Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2012, Article ID 142731, 11 pages
doi:10.1155/2012/142731

Research Article

A Class of Schur Multipliers on Some Quasi-Banach Spaces of Infinite Matrices

Nicolae Popa

Unit Research 1, Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

Correspondence should be addressed to Nicolae Popa, npopa@imar.ro

Received 31 July 2011; Accepted 16 November 2011

Academic Editor: Lars Erik Persson

We characterize the Schur multipliers of scalar type acting on scattered classes of infinite matrices.

In [1], Schur introduced a new product between two matrices $A = (a_{jk})$ and $B = (b_{jk})$ of the same size, finite or infinite. This product, known in the literature as *the Schur product or Hadamard product,* is defined to be the matrix of elementwise products

$$A * B = \left(a_{jk} b_{jk}\right). \tag{1}$$

This concept was used in different areas of analysis as complex function theory, Banach spaces, operator theory, and multivariate analysis.

Bennett studied in [2] the behaviour, under Schur multiplication, of the norm $\|\cdot\|_{p,q}$, $1 \le p, q \le \infty$,

$$\|A\|_{p\cdot q} = \sup_{\|x\|_p \le 1} \left(\sum_j \left| \sum_k a_{jk} x_k \right|^q \right)^{1/q}. \tag{2}$$

In particular, he was interested in characterizing the (p, q)-*multipliers:* the matrices M for which $M * A$ maps ℓ_p into ℓ_q whenever A does.

In his paper it is proved a theorem about Schur multipliers which are Toeplitz matrices, that is about the matrices of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_{-1} & a_0 & a_1 & a_2 & \cdots \\ a_{-2} & a_{-1} & a_0 & a_1 & \cdots \\ a_{-3} & a_{-2} & a_{-1} & a_0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{3}$$

where $(a_j)_{j=-\infty}^{\infty}$ is a sequence of complex numbers.

Theorem 8.1 in [2] reads as follows.

Theorem B. *A Toeplitz matrix A is a Schur multiplier if and only if $\mu = \sum_{j=-\infty}^{\infty} a_j e^{ijt}$ is a bounded Borel measure on $[0, 2\pi)$.*

This fact leads naturally to the idea of identifying the Schur multipliers with the noncommutative bounded Borel measures, see, for example, [3].

We denote by $M(\ell_2)$ the space of all (2,2) Schur multipliers from $B(\ell_2)$ into $B(\ell_2)$, where $B(\ell_2)$ is, as usual, the Banach space of linear and bounded operators on ℓ_2 with the usual operator norm.

The space $M(\ell_2)$ endowed with norm $\|A\|_{M(\ell_2)} = \sup_{\|B\|_{B(\ell_2)\leq 1}} \|A * B\|_{B(\ell_2)}$ becomes a Banach space.

Since we work with different quasi-Banach spaces of matrices X, Y we use the notation (X, Y) for the space of all Schur multipliers from X into Y equipped with the quasi-norm

$$\|A\|_{(X,Y)} = \sup_{\|B\|_Y \leq 1} \|A * B\|_X. \tag{4}$$

In this way (X, Y) becomes a quasi-Banach.

In [2] Bennett raised the problem of characterizing the Hankel matrices which are Schur multipliers.

We recall that a matrix A is called a *Hankel matrix* if it is defined by a sequence $(a_j)_{j=1}^{\infty}$ of complex numbers in the following way:

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{5}$$

Pisier in [4] solved the above problem. He proved the following theorem.

Theorem P. *A Hankel matrix is a Schur multiplier if and only if the Fourier multiplier* $\sum_{n=0}^{\infty} x_n e^{\mathrm{int}} \to \sum_{n=0}^{\infty} a_n x_n e^{\mathrm{int}}$ *maps boundedly* $H^1(S_1)$ *into itself.*

Here $H^1(S_1)$ is the Hardy space of the Schatten class S_1-valued analytic functions, endowed with the norm $\|f\|_{H^1(S_1)} = (1/2\pi)\int_0^{2\pi} \|\sum_{n=0}^{\infty} A_n e^{\mathrm{int}}\|_{S_1} dt < \infty$. For the definition of the Schatten classes S_p, see, for example, [5].

In [5], Aleksandrov and Peller characterized the Toeplitz matrices which are Schur multipliers for S_p, $0 < p < 1$. They proved the following theorem.

Theorem AP. *Let* $0 < p < 1$. *A Toeplitz matrix* T *given by the complex sequence* $(t_j)_{j=-\infty}^{\infty}$ *belongs to* (S_p, S_p) *if and only if there exists a measure* $\mu \in M_p$ *with the Fourier coefficients* $\widehat{\mu}(j) = t_j$. *Moreover, in this case*

$$\|T\|_{(S_p,S_p)} = \|\mu\|_{M_p}, \tag{6}$$

where $M_p = \{\mu : \mathbb{T} \to \mathbb{C} \mid \mu = \sum_j \alpha_j \delta_{t_j},\ t_j \in \mathbb{T},\ t_j \text{ distinct points}\}$, $\|\mu\|_{M_p} = \left(\sum_j |\alpha_j|^p\right)^{1/p} < \infty$, *and* δ_t *is the Dirac measure concentrated at the point* $t \in \mathbb{T}$.

The above-mentioned papers [4, 5] show that a complete description of general Schur multipliers, at least, either for $B(\ell_2)$ or S_p, $0 < p \leq 1$, is a difficult target. In this way it is natural to consider and study other classes of Schur multipliers than those which are Toeplitz matrices. In [6], the following notation, more apropriate for our aims, for the entries of a matrix B was introduced. Namely, we put

$$b_k^l = \begin{cases} b_{l,l+k}, & k \geq 0,\ l = 1,2,\ldots, \\ b_{l-k,l}, & k < 0,\ l = 1,2,\ldots, \end{cases} \tag{7}$$

and write $B = (b_k^l)_{l\geq 1,\, k\in\mathbb{Z}}$.

Let $B^{(l)} = (b_k^m)_{k\in\mathbb{Z},\, m\geq 1}$, where $l = 1,2,3,\ldots$, be the matrix given by

$$b_k^m = \begin{cases} b_k^l, & \text{if } m = l, \\ 0, & \text{if } m \neq l. \end{cases} \tag{8}$$

We call the matrix $B^{(l)}$, the lth corner matrix associated to B.

Now, we associate to each matrix B^l a periodical distribution on $\mathbb{T}$, denoted by f_l, such that $b_k^l = \widehat{f_l}(k)$, and we identify the matrix $B = (B^{(l)})_{l\in\mathbb{N}^*}$ with the sequence of associated distributions $(f_l)_{l\in\mathbb{N}^*}$.

Then for the sequence $\alpha = (\alpha^1, \alpha^2, \ldots)$ and the matrix $B = (f_l)_{l\in\mathbb{N}^*}$, we denote by $\alpha \odot B$ the matrix given by $(\alpha^l f_l)_{l\in\mathbb{N}^*}$.

In particular, if B is a Toeplitz matrix ($B \in \mathcal{T}$) and if α is the constant sequence then $\alpha \odot B$ coincides with the matrix αB.

Hence, if $[\alpha]$ is the matrix

$$[\alpha] = \begin{pmatrix} \alpha^1 & \alpha^1 & \alpha^1 & \cdots \\ \alpha^1 & \alpha^2 & \alpha^2 & \ddots \\ \alpha^1 & \alpha^2 & \alpha^3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{9}$$

it is clear that $\alpha \odot B = [\alpha] * B$.

We define ms to be the space of all sequences α such that $\alpha \odot B \in B(\ell_2)$ for all $B \in B(\ell_2)$, or equivalently $[\alpha] \in M(\ell_2)$.

On ms we consider the norm $\|\alpha\|_{ms} = \|[\alpha]\|_{M(\ell_2)}$. Then ms is a unital commutative Banach algebra with respect to the usual multiplication of sequences. As it was observed in [6], the multiplication of a function with a scalar corresponds to the multiplication $\odot$ of a sequence and an infinite matrix.

We call the matrices $[\alpha]$ *scalar matrices*. In this context, in [6] a theorem of Haar's type for infinite matrices was proved. The product $\odot$ appeared also in [7] in other contexts.

An important role in applications is played by the upper triangular projection applied to the matrix $[\alpha]$. For an infinite matrix $A = (a_{ij})_{i\geq 1, j\geq 1,}$ the upper triangular projection is

$$P_T(A) = \begin{cases} a_{i,j}, & \text{if } i \leq j, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

A sequence $b = (b_n)_{n\geq 1}$ belongs to pms if and only if

$$B = \{b\} = P_T([b]) \in M(\ell_2). \tag{11}$$

The space pms endowed with the norm $\|b\| = \|\{b\}\|_{M(\ell_2)}$ becomes a Banach algebra with respect to the usual product of sequences.

In [6] there were given sufficient and necessary conditions in order for matrices of the form $[\alpha]$ or $\{\alpha\}$, that is,

$$\{\alpha\} = \begin{pmatrix} \alpha^1 & \alpha^1 & \alpha^1 & \cdots \\ 0 & \alpha^2 & \alpha^2 & \ddots \\ 0 & 0 & \alpha^3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{12}$$

to be Schur multipliers. The following result was proved.

Theorem BLP 1. *Let $b = (b_n)_{n\geq 1}$ be a complex sequence.*

(1) *If $(i_n)_{n\geq 1}$ is a strictly increasing sequence of natural numbers with $i_1 = 0$, and $z_{i_n} = \max_{i_n<k\leq i_{n+1}} |b_k|$, then there is a constant $R > 0$ such that*

$$\|\{b\}\|_{M(\ell_2)} \leq R \inf_{(i_n)_{n\geq 1}} \left\{ \left\|(z_{i_n})_{n\geq 1}\right\|_2 + \left\|\left(z_{i_n} \log(i_{n+1} - i_n)\right)_n\right\|_\infty \right\}. \tag{13}$$

(2) *If $b \in pms$ then*

$$\sup_{n\geq 1;\, p\geq 1} \frac{(\log n)^2}{n} \sum_{k=p}^{n+p} |b_k|^2 < \infty. \tag{14}$$

(3) *If $(|b_k|)_{k\geq 1}$ is a decreasing sequence, then $b \in pms$ if and only if $|b_k| = \mathcal{O}(1/\log k)$.*

As an immediate consequence we have the following.

Corollary 1. (1) *One has $\ell_2 \subset ms \subset \ell_\infty$.*
(2) *One has $\{(b_n)_{n\geq 1} \mid |b_n| = \mathcal{O}(1/\log n)\} \subset ms$.*

A set of sufficient conditions in order for a matrix of the type $[\alpha]$ to be a Schur multiplier is given in [6], namely, the following theorem was proved.

Theorem BLP 2. *Let $b = (b_n)_{n\geq 1}$ a complex sequence. Then,*

(1) *if $\sup_{n\geq 1} \sum_{j=1}^{n} |b_j - b_n|^2 < \infty$, then $b \in ms$;*

(2) *if $\|b\|_{BV\mathbb{N}} = |b_1| + \sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$ then $b \in ms$.*

It is well known that $M(\ell_2)$ coincides with (S_1, S_1), the space of all Schur multipliers from S_1 into S_1, see, for example, [4]. Using this fact we give a simpler proof of the first statement of Corollary 1.

Theorem 2. *Let $c = (c_n)_{n\geq 1} \in \ell_2$. Then $c \in pms$.*

Proof. By using the Schmidt decomposition of a matrix A, it is enough to show that $A*\{c\} \in S_1$ for a matrix A of rank 1. Let $A = \alpha \otimes \beta$ with $\alpha = (\alpha_n)_{n\geq 1} \in \ell_2$ and $\beta = (\beta_n)_{n\geq 1} \in \ell_2$.

We have

$$A * \{c\} = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_1\beta_3 & \cdots \\ \alpha_2\beta_1 & \alpha_2\beta_2 & \alpha_2\beta_3 & \cdots \\ \alpha_3\beta_1 & \alpha_3\beta_2 & \alpha_3\beta_3 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} * \begin{pmatrix} c_1 & c_1 & c_1 & c_1 & \cdots \\ 0 & c_2 & c_2 & c_2 & \ddots \\ 0 & 0 & c_3 & c_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$
$$= \begin{pmatrix} c_1\alpha_1\beta_1 & c_1\alpha_1\beta_2 & c_1\alpha_1\beta_3 & \cdots \\ 0 & c_2\alpha_2\beta_2 & c_2\alpha_2\beta_3 & \cdots \\ 0 & 0 & c_3\alpha_3\beta_3 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{15}$$

By the definition of S_1 and Cauchy-Schwartz inequality we get

$$\|A * \{c\}\|_{S_1} \leq \sum_{j=1}^{\infty} \left\| \left(\alpha_j c_j \beta_{j+k}\right)_{k=0}^{\infty} \right\|_{\ell_2} = \sum_{j=1}^{\infty} \left(\sum_{k=0}^{\infty} |\alpha_j c_j|^2 |\beta_{j+k}|^2 \right)^{1/2}$$
$$\leq \left(\sum_{j=1}^{\infty} |\alpha_j c_j| \right) \|\beta_j\|_{\ell_2} \leq \|\alpha\|_{\ell_2} \|\beta\|_{\ell_2} \|c\|_{\ell_2} = \|c\|_{\ell_2} \|A\|_{S_1}, \tag{16}$$

that is, $\|A\|_{M(\ell_2)} \leq \|c\|_{\ell_2}$ and the proof is complete. □

We characterize now the upper triangular scalar matrices which are Schur multipliers, from the Hardy space H^2, respectively, from the Schatten class S_2 into $B(\ell_2)$.

Theorem 3. (1) *Let H^2 be the Hardy space of Toeplitz matrices generated by the classical Hardy space of functions. Then an upper triangular matrix $A = \{\alpha\}$ belongs to $(H^2, B(\ell_2))$ if and only if $\alpha \in \ell_2$. Moreover, one has equality of the norms.*

(2) *Let T_2 be the space of all upper triangular Hilbert-Schmidt matrices. Then $\{\alpha\} \in (T_2, B(\ell_2))$ if and only if $\alpha \in \ell_\infty$.*

Proof. (1) We use the following identity proved in [6]:

$$\|B\|_{B(\ell_2)} = \sup_{\|h\|_2 \leq 1;\, h \in H_0^2[0,1]} \left(\sum_{k=1}^{\infty} \left| \int_0^1 \sum_{j=k}^{\infty} b_{kj} e^{2\pi i j t} h(-t)\, dt \right|^2 \right)^{1/2}, \tag{17}$$

where B is an upper triangular matrix $B = (b_{kj})$.

Then, if $f(t) = \sum_{k=0}^{\infty} c_k e^{2\pi ikt} \in H^2$, $t \in [0,1]$, F is the Toeplitz matrix associated to f (i.e., F is given by $(c_k)_{k\geq 0}^{\infty}$), and $\alpha \in \ell_2$, we have

$$F * \{\alpha\} = \begin{pmatrix} \alpha_0 c_0 & \alpha_0 c_1 & \alpha_0 c_2 & \alpha_0 c_3 & \cdots \\ 0 & \alpha_1 c_0 & \alpha_1 c_1 & \alpha_1 c_2 & \ddots \\ 0 & 0 & \alpha_2 c_0 & \alpha_2 c_1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$\begin{aligned} \|F * \{\alpha\}\|_{B(\ell_2)} &= \sup_{\|h\|_2 \leq 1} \left(\sum_{k=1}^{\infty} \left| \int_0^1 \alpha_{k-1} \left(\sum_{j=0}^{\infty} c_j e^{2\pi ijt} \right) h(-t)\, dt \right|^2 \right)^{1/2} \\ &= \sup_{\|h\|_2 \leq 1} \left(\sum_{k=1}^{\infty} |\alpha_{k-1}|^2 \right)^{1/2} \left| \int_0^1 \left(\sum_{j=0}^{\infty} c_j e^{2\pi ijt} \right) h(-t)\, dt \right| = \|\alpha\|_{\ell_2} \|f\|_{H^2}. \end{aligned} \tag{18}$$

Hence $\|\{\alpha\}\|_{(H^2, B(\ell_2))} = \|\alpha\|_{\ell_2}$, and this completes the proof.

(2) Let $\alpha \in \ell_\infty$ and

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots \\ 0 & c_{22} & c_{23} & \cdots \\ 0 & 0 & c_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \in T_2. \tag{19}$$

Using formula (17) and Cauchy-Schwartz inequality we get

$$\begin{aligned} \|\{\alpha\} * C\|_{B(\ell_2)} &\leq \left(\sum_{k=1}^{\infty} |\alpha_{k-1}|^2 \sup_{\|h\|_2 \leq 1} \left| \int_0^1 \left(\sum_{j \geq k} c_{kj} e^{2\pi ijt} \right) h(-t) \right|^2 \right)^{1/2} \\ &\leq \left(\sum_{k=1}^{\infty} |\alpha_{k-1}|^2 \left(\sum_{j \geq k} |c_{kj}|^2 \right) \right)^{1/2} \\ &\leq \left(\sup_{k \geq 0} |\alpha_k| \right) \left(\sum_{k=1}^{\infty} \sum_{j \geq k} |c_{kj}|^2 \right)^{1/2} \\ &= \|\alpha\|_{\ell_\infty} \|C\|_{T_2}. \end{aligned} \tag{20}$$

Hence $\{\alpha\} \in (T_2, B(\ell_2))$, and this proves the first part of the theorem.

Conversely, let $\{\alpha\} \in (T_2, B(\ell_2))$, that is, $C * \{\alpha\} \in B(\ell_2)$ for all $C \in T_2$ and take $C = C_0$, that is, the matrix C is reduced to main diagonal. It is clear that the sequence of entries of

this diagonal belongs to ℓ_2. Consequently the sequence $(\alpha_{k-1}c_{kk})_{k\geq 0}$ belongs to ℓ_∞ for every sequence $(c_{kk})_{k\geq 1}^{\infty} \in \ell_2$. Hence $(\alpha_k)_{k=0}^{\infty} \in \ell_\infty$, and the proof is complete. □

Next we use the important results of Bennett proved in [8], in order to characterize the Schur multipliers of scalar type for some spaces of lower triangular infinite matrices contained in the Schatten classes S_p, $0 < p < \infty$. We denote these spaces by $\mathcal{LT}S_p$.

Next we get a general description of upper triangular Schur multipliers of scalar type for different quasi-Banach spaces.

In order to state the following result we need to recall some definitions (see [9]).

Let $\mathbf{f}$ be the space of all sequences with a finite number of nonzero elements. A norm Φ on $\mathbf{f}$ is called *symmetric* if $\Phi(a) = \Phi(a^*)$, for all $a \in \mathbf{f}$, that is, if Φ is invariant to permutations and to applications $a_n \to e^{i\theta_n} a_n$, where θ_n is a sequence of real numbers. Here $a^* = (a_n^*)_{n=1}^{\infty}$ is the decreasing rearrangement of the sequence (a_n) which converges to 0.

We say that the sequence $(a_n)_n$ belongs to the space s_Φ, if and only if $\lim_{n\to\infty}\Phi(a_1, \ldots, a_n, 0, 0, \ldots) = \Phi(a)$ exists.

We denote by S_Φ the space of all compact operators A on ℓ_2 with the sequence of their singular numbers $(\mu_n(A))$ belonging to s_Φ. For $A \in S_\Phi$ we put $\Phi(A) = \Phi((\mu_n(A))_n)$.

Then the following *noncommutative Hölder type inequality* proved in [9] holds.

Theorem AH. *Let Φ_1, Φ_2, Φ_3 be symmetric norms such that if $a \in s_{\Phi_2}$, $b \in s_{\Phi_3}$ then $ab \in s_{\Phi_1}$ and*

$$\Phi_1(ab) \leq \Phi_2(a)\Phi_3(b). \tag{21}$$

If $A \in S_{\Phi_2}$, $B \in S_{\Phi_3}$, then $AB \in S_{\Phi_1}$ and $\Phi_1(AB) \leq \Phi_2(A)\Phi_3(B)$.

Using this inequality we can state the following interesting result.

Theorem 4. *Let $s_{\Phi_1} = s_{\Phi_2}s_{\Phi_3}$ (i.e., for each $\alpha \in s_{\Phi_1}$, there exist $\beta \in s_{\Phi_2}$, $\gamma \in s_{\Phi_3}$ such that $\alpha = \beta\gamma$, and $\Phi_1(\alpha) \approx \inf_{\alpha=\beta\gamma}\Phi_2(\beta)\Phi_3(\gamma)$). Then a scalar matrix $[\alpha] \in (\mathcal{LT}S_{\Phi_2}, \mathcal{LT}S_{\Phi_1})$ if and only if $\alpha \in s_{\Phi_3}$.*

Proof. Let first $A \in \mathcal{LT}S_{\Phi_1}$ and $\alpha \in s_{\Phi_3}$. Then it is clear that

$$A * [\alpha] = A \cdot D_\alpha, \tag{22}$$

where

$$D_\alpha = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots \\ 0 & \alpha_2 & 0 & \ddots \\ 0 & 0 & \alpha_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{23}$$

By Theorem AH it follows that

$$\|A * [\alpha]\|_{S_{\Phi_1}} \leq \|A \cdot D_\alpha\|_{S_{\Phi_1}} \leq \|A\|_{S_{\Phi_2}}\|D_\alpha\|_{S_{\Phi_3}} \leq \|A\|_{S_{\Phi_2}}\|\alpha\|_{s_{\Phi_3}}. \tag{24}$$

Hence $[\alpha] \in (\mathcal{LT}S_{\Phi_2}, \mathcal{LT}S_{\Phi_1})$, and this completes the first part of the proof.

For the reverse implication, take A to be the main diagonal with the entries $(a_{jj})_{j=1}^{\infty} \in s_{\Phi_2}$ and $[\alpha] \in (\mathcal{LT}S_{\Phi_2}, \mathcal{LT}S_{\Phi_1})$.

Then

$$A * [\alpha] = A \cdot D_\alpha = \begin{pmatrix} a_{11}\alpha_1 & 0 & 0 & \cdots \\ 0 & a_{22}\alpha_2 & 0 & \ddots \\ 0 & 0 & a_{33}\alpha_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \in S_{\Phi_1} \tag{25}$$

and we get that $(a_{ii}\alpha_i)_{i=1}^{\infty} \in s_{\Phi_1}$ for all sequences $(a_{jj})_{j=1}^{\infty} \in s_{\Phi_2}$. Since $s_{\Phi_1} = s_{\Phi_2} s_{\Phi_3}$, it follows that $\alpha \in s_{\Phi_3}$, and this completes the proof of the theorem. □

Let $w = (w_n)$ be a positive decreasing sequence of numbers. Of course the Lorentz space of sequences $\ell_{p,w}$, $0 < p \leq \infty$, is a space of the previous type s_Φ, see, for example, [9]. By the well-known fact that $\ell_{p,w} \cdot \ell_{q,w} = \ell_{r,w}$, for $1/p + 1/q = 1/r;\ 0 < p, q, r < \infty$ we get the following result.

Corollary 2. (1) *Let $1/p + 1/q = 1/r,\ 0 < p, q, r < \infty$. Then $[\alpha] \in (S_p, S_r)$ if and only if $\alpha \in \ell_q$.*

(2) *Let w_n be a decreasing positive sequence, and let $0 < p,\ q < \infty$, be such that $1/p+1/q = 1$. Then $[\alpha] \in (S_{p,w}, S_{1,w})$ if and only if $\alpha \in \ell_{q,w}$, where $\ell_{p,w}$ is the weighted Lorentz space of sequences.*

We call *the Bergman-Schatten space of order p*, $0 < p < \infty$, and we denote by $L_a^p(\ell_2)$ the space of all upper triangular matrices A such that $\|A\|_{L_a^p(\ell_2)} = \left(\int_0^1 \| \sum_{k=0}^{\infty} A_k r^k \|_{S_p}^p 2r dr\right)^{1/p} < \infty$. See, for example, [10] for further notations and details.

By Hölder's inequality we get the following result.

Theorem 6. *Let $1 \leq p < \infty$. Then $[\alpha] \in (L_a^p(\ell_2), L_a^1(\ell_2))$ if and only if $\alpha \in \ell_q$, where $1/p + 1/q = 1$.*

Proof. Let $A \in L_a^p(\ell_2)$ and $\alpha \in \ell_q$. We clearly have that $A * [\alpha] = D_\alpha \cdot A$. By Theorem AH we get

$$\begin{aligned} \|A * [\alpha]\|_{L_a^1(\ell_2)} &= \int_0^1 \|D_\alpha \cdot A(r)\|_{S_1} 2r dr \\ &\leq \left(\int_0^1 \|A(r)\|_{S_p}^p 2r dr\right)^{1/p} \|\alpha\|_{\ell_q} = \|A\|_{L_a^p(\ell_2)} \|\alpha\|_{\ell_q}, \end{aligned} \tag{26}$$

that is, $[\alpha] \in (L_a^p(\ell_2), L_a^1(\ell_2))$, and this completes the first part of the proof.

Conversely, let $[\alpha] \in (L_a^p(\ell_2), L_a^1(\ell_2))$. By taking $A = A_0 = (a_{jj})_{j=1}^{\infty} \in L_a^p(\ell_2)$, that is, for $(a_{jj})_j \in \ell_p$, we get

$$A * [\alpha] = \begin{pmatrix} \alpha_1 a_{11} & 0 & \cdots \\ 0 & \alpha_2 a_{22} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} \in L_a^1(\ell_2), \tag{27}$$

or, equivalently, $(\alpha_j a_{jj})_j \in \ell_1$. Hence by Hölder's inequality it follows that $(\alpha_j)_j \in \ell_q$, and the proof is complete. □

Using the results of Bennett, proved in [8] we can also describe the Schur multipliers of scalar type also for others quasi-Banach spaces of matrices. The spaces of sequences $d(\mathbf{a}, p)$, $g(\mathbf{a}, p)$, and $\mathrm{ces}(p)$ were defined in [8].

We denote now by $d_M^q(\mathbf{a}, p)$, $g_M^q(\mathbf{a}, p)$, $\mathrm{ces}_M^q(p)$, and $\ell_M^q(p)$ the spaces of upper triangular infinite matrices $A = \sum_{k=0}^{\infty} A_k$, with all the sequences on the diagonals belonging to $d(\mathbf{a}, p)$ (resp., $g(\mathbf{a}, p)$, $\mathrm{ces}(p)$, ℓ_p,) and such that $\|A\| = \left(\sum_k \|A_k\|_{d(\mathbf{a},p)}^q\right)^{1/q} < \infty$ (resp., $\|A\| = \left(\sum_k \|A_k\|_{g(\mathbf{a},p)}^q\right)^{1/q} < \infty$ and so on) with the usual modification for $q = \infty$.

Using Theorems 4.5 and 3.8 in [8], we have the following.

Theorem 7. (1) *Let $1 < p < \infty$. Then $\alpha \in g(p^*)$, where $1/p + 1/p^* = 1$, if and only if $[\alpha] \in (\ell_M^{\infty}(p), \mathrm{ces}_M^{\infty}(p))$, where $g(p^*) = g(\mathbf{a}, p^*)$, with $\mathbf{a} = (1, 1, \ldots)$.*

(2) *Let $0 < p < \infty$. Then $[\alpha] \in (d_M^{\infty}(\mathbf{a}, p), \ell_M^{\infty}(p))$ if and only if $\alpha \in g(\mathbf{a}, p)$.*

Theorem 8. (1) *For $1 < p < \infty$, $[\alpha] \in (\ell_M^q(p), \mathrm{ces}_M^q(p))$ if and only if $\alpha \in g(p^*)$.*

(2) *For $0 < p < \infty$, $[\alpha] \in (d_M^q(\mathbf{a}, p), \ell_M^q(p))$ if and only if $\alpha \in g(\mathbf{a}, p)$.*

Acknowledgments

This paper was partially supported by the CNCSIS Grant ID-PCE 1905/2008. The author thanks the referee for his/her valuable remarks which improved considerably the presentation of this paper.

References

[1] J. Schur, "Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen," *Journal fur die Reine und Angewandte Mathematik*, vol. 140, pp. 1–28, 1911.
[2] G. Bennett, "Schur multipliers," *Duke Mathematical Journal*, vol. 44, no. 3, pp. 603–639, 1977.
[3] S. Barza, L.-E. Persson, and N. Popa, "A matriceal analogue of Fejer's theory," *Mathematische Nachrichten*, vol. 260, pp. 14–20, 2003.
[4] G. Pisier, *Similarity Problems and Completely Bounded Maps*, vol. 1618 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1995.
[5] A. B. Aleksandrov and V. V. Peller, "Hankel and Toeplitz-Schur multipliers," *Mathematische Annalen*, vol. 324, no. 2, pp. 277–327, 2002.
[6] S. Barza, V. Lie, and N. Popa, "Approximation of infinite matrices by matricial Haar polynomials," *Arkiv för Matematik*, vol. 43, no. 2, pp. 251–269, 2005.
[7] A. Marcoci, L. E. Persson, and N. Popa, "Schur multiplier characterization of a class of infinite matrices," *Czechoslovak Mathematical Journal*, vol. 60, no. 1, pp. 183–193, 2010.

[8] G. Bennett, "Factorizing the classical inequalities," *Memoirs of the American Mathematical Society*, vol. 120, no. 576, pp. 1–130, 1996.

[9] B. Simon, *Trace Ideals and Their Applications*, vol. 35 of *Lecture Notes in Mathematics*, Cambridge University Press, Cambridge, UK, 1979.

[10] L. G. Marcoci, L. E. Persson, I. Popa, and N. Popa, "A new characterization of Bergman-Schatten spaces and a duality result," *Journal of Mathematical Analysis and Applications*, vol. 360, no. 1, pp. 67–80, 2009.