n+1}}\right) \geq N'(\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0), \frac{t}{\alpha^{n+1}})$$

for all $x \in X$ and all positive real number t . Hence

$$\begin{aligned} & N\left(\frac{1}{m^{n+1}}f(m^{n+1}x) - \frac{1}{m^k}f(m^k x), \sum_{i=k}^n \frac{\alpha^{i+1}}{m^{i+1}}t\right) \\ &= N\left(\sum_{i=k}^n \frac{1}{m^{i+1}}f(m^{i+1}x) - \frac{1}{m^i}f(m^i x), \sum_{i=k}^n \frac{\alpha^{i+1}}{m^{i+1}}t\right) \\ (6) \quad & \geq \min \bigcup_{i=1}^n \left\{ N\left(\frac{1}{m^{i+1}}f(m^{i+1}x) - \frac{1}{m^i}f(m^i x), \frac{\alpha^{i+1}}{m^{i+1}}t\right) \right\} \\ & \geq N'(\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0), t). \end{aligned}$$

Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t \rightarrow \infty} N'(\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0), t) = 1$, there is some $t_0 > 0$ such that $N'(\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0), t_0) > 1 - \epsilon$. Since $\sum_{n=0}^{\infty} \frac{\alpha^n}{m^n} t_0 < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{i=k}^n \frac{\alpha^i}{m^i} t_0 < \frac{m\delta}{\alpha}$ for all $n > k \geq n_0$. It follows that

$$\begin{aligned}
 (7) \quad & N\left(\frac{1}{m^{n+1}}f(m^{n+1}x) - \frac{1}{m^k}f(m^kx), \delta\right) \\
 & \geq N\left(\frac{1}{m^{n+1}}f(m^{n+1}x) - \frac{1}{m^k}f(m^kx), \sum_{i=k}^n \frac{\alpha^{i+1}}{m^{i+1}}t_0\right) \\
 & \geq N'(\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0), t_0) > 1 - \epsilon
 \end{aligned}$$

for all $x \in X$ and all non-negative integers n and k with $n > k \geq n_0$. Therefore, the sequence $\{\frac{1}{m^n}f(m^n x)\}$ is a Cauchy sequence in (Y, N) for all $x \in X$. Since (Y, N) is complete, the sequence $\{\frac{1}{m^n}f(m^n x)\}$ converges in Y for all $x \in X$. So one can define the mapping $T : X \rightarrow Y$ by

$$(8) \quad T(x) := N - \lim_{n \rightarrow \infty} \frac{1}{m^n}f(m^n x)$$

for all $x \in X$. Now, we show that T is additive mapping. It follows from (2) and (8) that

$$\begin{aligned}
 N(DT(x_1, \dots, x_m), t) &= \lim_{n \rightarrow \infty} N\left(\frac{Df(m^n x_1, \dots, m^n x_m)}{m^n}, t\right) \\
 &\geq \lim_{n \rightarrow \infty} N'\left(\frac{\varphi(m^n x_1, \dots, m^n x_m)}{m^n}, t\right) \\
 &= \lim_{n \rightarrow \infty} N'(\varphi(x_1, \dots, x_m), \frac{m^n}{\alpha^n}t) \\
 &= 1
 \end{aligned}$$

for all $x_1, \dots, x_m \in X$ and all positive real numbers t . So $N(DT(x_1, \dots, x_m), t) = 1$ for all $x_1, \dots, x_m \in X$ and all positive real numbers t . Therefore $DT(x_1, \dots, x_m) = 0$. By Lemma (2.1), the mapping $T : X \rightarrow Y$ is additive. Moreover, if we put $k = 0$ in (6), we observe that

$$N\left(\frac{1}{m^{n+1}}f(m^{n+1}x) - f(x), \sum_{i=0}^n \frac{\alpha_j^{i+1}}{m^{i+1}}t\right) \geq N'(\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0), t).$$

Therefore,

$$(9) \quad N\left(\frac{1}{m^{n+1}}f(m^{n+1}x) - f(x), t\right) \geq N'\left(\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0), \frac{t}{\sum_{i=0}^n \frac{\alpha^{i+1}}{m^{i+1}}}\right)$$

It follows from (9) that, for large enough n ,

$$\begin{aligned} & N(T(x) - f(x), t) \\ & \geq \min \left\{ N\left(\frac{1}{m^{n+1}}f(m^{n+1}x) - f(x), \frac{t}{2}\right), N\left(T(x) - \frac{1}{m^{n+1}}f(m^{n+1}x), \frac{t}{2}\right) \right\} \\ & \geq N'\left(\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0), \frac{t}{2 \sum_{i=0}^n \frac{\alpha^{i+1}}{m^{i+1}}}\right) \\ & \geq N'\left(\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0), \frac{m - \alpha}{2\alpha}t\right). \end{aligned}$$

Now, we show that T is unique. Let T' be another additive mapping from X into Y , which satisfies the required inequality. Then for each $x \in X$ and $t > 0$, we have

$$\begin{aligned} N(T(x) - T'(x), t) & \geq \min \left\{ N\left(T(x) - f(x), \frac{t}{2}\right), N\left(f(x) - T'(x), \frac{t}{2}\right) \right\} \\ & \geq N'\left(\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0), \frac{m - \alpha}{4\alpha}t\right). \end{aligned}$$

So

$$\begin{aligned} N(T(x) - T'(x), t) & = N\left(\frac{T(m^n x)}{m^n} - \frac{T'(m^n x)}{m^n}, t\right) \\ & \geq N'\left(\frac{\alpha^n}{m^n}\varphi(0, \dots, \underbrace{x}_{jth}, \dots, 0), \frac{m - \alpha}{4\alpha}t\right). \end{aligned}$$

Hence, the right hand side of the above inequality tends to 1 as $n \rightarrow \infty$. It follows that $T(x) = T'(x)$ for all $x \in X$. $\square$

Theorem 2.3. Let (Z, N') be a fuzzy normed space and $\Phi : \underbrace{X \times X \times \dots \times X}_{m\text{-times}} \rightarrow Z$ be a mapping such that $\Phi(0, \dots, \underbrace{mx}_{jth}, \dots, 0) = \alpha\Phi(0, \dots, \underbrace{x}_{jth}, \dots, 0)$ for some α with $\alpha > m$. Assume that $f : X \rightarrow Y$ be mapping such that

$$(10) \quad N(Df(x_1, \dots, x_m), t) \geq N'(\Phi(x_1, \dots, x_m), t)$$

for all $x_i \in X$ ($1 \leq i \leq m$) and all positive real number t . Then there is a unique additive mapping $T : X \rightarrow Y$ such that $T(x) =$

$N - \lim_{n \rightarrow \infty} m^n f(\frac{x}{m^n})$ **and for all** $x \in X$

$$N(T(x) - f(x), t) \geq N'(\Phi(0, \dots, \underbrace{x}_{jth}, \dots, 0), \frac{\alpha - m}{2\alpha}t).$$

Proof. Similarly to the proof of Theorem 2.2, we have

$$(11) \quad N(f(mx) - mf(x), t) \geq N'(\Phi(0, \dots, \underbrace{mx}_{jth}, \dots, 0), t)$$

for all $x \in X$ and all positive real number t . Replacing x by $\frac{x}{m^{n+1}}$ in (11), we get

$$(12) \quad N(f(\frac{x}{m^n}) - mf(\frac{x}{m^{n+1}}), t) \geq N'(\Phi(0, \dots, \underbrace{\frac{x}{m^n}}_{jth}, \dots, 0), t).$$

Thus

$$(13) \quad N(m^n f(\frac{x}{m^n}) - m^{n+1} f(\frac{x}{m^{n+1}}), m^n t) \geq N'(\Phi(0, \dots, \underbrace{x}_{jth}, \dots, 0), \alpha^n t)$$

for all $x \in X$ and all positive real number t . Hence

$$(14) \quad \begin{aligned} N(m^{n+1} f(\frac{x}{m^{n+1}}) - m^k f(\frac{x}{m^k}), \sum_{i=k}^n \frac{m^i}{\alpha^i} t) &\geq N(\sum_{i=k}^n m^{i+1} f(\frac{x}{m^{i+1}}) - m^i f(\frac{x}{m^i}), \sum_{i=k}^n \frac{m^i}{\alpha^i} t) \\ &\geq \min \bigcup_{i=1}^m \{N(m^{i+1} f(\frac{x}{m^{i+1}}) - m^i f(\frac{x}{m^i}), \frac{m^i}{\alpha^i} t)\} \\ &\geq N'(\Phi(0, \dots, \underbrace{x}_{jth}, \dots, 0), t). \end{aligned}$$

Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t \rightarrow \infty} N'(\Phi(0, \dots, \underbrace{x}_{jth}, \dots, 0), t) = 1$, there is some $t_0 > 0$ such that $N'(\Phi(0, \dots, \underbrace{x}_{jth}, \dots, 0), t_0) > 1 - \epsilon$. Since

$\sum_{n=0}^{\infty} \frac{m^n}{\alpha^n} t_0 < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{i=k}^n \frac{m^i}{\alpha^i} t_0 < \delta$ for all $n > k \geq n_0$. It follows from (14) that

$$(15) \quad \begin{aligned} N(m^{n+1} f(\frac{x}{m^{n+1}}) - m^k f(\frac{x}{m^k}), \delta) &\geq N(m^{n+1} f(\frac{x}{m^{n+1}}) - m^k f(\frac{x}{m^k}), \sum_{i=k}^n \frac{m^i}{\alpha^i} t_0) \\ &\geq N'(\Phi(0, \dots, \underbrace{x}_{jth}, \dots, 0), t_0) > 1 - \epsilon \end{aligned}$$

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for all $x \in X$ and all non-negative integers n and k with $n > k \geq n_0$. Therefore, the sequence $\{m^n f(\frac{x}{m^n})\}$ is a Cauchy sequence in (Y, N) for all $x \in X$. Since (Y, N) is complete, the sequence $\{m^n f(\frac{x}{m^n})\}$ converges in Y for all $x \in X$. So one can define the mapping $T : X \rightarrow Y$ by

$$(16) \quad T(x) := N - \lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^n}\right), \quad x \in X.$$

The rest of the proof is similar to the proof of Theorem 2.2 $\square$

Corollary 2.4. *Let X be a normed space and (Z, N') be a fuzzy normed space. Assume that r_i ($1 \leq i \leq m$) be positive real numbers such that $r_i < 1$ or $r_i > 1$ ($1 \leq i \leq m$) and $f : X \rightarrow Y$ be a mapping such that*

$$(17) \quad N(Df(x_1, \dots, x_m), t) \geq N'\left(\sum_{i=1}^m \|x_i\|^{r_i} z_0, t\right)$$

for all $x_i \in X$ ($1 \leq i \leq m$) and all positive real number t , where z_0 is a fixed vector of Z . Then there is a unique additive mapping $T : X \rightarrow Y$ such that

$$N(T(x) - f(x), t) \geq N'(\|x\|^{r_j} z_0, \left|\frac{1}{m^{r_j-1}} - 1\right| \frac{t}{2})$$

for all $x \in X$.

Corollary 2.5. *Let X be a normed space and (Z, N') be a fuzzy normed space. Assume that r_i ($1 \leq i \leq m$) be positive real numbers such that $r_1 + \dots + r_m \neq 1$ and $f : X \rightarrow Y$ be a mapping such that*

$$(18) \quad N(Df(x_1, \dots, x_m), t) \geq N'\left(\prod_{i=1}^m \|x_i\|^{r_i} z_0, t\right)$$

for all $x_i \in X$ ($1 \leq i \leq m$) and all positive real number t , where z_0 is a fixed vector of Z . Then $f : X \rightarrow Y$ is additive.

Proof. Letting $x_1 = mx$ and $x_i = 0$ for all i ($2 \leq i \leq m$) in (18), we get

$$N(f(mx) - mf(x), t) = 1$$

for all $x \in X$ and all positive real number t . So

$$f(mx) = mf(x)$$

for all $x \in X$. Hence the result follows by Theorems 2.2 and 2.3. $\square$

3. Application of fuzzy stability to the stability of Ex. (1) in normed spaces

Hereafter, we will assume that X is a normed space and Y is a Banach space. Let $Z = \mathbb{R}$. As noted in Example 1.1

$$N'(z, t) = \begin{cases} \frac{t}{t+|z|} & t > 0, z \in Z \\ 0 & t \leq 0, z \in Z \end{cases}$$

defines a fuzzy norm on Z and

$$N(y, t) = \begin{cases} \frac{t}{t+\|y\|} & t > 0, y \in Y \\ 0 & t \leq 0, y \in Y \end{cases}$$

defines a fuzzy norm on Y . By using Corollaries 2.4 and 2.5, one can obtain the following corollaries.

Corollary 3.1. *Let θ, r_i ($1 \leq i \leq m$) be positive real numbers such that $r_i < 1$ or $r_i > 1$ ($1 \leq i \leq m$). Suppose that a mapping $f : X \rightarrow Y$ satisfies the inequality $\|Df(x_1, \dots, x_m)\| \leq \theta \sum_{i=1}^m \|x_i\|^{r_i}$ for all $x_i \in X$ ($1 \leq i \leq m$). Then there exists a unique additive mapping $T : X \rightarrow Y$ satisfying*

$$\|f(x) - T(x)\| \leq \frac{2m^{r_j}\theta}{|m - m^{r_j}|} \|x\|^{r_j}, \quad x \in X.$$

Corollary 3.2. *Let θ, r_i ($1 \leq i \leq m$) be positive real numbers such that $r_1 + \dots + r_m \neq 1$. Suppose that a mapping $f : X \rightarrow Y$ satisfies the inequality*

$$\|Df(x_1, \dots, x_m)\| \leq \theta \prod_{i=1}^m \|x_i\|^{r_i}$$

for all $x_i \in X$ ($1 \leq j \leq m$). Then the mapping $f : X \rightarrow Y$ is additive.

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