Research Article

Weighted Estimates for Maximal Commutators of Multilinear Singular Integrals

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This paper is concerned with the pointwise estimates for the sharp function of the maximal multilinear commutators $T_{\Sigma b}^*$ and maximal iterated commutator $T_{\Pi b}^*$, generalized by m -linear operator T and a weighted Lipschitz function b . The $(L^{p_1}(\mu) \times \cdots \times L^{p_m}(\mu), L^r(\mu^{1-r}))$ boundedness and the $(L^{p_1}(\mu) \times \cdots \times L^{p_m}(\mu), L^r(\mu^{1-mr}))$ boundedness are obtained for maximal multilinear commutator $T_{\Sigma b}^*$ and maximal iterated commutator $T_{\Pi b}^*$, respectively.

1. Introduction and Notation

The theory of multilinear Calderón-Zygmund singular integral operators, originated from the work of Coifman and Meyers', has an important role in harmonic analysis. Its study has been attracting a lot of attention in the last few decades. So far, a number of properties for multilinear operators are parallel to those of the classical linear Calderón-Zygmund operators but new interesting phenomena have also been observed. A systematic analysis of many basic properties of such multilinear operators can be found in the articles by Coifman and Meyer [1], Grafakos and Torres [2–4], and Lerner et al. [5]. So we first recall the definition and results of multilinear Calderón-Zygmund operators as well as the corresponding maximal multilinear operators.

Definition 1.1. Let T be a multilinear operator initially defined on the m -fold product of Schwartz space and taking values into the space of tempered distributions:

$$T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \quad (1.1)$$

Following [2], we say that T is an m -linear Calderón-Zygmund operator if for some $1 \leq q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q , where $1/q = (1/q_1) + \cdots + (1/q_m)$, and if there exists a function K , defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m, \quad (1.2)$$

for all $x \notin \bigcap_{j=1}^m \text{supp } f_j$.

And

$$|K(y_0, y_1, \dots, y_m)| \leq \frac{A}{\left(\sum_{k,l=0}^m |y_k - y_l|\right)^{mn}}, \quad (1.3)$$

$$\left| K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y'_j, \dots, y_m) \right| \leq \frac{A |y_j - y'_j|^\varepsilon}{\left(\sum_{k,l=0}^m |y_k - y_l|\right)^{mn+\varepsilon}}, \quad (1.4)$$

for some $\varepsilon > 0$ and all $0 \leq j \leq m$, where $|y_j - y'_j| \leq (1/2) \max_{0 \leq k \leq m} |y_j - y_k|$.

The maximal multilinear singular integral operator was defined by

$$T^*(\vec{f})(x) = \sup_{\delta > 0} |T_\delta(f_1, \dots, f_m)(x)|, \quad (1.5)$$

where T_δ is the smooth truncation of T given by

$$T_\delta(f_1, \dots, f_m)(x) = \int_{|x-y_1|^2 + \cdots + |x-y_m|^2 > \delta^2} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m. \quad (1.6)$$

As pointed in [4], $T^*(\vec{f})$ is pointwise well defined when $f_j \in L^{q_j}(\mathbb{R}^n)$ with $1 \leq q_j < \infty$.

The study for the multilinear singular integral operator and its maximal operators attracts many authors' attention. For maximal multilinear operator T^* , one can see [4] for details. We list some results for T^* as follows.

Theorem A (see [4]). *Let $1 \leq q_j < \infty$ and q such that $1/q = (1/q_1) + \cdots + (1/q_m)$ and $\omega \in A_{q_1} \cap \cdots \cap A_{q_m}$. Let T be an m -linear Calderón-Zygmund operator. Then there exists a constant $C_{n,q} < \infty$, such that for all $\vec{f} = (f_1, \dots, f_m)$ satisfying*

$$\|T^*(\vec{f})\|_{L^q(\omega)} \leq C_{n,q}(A + W) \prod_{j=1}^m \|f_j\|_{L^{q_j}(\omega)}, \quad (1.7)$$

where W is the norm of T in the mapping $T: L^1 \times \cdots \times L^1 \rightarrow L^{1/m, \infty}$.

Theorem B (see [4]). Let T be an m -linear Calderón-Zygmund operator. Then, for all exponents $p, p_1, \dots, p_m$, satisfying $(1/p_1) + \dots + (1/p_m) = 1/p$, one has

$$T^* : L^{p_1} \times \dots \times L^{p_m} \longrightarrow L^p, \quad (1.8)$$

when $1 < p_1, \dots, p_m \leq \infty$, one also has

$$T^* : L^{p_1} \times \dots \times L^{p_m} \longrightarrow L^{p, \infty}, \quad (1.9)$$

when at least one p_j is equal one. In either cases the norm of T^* is controlled by a constant multiple of $A + W$.

Definition 1.2 (see [5] (commutators in the j th entry)). Given a collection of locally integrable function $\vec{b} = (b_1, \dots, b_m)$, we define the commutators of the m -linear Calderón-Zygmund operator T to be

$$T_{\Sigma b}(f_1, \dots, f_m) = \sum_{j=1}^m T_{b_j}^j(\vec{f}), \quad (1.10)$$

where each term is the commutator of b_j and T in the j th entry of T , that is

$$T_{b_j}^j(\vec{f}) = b_j T(f_1, \dots, f_j, \dots, f_m) - T(f_1, \dots, b_j f_j, \dots, f_m). \quad (1.11)$$

In [6], the following more general iterated commutators of multilinear Calderón-Zygmund operators and pointwise multiplication with functions in BMO were defined and studied in products of Lebesgue spaces, including strong type and weak end-point estimates with multiple $A_{\vec{p}}$ weights. That is,

$$\begin{aligned} T_{\Pi b}(\vec{f})(x) &= [b_1, [b_2, \dots, [b_{m-1}, [b_m, T]_m]_{m-1}]_2]_1 \\ &= \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) \prod_{j=1}^m (b_j(x) - b_j(y_j)) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m. \end{aligned} \quad (1.12)$$

For the operator $[b, T]$, when T is the Calderón-Zygmund singular integral operator and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ (the homogeneous Lipschitz spaces), Paluszyński [7] established the (L^p, L^q) boundedness with $1 < p < 1/\beta$ and $1/q = 1/p + \beta/n$. Hu and Gu [8] extended this results to the case: $b \in \text{Lip}_{\beta, \mu}$ with $\mu \in A_1$.

Now we present the definitions of two classes of maximal commutators of multilinear singular integral operators. One is

$$\begin{aligned} T_{\Sigma b}^*(\vec{f})(x) &= \sup_{\delta > 0} \left| \sum_{j=1}^m \int_{\sum_{i=1}^m |x - y_i|^2 > \delta^2} K(x, y_1, \dots, y_m) f_1(y_1) \cdots (b_j(x) - b_j(y_j)) f_j(y_j) \cdots f_m(y_m) d\vec{y} \right|, \end{aligned} \quad (1.13)$$

the other is

$$\begin{aligned} T_{\Pi b}^*(\vec{f})(x) &= \sup_{\delta > 0} \left| [b_1, [b_2, \dots, [b_{m-1}, [b_m, T_\delta]_m]_{m-1}, \dots]_2]_1(\vec{f})(x) \right| \\ &= \sup_{\delta > 0} \left| \int_{\sum_{i=1}^m |x - y_i|^2 > \delta^2} K(x, y_1, \dots, y_m) \prod_{j=1}^m (b_j(x) - b_j(y_j)) f_1(y_1) \cdots f_m(y_m) d\vec{y} \right|, \end{aligned} \quad (1.14)$$

where $d\vec{y} = dy_1 \cdots dy_m$. It is obvious to see that

$$T_{\Sigma b}^*(\vec{f})(x) \leq \sum_{j=1}^m T_{b_j}^{*,j}(\vec{f})(x). \quad (1.15)$$

The main purpose of this paper is to extend the results in [8] to the maximal commutators generated by multilinear singular integrals T and $\text{Lip}_{\beta, \mu}$ functions $\vec{b}$.

We can formulate our result as following.

Theorem 1.3. Assume that the kernel K satisfies (1.3) and (1.4). Let $1 < q_1, \dots, q_m, q < \infty$ be given numbers satisfying $1/q = (1/q_1) + \cdots + (1/q_m)$. And assume that T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. For $j \in \{1, \dots, m\}$ and let $1/r = (1/p) - (\beta/n)$, $1 < p < r < \infty$, $0 < \beta < 1$, and $1/p = 1/p_1 + \cdots + 1/p_m$ with $1 < p_i < \infty$, $i = 1, \dots, m$. Given μ such that $\mu \in A_1(\mathbb{R}^n)$ and $b_j \in \text{Lip}_{\beta, \mu}(\mathbb{R}^n)$ ($j = 1, \dots, m$), then one has

$$\|T_{b_j}^{*,j}(\vec{f})\|_{L^r(\mu^{1-r})} \leq C \|b_j\|_{\text{Lip}_{\beta, \mu}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mu)}, \quad j = 1, \dots, m. \quad (1.16)$$

From (1.15) and (1.16), one can get

$$\|T_{\Sigma b}^*(\vec{f})\|_{L^r(\mu^{1-r})} \leq C \sum_{j=1}^m \|b_j\|_{\text{Lip}_{\beta, \mu}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mu)}. \quad (1.17)$$

If $\mu = 1$, one can get the following.

Theorem 1.4. Assume that the kernel K satisfies (1.3) and (1.4). Let $1 < q_1, \dots, q_m, q < \infty$ be given numbers satisfying $1/q = (1/q_1) + \cdots + (1/q_m)$. And assume that T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. For $j \in \{1, \dots, m\}$ and let $1/r = (1/p) - (\beta/n)$, $1 < p < r < \infty$, $0 < \beta < 1$ and $1/p = (1/p_1) + \cdots + (1/p_m)$ with $1 < p_i < \infty$, $i = 1, \dots, m$. Set $b_j \in \text{Lip}_\beta(\mathbb{R}^n)$ ($j = 1, \dots, m$), then one has

$$\|T_{b_j}^{*,j}(\vec{f})\|_{L^r(\mathbb{R}^n)} \leq C \|b_j\|_{\text{Lip}_\beta} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}, \quad j = 1, \dots, m. \quad (1.18)$$

From (1.15), one can get

$$\left\| T_{\Sigma b}^*(\vec{f}) \right\|_{L^r(\mathbb{R}^n)} \leq C \sum_{j=1}^m \|b_j\|_{Lip_\beta} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}. \quad (1.19)$$

The following theorem states the weighted estimates with two different weights for maximal iterated commutator of multilinear singular integrals.

Theorem 1.5. Assume that the kernel K satisfies (1.3) and (1.4). Let $1 < q_1, \dots, q_m, q < \infty$ be given numbers satisfying $1/q = (1/q_1) + \dots + (1/q_m)$. And assume that T maps $L^{q_1}(\mathbb{R}^n) \times \dots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Let $1/r_i = (1/p_i) - (\beta_i/n)$, $1 < p_i < r_i < \infty$, $0 < \beta_i < 1$, $i = 1, \dots, m$ with $1/p = (1/p_1) + \dots + (1/p_m)$, $1/r = (1/r_1) + \dots + (1/r_m)$, and $\beta = \beta_1 + \dots + \beta_m$, $0 < \beta < 1$. Given μ such that $\mu \in A_1(\mathbb{R}^n)$ and $b_i \in Lip_{\beta_i, \mu}(\mathbb{R}^n)$ ($i = 1, \dots, m$), then one has

$$\left\| T_{\Pi b}^*(\vec{f}) \right\|_{L^r(\mu^{1-mr})} \leq C \prod_{i=1}^m \|b_i\|_{Lip_{\beta_i, \mu}} \|f_i\|_{L^{p_i}(\mu)}. \quad (1.20)$$

Similarly as Theorem 1.4, one also obtains the unweighted estimates of maximal iterated commutators.

Theorem 1.6. Assume that the kernel K satisfies (1.3) and (1.4). Let $1 < q_1, \dots, q_m, q < \infty$ be given numbers satisfying $1/q = (1/q_1) + \dots + (1/q_m)$. And assume that T maps $L^{q_1}(\mathbb{R}^n) \times \dots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Let $1/r_i = (1/p_i) - (\beta_i/n)$, $1 < p_i < r_i < \infty$, $0 < \beta_i < 1$, $i = 1, \dots, m$ with $1/p = (1/p_1) + \dots + (1/p_m)$, $1/r = (1/r_1) + \dots + (1/r_m)$, and $\beta = \beta_1 + \dots + \beta_m$, $0 < \beta < 1$. Set $b_i \in Lip_{\beta_i}(\mathbb{R}^n)$ ($i = 1, \dots, m$), then one has

$$\left\| T_{\Pi b}^*(\vec{f}) \right\|_{L^r(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|b_i\|_{Lip_{\beta_i}} \|f_i\|_{L^{p_i}(\mathbb{R}^n)}. \quad (1.21)$$

The rest of this paper is organized as follows. In Section 2, we recall some standard definitions and lemmas. Section 3 is devoted to the proof of our theorems. Throughout this paper, we use the letter C to denote a positive constant that varies line to line, but it is independent of the essential variable. For any $1 \leq p \leq \infty$, the p' is always used to denote the dual index such that $(1/p) + (1/p') = 1$.

2. Preliminaries

A nonnegative function μ defined on $\mathbb{R}^n$ is called weight if it is locally integrable. A weight μ is said to belong to the Muckenhoupt class $A_p(\mathbb{R}^n)$, $1 < p < \infty$, if there exists a constant C such that

$$\sup_B \left(\frac{1}{|B|} \int_B \mu(x) dx \right) \left(\frac{1}{|B|} \int_B \mu(x)^{-1/(p-1)} dx \right)^{p-1} \leq C < \infty, \quad (2.1)$$

for every ball $B \subset \mathbb{R}^n$. A weight μ is said to belong to class $A_1(\mathbb{R}^n)$ if

$$\left(\frac{1}{|B|} \int_B \mu(x) dx \right) \leq C \inf_{x \in B} \mu(x), \quad \text{almost all } x \in \mathbb{R}^n, \quad (2.2)$$

for every ball $B \ni x$. The class $A_\infty(\mathbb{R}^n)$ can be characterized as $A_\infty = \bigcup_{1 \leq p < \infty} A_p$.

Many properties of weights can be found in the book [9], we only collect some of them in the following lemma which will be used below.

Lemma 2.1. (i) $A_p \subset A_q$ for $1 \leq p < q \leq \infty$;
(ii) if $\mu \in A_1$, then $\mu^\theta \in A_1$ for $0 \leq \theta \leq 1$;
(iii) for $1 < p < \infty$, $\mu \in A_p$ if and only if $\mu^{1-p'} \in A_{p'}$.

A locally integrable function f belongs to the weighted Lipschitz space $\text{Lip}_{\beta, \mu}^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$, $0 < \beta < 1$ and $\mu \in A_\infty$ if

$$\sup_{B \ni x} \frac{1}{\mu(B)^{\beta/n}} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p \mu(x)^{1-p} dx \right)^{1/p} \leq C < \infty. \quad (2.3)$$

The smallest bound C satisfying (1.19) is then taken to be the norm of f denoted by $\|f\|_{\text{Lip}_{\beta, \mu}^p}$.

Put $\text{Lip}_{\beta, \mu} = \text{Lip}_{\beta, \mu}^1$.

If $\mu \in A_1$, $b \in \text{Lip}_{\beta, \mu}$ ($0 < \beta < 1$), from the definition of $\|f\|_{\text{Lip}_{\beta, \mu}}$, it is obvious to see

$$|b_B - b_{2^{k+1}B}| \leq Ck\mu(x)\mu\left(B\left(x, 2^{k+1}R\right)\right)^{\beta/n} \|b\|_{\text{Lip}_{\beta, \mu}}, \quad (2.4)$$

where $b_B = (1/|B|) \int_B b(y) dy$.

The important properties of the weights are the weighted estimates for the maximal function, the sharp maximal function and their variants. One first recalls the maximal function defined by

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy, \quad (2.5)$$

It is well known that for $1 < p < \infty$, M maps $L^p(\mu)$ into itself if and only if $\mu \in A_p$, see [10].

The sharp maximal function is defined by

$$M^\#(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(x) - f_B| dy \approx \sup_{B \ni x} \inf_c \frac{1}{|B|} \int_B |f(x) - c| dy. \quad (2.6)$$

One also recalls the variants $M_\delta(f)(x) = (M(|f|^\delta)(x))^{1/\delta}$, and $M_\delta^\#(f)(x) = (M^\#(|f|^\delta)(x))^{1/\delta}$. We denote the weighted fractional maximal operators by

$$M_{\alpha, \mu, s}(f)(x) = \sup_{B \ni x} \left(\frac{1}{\mu(B)^{1-(s\alpha/n)}} \int_B |f(y)|^s \mu(y) dy \right)^{1/s}. \quad (2.7)$$

Recall that $M_\alpha := M_{\alpha,1,1}$ is the weighted fractional maximal operators, that is

$$M_\alpha(f)(x) = \sup_{B \ni x} \left(\frac{1}{\mu(B)^{1-(\alpha/n)}} \int_B |f(y)| \mu(y) dy \right). \quad (2.8)$$

The following lemmas are all from [11].

Lemma 2.2 (Kolmogorov's inequality). *Let (X, μ) be a probability measure space and let $0 < p < q < \infty$ then there exists a constant $C = C_{p,q}$ such that for any measurable function f*

$$\|f\|_{L^p(\mu)} \leq C \|f\|_{L^{q,\infty}(\mu)}. \quad (2.9)$$

Lemma 2.3. *Let $0 < p, \delta < \infty$ and $\mu \in A_\infty(\mathbb{R}^n)$, there exists $C > 0$ depending on the $A_\infty(\mathbb{R}^n)$ constant of μ such that*

$$\|M_\delta(f)\|_{L^p(\mu)} \leq C \|M_\delta^\#(f)\|_{L^p(\mu)}, \quad (2.10)$$

for any function f for which the left side of the above inequality is finite.

Lemma 2.4. *Suppose that $0 < \alpha < n$, $0 < s < p < \alpha/n$, $1/q = (1/p) - (\alpha/n)$. If $\mu \in A_\infty(\mathbb{R}^n)$, then there exists a constant $C = C_{p,q}$ such that for any measurable function f*

$$\|M_{\alpha,\mu,s}(f)\|_{L^q(\mu)} \leq C \|f\|_{L^p(\mu)}. \quad (2.11)$$

Lemma 2.5. *Suppose that $0 < \alpha < n$, $0 < s < p < \alpha/n$, $1/q = (1/p) - (\alpha/n)$. If $\mu \in A_{1+(q/p')}(\mathbb{R}^n)$, then there exists a constant $C = C_{p,q}$ such that for any measurable function f*

$$\|M_\alpha(f)\|_{L^q(\mu)} \leq C \|f\|_{L^p(\mu^{p/q})}. \quad (2.12)$$

3. Two Estimates for Maximal Multilinear Commutators

We will prove our theorems in this section. To begin, we prepare another two iterated operators to control the commutators.

Let $\varphi, \psi \in C^\infty([0, +\infty))$ such that $|\varphi'(t)| \leq C/t$, $|\psi'(t)| \leq C/t$ and satisfying

$$\chi_{[2,\infty)}(t) \leq \varphi(t) \leq \chi_{[1,\infty)}(t), \chi_{[1,2]}(t) \leq \psi(t) \leq \chi_{[1/2,3]}(t). \quad (3.1)$$

We define the maximal operators

$$\begin{aligned}\Phi^*(\vec{f})(x) &= \sup_{\eta > 0} \left| \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) \varphi\left(\frac{\sqrt{|x - y_1| + \dots + |x - y_m|}}{\eta}\right) \prod_{i=1}^m f_i(y_i) d\vec{y} \right|, \\ \Psi^*(\vec{f})(x) &= \sup_{\eta > 0} \left| \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) \psi\left(\frac{\sqrt{|x - y_1| + \dots + |x - y_m|}}{\eta}\right) \prod_{i=1}^m f_i(y_i) d\vec{y} \right|.\end{aligned}\quad (3.2)$$

For simplicity, we denote $K_{\varphi, \eta}(x, y_1, \dots, y_m) = K(x, y_1, \dots, y_m) \varphi(\sqrt{|x - y_1| + \dots + |x - y_m|}/\eta)$, $K_{\psi, \eta}(x, y_1, \dots, y_m) = K(x, y_1, \dots, y_m) \psi(\sqrt{|x - y_1| + \dots + |x - y_m|}/\eta)$ and

$$\begin{aligned}\Phi_{\eta}(\vec{f})(x) &= \int_{(\mathbb{R}^n)^m} K_{\varphi, \eta}(x, y_1, \dots, y_m) \prod_{i=1}^m f_i(y_i) d\vec{y}, \\ \Psi_{\eta}(\vec{f})(x) &= \int_{(\mathbb{R}^n)^m} K_{\psi, \eta}(x, y_1, \dots, y_m) \prod_{i=1}^m f_i(y_i) d\vec{y}.\end{aligned}\quad (3.3)$$

The kernels of Φ_{η} and Ψ_{η} satisfy conditions (1.3) and (1.4) uniformly in η , respectively. And by the same argument in [4], both Φ^* and Ψ^* have the same weighted estimates to T^* that appeared in Theorems A and B.

It is easy to see that $T^*(\vec{f}) \leq \Phi^*(\vec{f}) + \Psi^*(\vec{f})$. Moreover,

$$T_{\Sigma b}^*(\vec{f}) \leq \Phi_{\Sigma b}^*(\vec{f}) + \Psi_{\Sigma b}^*(\vec{f}), \quad T_{\Pi b}^*(\vec{f}) \leq \Phi_{\Pi b}^*(\vec{f}) + \Psi_{\Pi b}^*(\vec{f}), \quad (3.4)$$

where

$$\begin{aligned}\Phi_{\Sigma b}^*(\vec{f})(x) &= \sup_{\eta > 0} \left| \sum_{j=1}^m \int_{(\mathbb{R}^n)^m} K_{\varphi, \eta}(x, y_1, \dots, y_m) f_1(y_1) \right. \\ &\quad \left. \dots (b_j(x) - b_j(y_j)) f_j(y_j) \dots f_m(y_m) d\vec{y} \right| \\ &\leq \sum_{j=1}^m \sup_{\eta > 0} \left| \int_{(\mathbb{R}^n)^m} K_{\varphi, \eta}(x, y_1, \dots, y_m) f_1(y_1) \right. \\ &\quad \left. \dots (b_j(x) - b_j(y_j)) f_j(y_j) \dots f_m(y_m) d\vec{y} \right| \\ &= \sum_{j=1}^m \Phi_{b_j}^{*,j}(\vec{f})(x),\end{aligned}$$

$$\begin{aligned}
\Psi_{\Sigma b}^*(\vec{f})(x) &= \sup_{\eta > 0} \left| \sum_{j=1}^m \int_{(\mathbb{R}^n)^m} K_{\varphi, \eta}(x, y_1, \dots, y_m) f_1(y_1) \right. \\
&\quad \left. \dots (b_j(x) - b_j(y_j)) f_j(y_j) \dots f_m(y_m) d\vec{y} \right| \\
&\leq \sum_{j=1}^m \sup_{\eta > 0} \left| \int_{(\mathbb{R}^n)^m} K_{\varphi, \eta}(x, y_1, \dots, y_m) f_1(y_1) \right. \\
&\quad \left. \dots (b_j(x) - b_j(y_j)) f_j(y_j) \dots f_m(y_m) d\vec{y} \right| \\
&= \sum_{j=1}^m \Psi_{b_j}^{*,j}(\vec{f})(x), \\
\Phi_{\Pi b}^*(\vec{f})(x) &= \sup_{\eta > 0} \left| [b_1, [b_2, \dots, [b_{m-1}, [b_m, \Phi_\eta]_m]_{m-1} \dots]_2]_1(\vec{f})(x) \right| \\
&= \sup_{\eta > 0} \left| \int_{(\mathbb{R}^n)^m} K_{\varphi, \eta}(x, y_1, \dots, y_m) \prod_{j=1}^m (b_j(x) - b_j(y_j)) \prod_{i=1}^m f_i(y_i) d\vec{y} \right|, \\
\Psi_{\Pi b}^*(\vec{f})(x) &= \sup_{\eta > 0} \left| [b_1, [b_2, \dots, [b_{m-1}, [b_m, \Psi_\eta]_m]_{m-1} \dots]_2]_1(\vec{f})(x) \right| \\
&= \sup_{\eta > 0} \left| \int_{(\mathbb{R}^n)^m} K_{\varphi, \eta}(x, y_1, \dots, y_m) \prod_{j=1}^m (b_j(x) - b_j(y_j)) \prod_{i=1}^m f_i(y_i) d\vec{y} \right|.
\end{aligned} \tag{3.5}$$

For simplicity, we will only prove for the case $m = 2$. The arguments for the case $m > 2$ are similar. For the similarity to the two commutators $\Phi_{\Sigma b}^*(\vec{f})$ and $\Psi_{\Sigma b}^*(\vec{f})$, we might as well consider the former. We only consider the former. And we establish the following crucial lemma.

Lemma 3.1. *Let $\mu \in A_1(\mathbb{R}^n)$ and $b_j \in Lip_{\beta, \mu}$, with $0 < \beta < 1$, $j = 1, 2$. Let $0 < \delta < 1/2 < 1 < s < n/\beta$. Then one has*

$$\begin{aligned}
M_\delta^\# [\Phi_{b_j}^{*,j}(f_1, f_2)](x) &\leq C\mu(x) \|b_j\|_{Lip_{\beta, \mu}} M_{\beta, \mu, s}(\Phi^*(f_1, f_2))(x) + C\mu(x) \|b_j\|_{Lip_{\beta, \mu}} \\
&\quad \times (M_{\beta, \mu, s}(f_1)(x) M(f_2)(x) + M(f_1)(x) M_{\beta, \mu, s}(f_2)(x)),
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
M_\delta^\# [\Psi_{b_j}^{*,j}(f_1, f_2)](x) &\leq C\mu(x) \|b_j\|_{Lip_{\beta, \mu}} M_{\beta, \mu, s}(\Psi^*(f_1, f_2))(x) + C\mu(x) \|b_j\|_{Lip_{\beta, \mu}} \\
&\quad \times (M_{\beta, \mu, s}(f_1)(x) M(f_2)(x) + M(f_1)(x) M_{\beta, \mu, s}(f_2)(x)).
\end{aligned} \tag{3.7}$$

Proof. Without loss of generality, we only consider the case $j = 1$ and denote b_1 by b for convenience. Fix $x \in \mathbb{R}^n$ and let $B = B(x, R)$, $\lambda = b_{B^*}$ be the average of b on B^* , where $B^* = B(x, 2R)$. To proceed, we decompose $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{B^*}$, $i = 1, 2$. Let c be a constant to be fixed along the proof.

Since $0 < \delta < 1$, we have

$$\begin{aligned}
 \left(\frac{1}{|B|} \int_B \left| \Phi_b^{*,1}(f_1, f_2)(y) \right|^\delta - |c|^\delta dy \right)^{1/\delta} &\leq \left(\frac{1}{|B|} \int_B \left| \Phi_b^{*,1}(f_1, f_2)(y) - c \right|^\delta dy \right)^{1/\delta} \\
 &\leq \left(\frac{1}{|B|} \int_B |(b(y) - \lambda) \Phi^*(f_1, f_2)(y)|^\delta dy \right)^{1/\delta} \\
 &\quad + \left(\frac{1}{|B|} \int_B \left| \Phi^*((b - \lambda)f_1^0, f_2^0)(y) \right|^\delta dy \right)^{1/\delta} \\
 &\quad + \left(\frac{1}{|B|} \int_B \left| \Phi^*((b - \lambda)f_1^0, f_2^\infty)(y) \right|^\delta dy \right)^{1/\delta} \\
 &\quad + \left(\frac{1}{|B|} \int_B \left| \Phi^*((b - \lambda)f_1^\infty, f_2^0)(y) \right|^\delta dy \right)^{1/\delta} \\
 &\quad + \left(\frac{1}{|B|} \int_B \sup_{\eta > 0} |\Phi_\eta((b - \lambda)f_1^\infty, f_2^\infty)(y) - c|^\delta dy \right)^{1/\delta} \\
 &:= I + II + III + IV + V.
 \end{aligned} \tag{3.8}$$

For I , since $0 < \delta < 1$, $\mu \in A_1$ and $b \in \text{Lip}_{\beta, \mu'}$ by Hölder' inequality, we have

$$\begin{aligned}
 I &\leq \frac{1}{|B|} \int_B |(b(y) - \lambda) \Phi^*(f_1, f_2)(y)| dy \\
 &\leq \frac{C}{|B^*|} \int_{B^*} |b(y) - b_{B^*}| \mu(y)^{-1/s} \Phi^*(f_1, f_2)(y) \mu(y)^{1/s} dy \\
 &\leq \frac{C}{|B^*|} \int_{B^*} (|b(y) - b_{B^*}|^{s'} \mu(y)^{1-s'})^{1/s'} \left(\int_{B^*} \Phi^*(f_1, f_2)(y)^s \mu(y) dy \right)^{1/s} \\
 &\leq C \frac{1}{\mu(B^*)^{\beta/n}} \left(\frac{1}{\mu(B^*)} \int_{B^*} |b(y) - b_{B^*}|^{s'} \mu(y)^{1-s'} dy \right)^{1/s'} \frac{\mu(B^*)^{(\beta/n)+(1/s')}}{|B^*|} \\
 &\quad \times \left(\frac{1}{\mu(B^*)^{1-(s\beta/n)}} \int_{B^*} \Phi^*(f_1, f_2)(y)^s \mu(y) dy \right)^{1/s} \mu(B^*)^{(1/s)-(\beta/n)} \\
 &\leq C \|b\|_{\text{Lip}_{\beta, \mu}} M_{\beta, \mu, s}(\Phi^*(f_1, f_2))(x) \frac{\mu(B^*)}{|B^*|} \\
 &\leq C \mu(x) \|b\|_{\text{Lip}_{\beta, \mu}} M_{\beta, \mu, s}(\Phi^*(f_1, f_2))(x).
 \end{aligned} \tag{3.9}$$

To estimate the second term II . Since $0 < \delta < 1/2$, using Kolmogorov's inequality with $p = \delta$, $q = 1/2$, $X = B$, $\omega = dx/|B|$ and the $(L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n), L^{1/2,\infty}(\mathbb{R}^n))$ -boundedness of Φ^* , we derive that

$$\begin{aligned}
 II &\leq \left\| \Phi^* \left((b - \lambda) f_1^0, f_2^0 \right) \right\|_{L^{1/2,\infty}(dy/|B|)} \\
 &\leq C \left(\frac{1}{|B|} \int_{B^*} |(b(y_1) - b_{B^*}) f_1(y_1)| dy_1 \right) \left(\frac{1}{|B|} \int_{B^*} |f_2(y_2)| dy_2 \right) \\
 &\leq C \left(\frac{1}{|B^*|} \int_{B^*} |(b(y_1) - b_{B^*}) f_1(y_1)| dy_1 \right) \left(\frac{1}{|B^*|} \int_{B^*} |f_2(y_2)| dy_2 \right) \\
 &\leq C \mu(x) \|b\|_{\text{Lip}_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x) M(f_2)(x),
 \end{aligned} \tag{3.10}$$

where we have used the analogous technique in I to get the last inequality.

For the term III , using the fact $|y - y_2| \sim |y_2 - x|$ for any $y_2 \in (B^*)^c$, $y \in B$, and note that $K_{\varphi,\eta}$ satisfies (1.3) uniformly in η , we obtain

$$\begin{aligned}
 III &\leq \frac{1}{|B|} \int_B \left| \Phi^* \left((b - \lambda) f_1^0, f_2^\infty \right) (y) \right| dy \\
 &\leq \frac{1}{|B|} \int_B \int_{B^* \times (\mathbb{R}^n \setminus B^*)} \frac{A |b(y_1) - \lambda| |f_1(y_1)| |f_2(y_2)|}{(|y - y_1| + |y - y_2|)^{2n}} dy_1 dy_2 dy \\
 &\leq \int_{B^*} |b(y_1) - b_{B^*}| |f_1(y_1)| dy_1 \int_{\mathbb{R}^n \setminus B^*} \frac{|f_2(y_2)|}{|y_2 - x|^{2n}} dy_2 \\
 &\leq C \left(\int_{B^*} |b(y_1) - b_{B^*}| |f_1(y_1)| dy_1 \right) \left(\sum_{k=1}^{\infty} \int_{2^k B^* \setminus 2^{k-1} B^*} \frac{|f_2(y_2)|}{|y_2 - x|^{2n}} dy_2 \right) \\
 &\leq C \frac{1}{|B^*|} \left(\int_{B^*} |b(y_1) - b_{B^*}| |f_1(y_1)| dy_1 \right) \left(\sum_{k=1}^{\infty} 2^{-kn} \frac{1}{|2^k B^*|} \int_{2^k B^*} |f_2(y_2)| dy_2 \right) \\
 &\leq C \mu(x) \|b\|_{\text{Lip}_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x) M(f_2)(x).
 \end{aligned} \tag{3.11}$$

For the term IV , using the fact $|y - y_1| \sim |y_1 - x|$ for any $y_1 \in (B^*)^c$, $y \in B$, and note that $K_{\varphi,\eta}$ satisfies (1.3) uniformly in η , and using (2.4), we obtain

$$\begin{aligned}
 IV &\leq \frac{1}{|B|} \int_B \left| \Phi^* \left((b - \lambda) f_1^\infty, f_2^0 \right) (y) \right| dy \\
 &\leq \frac{1}{|B|} \int_B \int_{(\mathbb{R}^n \setminus B^*) \times B^*} \frac{A |b(y_1) - \lambda| |f_1(y_1)| |f_2(y_2)|}{(|y - y_1| + |y - y_2|)^{2n}} dy_1 dy_2 dy \\
 &\leq C \int_{\mathbb{R}^n \setminus B^*} \frac{|b(y_1) - b_{B^*}| |f_1(y_1)|}{|x - y_1|^{2n}} dy_1 \int_{B^*} |f_2(y_2)| dy_2
 \end{aligned}$$

$$\begin{aligned}
&\leq C \left(\sum_{k=0}^{\infty} \int_{2^{k+1}B^* \setminus 2^k B^*} \frac{|b(y_1) - b_{B^*}| |f_1(y_1)|}{|x - y_1|^{2n}} dy_1 \right) \left(\int_{B^*} |f_2(y_2)| dy_2 \right) \\
&\leq C \left(\sum_{k=0}^{\infty} \frac{1}{|2^k B^*|^2} \int_{2^{k+1}B^*} |b(y_1) - b_{B^*}| |f_1(y_1)| dy_1 \right) \left(\int_{B^*} |f_2(y_2)| dy_2 \right) \\
&\leq C \left(\sum_{k=0}^{\infty} 2^{-kn} \frac{1}{|2^{k+1}B^*|} \int_{2^{k+1}B^*} |b(y_1) - b_{B^*}| |f_1(y_1)| dy_1 \right) \left(\frac{1}{|B^*|} \int_{B^*} |f_2(y_2)| dy_2 \right) \\
&\leq CM(f_2)(x) \sum_{k=0}^{\infty} 2^{-kn} \frac{1}{|2^{k+1}B^*|} \left(\int_{2^{k+1}B^*} |b(y_1) - b_{2^{k+1}B^*}| |f_1(y_1)| dy_1 \right. \\
&\quad \left. + |b_{B^*} - b_{2^{k+1}B^*}| \int_{2^{k+1}B^*} |f_1(y_1)| dy_1 \right) \\
&\leq CM(f_2)(x) \sum_{k=0}^{\infty} 2^{-kn} \left(\mu(x) \|b\|_{Lip_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x) \right. \\
&\quad \left. + k \mu(x) \|b\|_{Lip_{\beta,\mu}} \frac{\mu(B(x, 2^{k+1}R))^{\beta/n}}{|2^{k+1}B^*|} \int_{2^{k+1}B^*} |f_1(y_1)| dy_1 \right) \\
&\leq C \mu(x) \|b\|_{Lip_{\beta,\mu}} M(f_2)(x) M_{\beta,\mu,s}(f_1)(x).
\end{aligned} \tag{3.12}$$

For V , fix the value of c by taking $c = \Phi^*((b - \lambda)f_1^\infty, f_2^\infty)(x)$, recall that $K_{\varphi,\eta}$ satisfies (1.4) uniformly in η , then we can obtain

$$\begin{aligned}
V &\leq \frac{1}{|B|} \int_B |\Phi^*((b - \lambda)f_1^\infty, f_2^\infty)(y) - \Phi^*((b - \lambda)f_1^\infty, f_2^\infty)(x)| dy \\
&\leq \frac{1}{|B|} \int_B \sup_{\eta>0} |\Phi_\eta((b - \lambda)f_1^\infty, f_2^\infty)(y) - \Phi_\eta((b - \lambda)f_1^\infty, f_2^\infty)(x)| dy \\
&\leq \frac{1}{|B|} \int_B \int_{(\mathbb{R}^n \setminus B^*)^2} \sup_{\eta>0} |K_{\varphi,\eta}(y, y_1, y_2) - K_{\varphi,\eta}(x, y_1, y_2)| |b(y_1 - \lambda)| |f_1(y_1)| \\
&\quad \times |f_2(y_2)| dy_1 dy_2 dy \\
&\leq \frac{C}{|B|} \int_B \int_{(\mathbb{R}^n \setminus B^*)^2} \frac{|x - y|^\varepsilon}{(|y - y_1| + |y - y_2|)^{2n+\varepsilon}} |(b(y_1) - b_{B^*}) f_1(y_1) f_2(y_2)| dy_1 dy_2 dy \\
&\leq \frac{C}{|B|} \sum_{k=0}^{\infty} \int_{2^{k+1}B^* \setminus 2^k B^*} \frac{|x - y|^\varepsilon}{|y_1 - x|^{2n+\varepsilon}} |(b(y_1) - b_{B^*}) f_1(y_1)| |f_2(y_2)| dy_1 dy_2 dy \\
&\leq C \left(\sum_{k=0}^{\infty} \frac{|B^*|^{\varepsilon/n}}{|2^k B^*|^{2+\varepsilon/n}} \int_{2^{k+1}B^*} |(b(y_1) - b_{B^*}) f_1(y_1)| dy_1 \right) \left(\int_{2^{k+1}B^*} |f_2(y_2)| dy_2 \right)
\end{aligned}$$

$$\begin{aligned}
&\leq C \left(\sum_{k=0}^{\infty} 2^{-k\varepsilon} \frac{1}{|2^{k+1}B^*|} \int_{2^{k+1}B^*} |(b(y_1) - b_{B^*}) f_1(y_1)| dy_1 \right) \left(\frac{1}{|2^{k+1}B^*|} \int_{2^{k+1}B^*} |f_2(y_2)| dy_2 \right) \\
&\leq C \mu(x) \|b\|_{\text{Lip}_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x) M(f_2)(x),
\end{aligned} \tag{3.13}$$

where in the last inequality, we use the same computation in the IV term.

Consequently, combining the estimates of I, II, III , and V , we conclude the proof of Lemma 3.1. $\square$

Now we are ready to return to prove Theorem 1.3.

Proof. First, by Lemma 2.1, we have that $\mu \in A_1 \subset A_{r'}$, and hence $\mu^{1-r} \in A_r \subset A_{\infty}$. Then by Lemma 2.3, we obtain

$$\begin{aligned}
\|\Phi_{b_j}^{*,j}(f_1, f_2)(x)\|_{L^r(\mu^{1-r})} &\leq \|M_{\delta}(\Phi_{b_j}^{*,j}(f_1, f_2))(x)\|_{L^r(\mu^{1-r})} \\
&\leq \|M_{\delta}^{\#}(\Phi_{b_j}^{*,j}(f_1, f_2))(x)\|_{L^r(\mu^{1-r})}.
\end{aligned} \tag{3.14}$$

For $j = 1, 2$, by Lemma 3.1, we reduce to bound the $\|\cdot\|_{L^r(\mu^{1-r})}$ norm of the right-hand side of (3.6). For the first term, since $1/r = (1/p) - (\beta/n)$ and taking s such that $1 < s < p < n/\beta$, by Lemma 2.4, and Theorem B(ii), we have

$$\begin{aligned}
\|\mu M_{\beta,\mu,s}(\Phi^*(f_1, f_2))\|_{L^r(\mu^{1-r})} &= \|M_{\beta,\mu,s}(\Phi^*(f_1, f_2))\|_{L^r(\mu)} \\
&\leq C \|\Phi^*(f_1, f_2)\|_{L^p(\mu)} \\
&\leq C \|f_1\|_{L^{p_1}(\mu)} \|f_2\|_{L^{p_2}(\mu)}.
\end{aligned} \tag{3.15}$$

For the second term, we let $1/r = 1/p_2 + 1/l$, and $1/l = 1/p_1 - \beta/n$. Then by Lemma 2.4 again, together with Hölder's inequality, we obtain

$$\begin{aligned}
\|\mu M_{\beta,\mu,s}(f_1) M(f_2)\|_{L^r(\mu^{1-r})} &= \|M_{\beta,\mu,s}(f_1) M(f_2)\|_{L^r(\mu)} \\
&\leq \|M_{\beta,\mu,s}(f_1)\|_{L^l(\mu)} \|M(f_2)\|_{L^{p_2}(\mu)} \\
&\leq C \|f_1\|_{L^{p_1}(\mu)} \|f_2\|_{L^{p_2}(\mu)}.
\end{aligned} \tag{3.16}$$

We can obtain that

$$\|\Phi_b^{*,1}(f_1, f_2)\|_{L^r(\mu^{1-r})} \leq C \|b\|_{\text{Lip}_{\beta,\mu}} \|f_1\|_{L^{p_1}(\mu)} \|f_2\|_{L^{p_2}(\mu)}. \tag{3.17}$$

Similarly, we have

$$\|\Psi_b^{*,1}(f_1, f_2)\|_{L^r(\mu^{1-r})} \leq C \|b\|_{\text{Lip}_{\beta,\mu}} \|f_1\|_{L^{p_1}(\mu)} \|f_2\|_{L^{p_2}(\mu)}. \tag{3.18}$$

Consequently, by the above arguments, we conclude the proof of Theorem 1.3. $\square$

Similarly as the proof of Lemma 3.1 and that of Theorem 1.3, we only consider the case $m = 2$ and establish the following sharp maximal function for $\Phi_{\Pi b}^*$.

Lemma 3.2. *Let $\mu \in A_1(\mathbb{R}^n)$ and $b_i \in Lip_{\beta_i, \mu}$, $i = 1, 2$; $\beta = \beta_1 + \beta_2$, and $0 < \beta < 1$. And let $0 < \delta < 1/3 < 1 < s_i < n/\beta_i$, $i = 1, 2$. Then one has*

$$M_\delta^\#(\Phi_{\Pi b}^*(f_1, f_2))(x) \leq \mu(x)^2 \prod_{j=1}^2 \|b_j\|_{Lip_{\beta_j, \mu}} \times (M_{\beta, \mu, s}(\Phi^*(f_1, f_2))(x) + M_{\beta, \mu, s}(f_1)(x)M_{\beta, \mu, s}(f_2)(x)), \quad (3.19)$$

$$M_\delta^\#(\Psi_{\Pi b}^*(f_1, f_2))(x) \leq \mu(x)^2 \prod_{j=1}^2 \|b_j\|_{Lip_{\beta_j, \mu}} \times (M_{\beta, \mu, s}(\Phi^*(f_1, f_2))(x) + M_{\beta, \mu, s}(f_1)(x)M_{\beta, \mu, s}(f_2)(x)). \quad (3.20)$$

Proof. Fix $x \in \mathbb{R}^n$ and let $B = B(x, R)$ with $n > 0$. Taking $\lambda_i = (b_i)_{B^*}$, the average of b_i on B^* , $i = 1, 2$, where $B^* = B(x, 2R)$. Let c be a constant to be fixed along the proof. We split $\Phi_{\Pi b}^*(f_1, f_2)(y)$ in the following way:

$$\begin{aligned} \Phi_{\Pi b}^*(f_1, f_2)(y) &= \sup_{\eta > 0} |(b_1(y) - \lambda_1)(b_2(y) - \lambda_2)\Phi_\eta(f_1, f_2)(y) \\ &\quad - (b_1(y) - \lambda_1)\Phi_\eta(f_1, (b_2 - \lambda_2)f_2)(y) - (b_2(y) - \lambda_2)\Phi_\eta \\ &\quad \times ((b_1 - \lambda_1)f_1, f_2)(y) + \Phi_\eta((b_1 - \lambda_1)f_1, (b_2 - \lambda_2)f_2)(y)|. \end{aligned} \quad (3.21)$$

Since $0 < \delta < 1/3$, then we have

$$\begin{aligned} &\left(\frac{1}{|B|} \int_B |\Phi_{\Pi b}^*(f_1, f_2)(y)|^\delta - |c|^\delta dy \right)^{1/\delta} \\ &\leq \left(\frac{1}{|B|} \int_B |\Phi_{\Pi b}^*(f_1, f_2)(y) - c|^\delta dy \right)^{1/\delta} \\ &\leq \left(\frac{1}{|B|} \int_B |(b_1(y) - \lambda_1)(b_2(y) - \lambda_2)\Phi^*(f_1, f_2)(y)|^\delta dy \right)^{1/\delta} \\ &\quad + \left(\frac{1}{|B|} \int_B \left(\sup_{\eta > 0} |(b_1(y) - \lambda_1)\Phi_\eta(f_1, (b_2 - \lambda_2)f_2)(y)| \right)^\delta dy \right)^{1/\delta} \\ &\quad + \left(\frac{1}{|B|} \int_B \left(\sup_{\eta > 0} |(b_2(y) - \lambda_2)\Phi_\eta((b_1 - \lambda_1)f_1, f_2)(y)| \right)^\delta dy \right)^{1/\delta} \\ &\quad + \left(\frac{1}{|B|} \int_B \sup_{\eta > 0} |\Phi_\eta((b_1 - \lambda_1)f_1, (b_2 - \lambda_2)f_2) - c|^\delta dy \right)^{1/\delta} \\ &:= U_1 + U_2 + U_3 + U_4. \end{aligned} \quad (3.22)$$

For the term U_1 , since $0 < \delta < 1/3$, and $\beta = \beta_1 + \beta_2$, then by Hölder's inequality, we have

$$\begin{aligned}
U_1 &\leq \left(\frac{1}{|B|} \int_B |b_1(y) - \lambda_1|^{3\delta} dy \right)^{1/3\delta} \left(\frac{1}{|B|} \int_B |b_2(y) - \lambda_2|^{3\delta} dy \right)^{1/3\delta} \\
&\quad \times \left(\frac{1}{|B|} \int_B |\Phi^*(f_1, f_2)(y)|^{3\delta} dy \right)^{1/3\delta} \\
&\leq C \left(\frac{1}{|B^*|} \int_{B^*} |b_1(y) - (b_1)_{B^*}| dy \right) \left(\frac{1}{|B^*|} \int_{B^*} |b_2(y) - (b_2)_{B^*}| dy \right) \\
&\quad \times \left(\frac{1}{|B^*|} \int_{B^*} |\Phi^*(f_1, f_2)(y)| dy \right) \\
&\leq C \left(\frac{1}{|B^*|} \int_{B^*} |b_1(y) - (b_1)_{B^*}|^{s'} \mu(y)^{1-s'} dy \right)^{1/s'} \left(\frac{1}{|B^*|} \int_{B^*} \mu(y) dy \right)^{1/s} \\
&\quad \times \left(\frac{1}{|B^*|} \int_{B^*} |b_2(y) - (b_2)_{B^*}|^{s'} \mu(y)^{1-s'} dy \right)^{1/s'} \left(\frac{1}{|B^*|} \int_{B^*} \mu(y) dy \right)^{1/s} \\
&\quad \times \left(\frac{1}{|B^*|} \int_{B^*} |\Phi^*(f_1, f_2)(y)|^s \mu(y) dy \right)^{1/s} \left(\frac{1}{|B^*|} \int_{B^*} \mu(y)^{-s'/s} dy \right)^{1/s'} \\
&\leq C \frac{1}{\mu(B^*)^{\beta_1/n}} \left(\frac{1}{\mu(B^*)} \int_{B^*} |b_1(y) - (b_1)_{B^*}|^{s'} \mu(y)^{1-s'} dy \right)^{1/s'} \frac{\mu(B^*)^{(\beta_1/n)+1}}{|B^*|} \\
&\quad \times \frac{1}{\mu(B^*)^{\beta_2/n}} \left(\frac{1}{\mu(B^*)} \int_{B^*} |b_2(y) - (b_2)_{B^*}|^{s'} \mu(y)^{1-s'} dy \right)^{1/s'} \frac{\mu(B^*)^{(\beta_2/n)+1}}{|B^*|} \\
&\quad \times \left(\frac{1}{\mu(B^*)^{1-(s\beta/n)}} \int_{B^*} \Phi^*(f_1, f_2)(y)^s \mu(y) dy \right)^{1/s} \frac{\mu(B^*)^{(1/s)-(\beta/n)} \mu(B^*)^{(1/s')-1}}{|B^*|} \\
&\leq C \mu(x)^2 \|b_1\|_{\text{Lip}_{\beta_1, \mu}} \|b_2\|_{\text{Lip}_{\beta_2, \mu}} M_{\beta, \mu, s}(\Phi^*(f_1, f_2))(x).
\end{aligned} \tag{3.23}$$

For the term U_2 , noting that $0 < \delta < 1/3$, we use the facts $1 = \delta + (1 - \delta)$ and $0 < \delta/(1 - \delta) < 1/2$, then by Hölder's inequality and Komolgorov's inequality (Lemma 2.2) and Theorem B, we have

$$\begin{aligned}
U_2 &\leq C \left(\frac{1}{|B^*|} \int_{B^*} |b_1(y) - (b_1)_{B^*}| dy \right) \left(\frac{1}{|B^*|} \int_{B^*} \sup_{\eta > 0} |\Phi_\eta(f_1, (b_2 - \lambda_2)f_2)(y)|^{\delta/(1-\delta)} dy \right)^{(1-\delta)/\delta} \\
&\leq \left(\frac{1}{|B^*|} \int_{B^*} |b_1(y) - (b_1)_{B^*}| dy \right) \left(\frac{1}{|B^*|} \int_{B^*} |\Phi^*(f_1, (b_2 - \lambda_2)f_2)(y)|^{\delta/(1-\delta)} dy \right)^{1-\delta/\delta} \\
&\leq \left(\frac{1}{|B^*|} \int_{B^*} |b_1(y) - (b_1)_{B^*}| dy \right) \|\Phi^*(f_1, (b_2 - \lambda_2)f_2)\|_{L^{1/2, \infty}(dy/|B^*|)} \\
&\leq \left(\frac{1}{|B^*|} \int_{B^*} |b_1(y) - (b_1)_{B^*}| dy \right) \left(\frac{1}{|B^*|} \int_{B^*} |f_1(y_1)| dy_1 \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{1}{|B^*|} \int_{B^*} |(b_2(y_2) - (b_2)_{B^*}) f_2(y_2)| dy_2 \right) \\
& \leq \left(\frac{1}{|B^*|} \int_{B^*} |b_1(y) - (b_1)_{B^*}|^{s'} \mu(y)^{1-s'} dy \right)^{1/s'} \left(\int_{B^*} \mu(y) dy \right)^{1/s} \\
& \quad \times \left(\frac{1}{|B^*|} \int_{B^*} |f_1(y_1)|^s \mu(y_1) dy_1 \right)^{1/s} \left(\frac{1}{|B^*|} \int_{B^*} \mu(y_1)^{-s'/s} dy_1 \right)^{1/s'} \\
& \quad \times \frac{1}{|B^*|} \left(\int_{B^*} |b_2(y_2) - (b_2)_{B^*}|^{s'} \mu(y_2)^{1-s'} dy_2 \right)^{1/s'} \left(\int_{B^*} |f_2(y_2)|^s \mu(y_2) dy_2 \right)^{1/s} \\
& \leq \frac{1}{\mu(B^*)^{\beta_1/n}} \left(\frac{1}{\mu(B^*)} \int_{B^*} |b(y) - b_{B^*}|^{s'} \mu(y)^{1-s'} dy \right)^{1/s'} \frac{\mu(B^*)^{(\beta_1/n)+1}}{|B^*|} \\
& \quad \times \left(\frac{1}{\mu(B^*)^{1-(s\beta/n)}} \int_{B^*} |f_1(y_1)|^s \mu(y_1) dy_1 \right)^{1/s} \frac{\mu(B^*)^{(1/s)-(\beta/n)} \mu(B^*)^{-1/s}}{|B^*|} \\
& \quad \times \frac{1}{\mu(B^*)^{\beta_2/n}} \left(\frac{1}{\mu(B^*)} \int_{B^*} |b_2(y) - (b_2)_{B^*}|^{s'} \mu(y)^{1-s'} dy_2 \right)^{1/s'} \frac{\mu(B^*)^{(\beta_2/n)+(1/s')}}{|B^*|} \\
& \quad \times \left(\frac{1}{\mu(B^*)^{1-(s\beta/n)}} \int_{B^*} |f_2(y_2)|^s \mu(y_2) dy_2 \right)^{1/s} \mu(B^*)^{(1/s)-(\beta/n)} \\
& \leq C\mu(x)^2 \|b_1\|_{\text{Lip}_{\beta_1,\mu}} \|b_2\|_{\text{Lip}_{\beta_2,\mu}} M_{\beta,\mu,s}(f_1)(x) M_{\beta,\mu,s}(f_2)(x).
\end{aligned} \tag{3.24}$$

Similarly, for the term U_3 , we have

$$U_3 \leq C\mu(x)^2 \|b_1\|_{\text{Lip}_{\beta,\mu}} \|b_2\|_{\text{Lip}_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x) M_{\beta,\mu,s}(f_2)(x). \tag{3.25}$$

Now we turn to estimate the last term U_4 . To proceed, we denote that $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{B^*}$, $i = 1, 2$. Let $c = c_1 + c_2 + c_3$, where

$$\begin{aligned}
c_1 &= \Phi_\eta \left((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^\infty \right)(x), \\
c_2 &= \Phi_\eta \left((b_1 - \lambda_1) f_1^\infty, (b_2 - \lambda_2) f_2^0 \right)(x), \\
c_3 &= \Phi_\eta \left((b_1 - \lambda_1) f_1^\infty, (b_2 - \lambda_2) f_2^\infty \right)(x).
\end{aligned} \tag{3.26}$$

We split IV in the following way:

$$U_4 \leq U_{41} + U_{42} + U_{43} + U_{44}, \tag{3.27}$$

where

$$\begin{aligned}
U_{41} &= \left(\frac{1}{|B|} \int_B \sup_{\eta > 0} \left| \Phi_\eta \left((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^0 \right) (y) \right|^\delta dy \right)^{1/\delta}, \\
U_{42} &= \left(\frac{1}{|B|} \int_B \sup_{\eta > 0} \left| \Phi_\eta \left((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^\infty \right) (y) \right. \right. \\
&\quad \left. \left. - \Phi_\eta \left((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^\infty \right) (x) \right|^\delta dy \right)^{1/\delta}, \\
U_{43} &= \left(\frac{1}{|B|} \int_B \sup_{\eta > 0} \left| \Phi_\eta \left((b_1 - \lambda_1) f_1^\infty, (b_2 - \lambda_2) f_2^0 \right) (y) \right. \right. \\
&\quad \left. \left. - \Phi_\eta \left((b_1 - \lambda_1) f_1^\infty, (b_2 - \lambda_2) f_2^0 \right) (x) \right|^\delta dy \right)^{1/\delta}, \\
U_{44} &= \left(\frac{1}{|B|} \int_B \sup_{\eta > 0} \left| \Phi_\eta \left((b_1 - \lambda_1) f_1^\infty, (b_2 - \lambda_2) f_2^\infty \right) (y) \right. \right. \\
&\quad \left. \left. - \Phi_\eta \left((b_1 - \lambda_1) f_1^\infty, (b_2 - \lambda_2) f_2^\infty \right) (x) \right|^\delta dy \right)^{1/\delta}.
\end{aligned} \tag{3.28}$$

For the term U_{41} , we choose $1 < p_0 < 1/2\delta$ and use Kolmogorov's inequality and Theorem B, then we use the same computation as U_2 to deduce that

$$\begin{aligned}
U_{41} &\leq \left(\frac{1}{|B|} \int_B \left| \Phi^* \left((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^0 \right) (y) \right|^{p_0 \delta} dy \right)^{1/p_0 \delta} \\
&\leq \left\| \Phi^* \left((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^0 \right) \right\|_{L^{1/2, \infty}(dy/|B|)} \\
&\leq \left(\frac{1}{|B|} \int_B \left| (b_1(y_1) - \lambda_1) f_1^0(y_1) \right| dy_1 \right) \left(\frac{1}{|B|} \int_B \left| (b_2(y_2) - \lambda_2) f_2^0(y_2) \right| dy_2 \right) \\
&\leq C \left(\frac{1}{|B^*|} \int_{B^*} \left| (b_1(y_1) - (b_1)_{B^*}) f_1(y_1) \right| dy_1 \right) \left(\frac{1}{|B^*|} \int_{B^*} \left| (b_2(y_2) - (b_2)_{B^*}) f_2(y_2) \right| dy_2 \right) \\
&\leq C \mu(x)^2 \|b_1\|_{\text{Lip}_{\beta_1, \mu}} \|b_2\|_{\text{Lip}_{\beta_2, \mu}} M_{\beta, \mu, s}(f_1)(x) M_{\beta, \mu, s}(f_2)(x).
\end{aligned} \tag{3.29}$$

For U_{42} , by the fact $|y - y_2| \sim |y_2 - x|$, for any $y_2 \in (B^*)^c$, $y \in B$, and note that $K_{\varphi, \eta}$ satisfies (1.4) uniformly in η , then we get that

$$\begin{aligned}
U_{42} &\leq \left(\frac{1}{|B|} \int_B \sup_{\eta > 0} \left| \Phi_\eta \left((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^\infty \right) (y) \right. \right. \\
&\quad \left. \left. - \Phi_\eta \left((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^\infty \right) (x) \right| dy \right) \\
&\leq \frac{C}{|B|} \int_B \left(\int_{B^*} \left| (b_1(y_1) - \lambda_1) f_1(y_1) \right| dy_1 \int_{(B^*)^c} \frac{|y - x|^\varepsilon |b_2(y_2) - \lambda_2| |f_2(y_2)|}{(|y - y_1| + |y - y_2|)^{2n+\varepsilon}} dy_2 \right) dy
\end{aligned}$$

$$\begin{aligned}
& \leq \left(\int_{B^*} |(b_1(y_1) - (b_1)_{B^*}) f_1(y_1)| dy_1 \right) \\
& \quad \times \left(\sum_{k=0}^{\infty} \int_{2^{k+1}B^* \setminus 2^k B^*} \frac{|y-x|^\varepsilon |b_2(y_2) - (b_2)_{B^*}| |f_2(y_2)|}{|x-y_2|^{2n+\varepsilon}} dy_2 \right) \\
& \leq \left(\int_{B^*} |(b_1(y_1) - (b_1)_{B^*}) f_1(y_1)| dy_1 \right) \\
& \quad \times \left(\sum_{k=0}^{\infty} \frac{|B^*|^{\varepsilon/n}}{|2^k B^*|^{2+\varepsilon/n}} \int_{2^{k+1}B^*} |b_2(y_2) - (b_2)_{B^*}| |f_2(y_2)| dy_2 \right) \\
& \leq \left(\frac{1}{|B^*|} \int_{B^*} |(b_1(y_1) - (b_1)_{B^*}) f_1(y_1)| dy_1 \right) \\
& \quad \times \left(\sum_{k=0}^{\infty} 2^{-k\varepsilon} \frac{1}{|2^{k+1}B^*|} \int_{2^{k+1}B^*} |b_2(y_2) - (b_2)_{B^*}| |f_2(y_2)| dy_2 \right) \\
& \leq C\mu(x) \|b_1\|_{\text{Lip}_{\beta_1, \mu}} M_{\beta, \mu, s}(f_1)(x) \sum_{k=1}^{\infty} 2^{-k\varepsilon} \frac{1}{|2^{k+1}B^*|} \left(\int_{2^{k+1}B^*} |b_2(y_2) - (b_2)_{2^{k+1}B^*}| |f_2(y_2)| dy_2 \right. \\
& \quad \left. + |(b_2)_{B^*} - (b_2)_{2^{k+1}B^*}| \int_{2^{k+1}B^*} |f_2(y_2)| dy_2 \right) \\
& \leq C\mu(x)^2 \|b_1\|_{\text{Lip}_{\beta_1, \mu}} \|b_2\|_{\text{Lip}_{\beta_2, \mu}} M_{\beta, \mu, s}(f_1)(x) M_{\beta, \mu, s}(f_2)(x),
\end{aligned} \tag{3.30}$$

where we have used the same computation of IV to gain the last inequality.

Similarly as U_{42} , we can get the estimates for U_{43} ,

$$U_{43} \leq C\mu(x)^2 \|b_1\|_{\text{Lip}_{\beta, \mu}} \|b_2\|_{\text{Lip}_{\beta, \mu}} M_{\beta, \mu, s}(f_1)(x) M_{\beta, \mu, s}(f_2)(x). \tag{3.31}$$

Now we turn to U_{44} , by the fact $|y-y_1| \sim |y_1-x|$, $|y-y_2| \sim |y_2-x|$ for any $y_1, y_2 \in (B^*)^c$, $y \in B$, and recalling that $K_{\varphi, \eta}$ satisfies (1.4) uniformly in η , then we can obtain

$$\begin{aligned}
& |\Phi_\eta((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty)(y) - \Phi_\eta((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty)(x)| \\
& \leq C \sum_{k=0}^{\infty} \int_{(2^{k+1}B^* \setminus 2^k B^*)^2} \frac{|x-y|^\varepsilon}{(|y-y_1| + |y-y_2|)^{2n+\varepsilon}} \\
& \quad |(b_1(y_1) - \lambda_1)f_1(y_1)(b_2(y_2) - \lambda_2)f_2(y_2)| dy_1 dy_2 \\
& \leq C \sum_{k=0}^{\infty} \int_{(2^{k+1}B^* \setminus 2^k B^*)^2} \frac{|x-y|^\varepsilon}{|y-y_1|^{2n+\varepsilon}} |(b_1(y_1) - \lambda_1)f_1(y_1)(b_2(y_2) - \lambda_2)f_2(y_2)| dy_1 dy_2 \\
& \leq C \sum_{k=0}^{\infty} \frac{|B^*|^{\varepsilon/n}}{|2^k B^*|^{2+\varepsilon/n}} \int_{2^{k+1}B^*} |(b_1(y_1) - \lambda_1)f_1(y_1)| dy_1 \int_{2^{k+1}B^*} |(b_2(y_2) - \lambda_2)f_2(y_2)| dy_2
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{k=0}^{\infty} 2^{-k\varepsilon} \left(\frac{1}{|2^{k+1}B^*|} \int_{2^{k+1}B^*} |(b_1(y_1) - \lambda_1)f_1(y_1)| dy_1 \right) \\
&\quad \times \left(\frac{1}{|2^{k+1}B^*|} \int_{2^{k+1}B^*} |(b_2(y_2) - \lambda_2)f_2(y_2)| dy_2 \right) \\
&\leq C\mu(x)^2 \|b_1\|_{\text{Lip}_{\beta_1,\mu}} \|b_2\|_{\text{Lip}_{\beta_2,\mu}} M_{\beta,\mu,s}(f_1)(x) M_{\beta,\mu,s}(f_2)(x).
\end{aligned} \tag{3.32}$$

Therefore,

$$\begin{aligned}
U_{44} &\leq \frac{1}{|B|} \int_B \sup_{\eta>0} |\Phi_{\eta}((b_1 - \lambda_1)f_1^{\infty}, (b_2 - \lambda_2)f_2^{\infty})(y) - \Phi_{\eta}((b_1 - \lambda_1)f_1^{\infty}, (b_2 - \lambda_2)f_2^{\infty})(x)| dy \\
&\leq C\mu(x)^2 \|b_1\|_{\text{Lip}_{\beta_1,\mu}} \|b_2\|_{\text{Lip}_{\beta_2,\mu}} M_{\beta,\mu,s}(f_1)(x) M_{\beta,\mu,s}(f_2)(x).
\end{aligned} \tag{3.33}$$

Consequently, the estimate for U_1 , together with those of U_2, U_3 , and U_4 , can conclude the proof of Lemma 3.2. $\square$

Now we return to prove Theorem 1.5.

Proof. Similarly as the proof of Theorem 1.3,

$$\|\Phi_{\Pi b}^*(f_1, f_2)\|_{L^r(\mu^{1-2r})} \leq \|M_{\delta}(\Phi_{\Pi b}^*(f_1, f_2))\|_{L^r(\mu^{1-2r})} \leq \|M_{\delta}^{\#}(\Phi_{\Pi b}^*(f_1, f_2))\|_{L^r(\mu^{1-2r})}. \tag{3.34}$$

We reduce to bound the $\|\cdot\|_{L^r(\mu^{1-r})}$ norm of the right-hand side of (3.19). We estimate each term as follows. For the first term, since $1/r = (1/p) - (\beta/n)$ and choosing s such that $1 < s < p < n/\beta$, by Lemma 2.4 and Theorem A and observe that $\mu \in A_1$, we obtain

$$\begin{aligned}
\|\mu^2 M_{\beta,\mu,s}(\Phi^*(f_1, f_2))\|_{L^r(\mu^{1-2r})} &= \|M_{\beta,\mu,s}(\Phi^*(f_1, f_2))\|_{L^r(\mu)} \\
&\leq C \|\Phi^*(f_1, f_2)\|_{L^p(\mu)} \\
&\leq C \|f_1\|_{L^{p_1}(\mu)} \|f_2\|_{L^{p_2}(\mu)}.
\end{aligned} \tag{3.35}$$

For the second term, since $1/r = (1/r_1) + (1/r_2)$, by Hölder's inequality and Lemma 2.4, we get

$$\begin{aligned}
\|\mu^2 M_{\beta,\mu,s}(f_1) M_{\beta,\mu,s}(f_2)\|_{L^r(\mu^{1-2r})} &= \|M_{\beta,\mu,s}(f_1) M_{\beta,\mu,s}(f_2)\|_{L^r(\mu)} \\
&\leq \|M_{\beta,\mu,s}(f_1)\|_{L^{r_1}(\mu)} \|M_{\beta,\mu,s}(f_2)\|_{L^{r_2}(\mu)} \\
&\leq C \|f_1\|_{L^{p_1}(\mu)} \|f_2\|_{L^{p_2}(\mu)}.
\end{aligned} \tag{3.36}$$

Similarly, we also have

$$\|\Psi_{\Pi b}^*(f_1, f_2)\|_{L^r(\mu^{1-2r})} \leq C \|f_1\|_{L^{p_1}(\mu)} \|f_2\|_{L^{p_2}(\mu)}. \quad (3.37)$$

This estimate together with that for $\Phi_{\Pi b}^*(f_1, f_2)$ finishes the proof of Theorem 1.5. $\square$

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