Research Article

A Class of Schur Multipliers on Some Quasi-Banach Spaces of Infinite Matrices

Nicolae Popa

Unit Research 1, Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

Correspondence should be addressed to Nicolae Popa, npopa@imar.ro

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We characterize the Schur multipliers of scalar type acting on scattered classes of infinite matrices.

In [1], Schur introduced a new product between two matrices \( A = (a_{jk}) \) and \( B = (b_{jk}) \) of the same size, finite or infinite. This product, known in the literature as the Schur product or Hadamard product, is defined to be the matrix of elementwise products

\[
A \ast B = (a_{jk}b_{jk}).
\]  

This concept was used in different areas of analysis as complex function theory, Banach spaces, operator theory, and multivariate analysis.

Bennett studied in [2] the behaviour, under Schur multiplication, of the norm \( \| \cdot \|_{p,q}, \ 1 \leq p,q \leq \infty \),

\[
\|A\|_{p,q} = \sup_{\|x\|_p \leq 1} \left( \sum_j \left( \sum_k |a_{jk}x_k|^q \right)^{1/q} \right).
\]  

In particular, he was interested in characterizing the \((p,q)\)-multipliers: the matrices \( M \) for which \( M \ast A \) maps \( \ell_p \) into \( \ell_q \) whenever \( A \) does.
In his paper it is proved a theorem about Schur multipliers which are Toeplitz matrices, that is about the matrices of the form

\[ A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_{-1} & a_0 & a_1 & a_2 & \cdots \\ a_{-2} & a_{-1} & a_0 & a_1 & \cdots \\ a_{-3} & a_{-2} & a_{-1} & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \]  

where \( (a_j)_{j=-\infty}^{\infty} \) is a sequence of complex numbers.

Theorem 8.1 in [2] reads as follows.

**Theorem B.** A Toeplitz matrix \( A \) is a Schur multiplier if and only if \( \mu = \sum_{j=-\infty}^{\infty} a_j e^{ijt} \) is a bounded Borel measure on \([0, 2\pi)\).

This fact leads naturally to the idea of identifying the Schur multipliers with the noncommutative bounded Borel measures, see, for example, [3].

We denote by \( M(\ell_2) \) the space of all \((2,2)\) Schur multipliers from \( B(\ell_2) \) into \( B(\ell_2) \), where \( B(\ell_2) \) is, as usual, the Banach space of linear and bounded operators on \( \ell_2 \) with the usual operator norm.

The space \( M(\ell_2) \) endowed with norm \( \|A\|_{M(\ell_2)} = \sup_{\|B\|_{B(\ell_2)}} \|A \ast B\|_{B(\ell_2)} \) becomes a Banach space.

Since we work with different quasi-Banach spaces of matrices \( X, Y \) we use the notation \( (X,Y) \) for the space of all Schur multipliers from \( X \) into \( Y \) equipped with the quasi-norm

\[ \|A\|_{(X,Y)} = \sup_{\|B\|_X \leq 1} \|A \ast B\|_X. \]  

In this way \( (X,Y) \) becomes a quasi-Banach.

In [2] Bennett raised the problem of characterizing the Hankel matrices which are Schur multipliers.

We recall that a matrix \( A \) is called a *Hankel matrix* if it is defined by a sequence \( (a_j)_{j=1}^{\infty} \) of complex numbers in the following way:

\[ A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]  

Pisier in [4] solved the above problem. He proved the following theorem.
Theorem P. A Hankel matrix is a Schur multiplier if and only if the Fourier multiplier \( \sum_{n=0}^{\infty} x_n e^{int} \rightarrow \sum_{n=0}^{\infty} a_n x_n e^{int} \) maps boundedly \( H^1(S_1) \) into itself.

Here \( H^1(S_1) \) is the Hardy space of the Schatten class \( S_1 \)-valued analytic functions, endowed with the norm \( \|f\|_{H^1(S_1)} = (1/2\pi) \int_0^{2\pi} \| \sum_{n=0}^{\infty} A_n e^{int} \|_{S_1} dt < \infty \). For the definition of the Schatten classes \( S_p \), see, for example, [5].

In [5], Aleksandrov and Peller characterized the Toeplitz matrices which are Schur multipliers for \( S_p, 0 < p < 1 \). They proved the following theorem.

Theorem AP. Let \( 0 < p < 1 \). A Toeplitz matrix \( T \) given by the complex sequence \( (t_n)_{n=0}^{\infty} \) belongs to \( (S_p, S_p) \) if and only if there exists a measure \( \mu \in M_p \) with the Fourier coefficients \( \hat{\mu}(j) = t_j \). Moreover, in this case

\[
\|T\|_{(S_p, S_p)} = \|\mu\|_{M_p},
\]

where \( M_p = \{ \mu : \mathbb{T} \rightarrow \mathbb{C} | \mu = \sum_{j} \alpha_j \delta_{t_j}, t_j \in \mathbb{T}, t_j \text{ distinct points} \}, \|\mu\|_{M_p} = (\sum_{|\alpha_j|^p})^{1/p} < \infty \), and \( \delta_t \) is the Dirac measure concentrated at the point \( t \in \mathbb{T} \).

The above-mentioned papers [4, 5] show that a complete description of general Schur multipliers, at least, either for \( B(\ell_2) \) or \( S_p, 0 < p \leq 1 \), is a difficult target. In this way it is natural to consider and study other classes of Schur multipliers than those which are Toeplitz matrices. In [6], the following notation, more appropriate for our aims, for the entries of a matrix \( B \) was introduced. Namely, we put

\[
b_k^l = \begin{cases} b_{l+k}, & k \geq 0, \ l = 1, 2, \ldots, \\ b_{l-k}, & k < 0, \ l = 1, 2, \ldots, \end{cases}
\]

and write \( B = (b_k^l)_{l \geq 1, k \in \mathbb{Z}} \).

Let \( B^{(1)} = (b_k^m)_{k \in \mathbb{Z}, m \geq 1} \), where \( l = 1, 2, 3, \ldots \), be the matrix given by

\[
b_k^m = \begin{cases} b_k^l, & m = l, \\ 0, & m \neq l. \end{cases}
\]

We call the matrix \( B^{(1)} \), the \( l \)th corner matrix associated to \( B \).

Now, we associate to each matrix \( B^l \) a periodical distribution on \( \mathbb{T} \), denoted by \( f_l \), such that \( \hat{b}_k^l = \hat{f}_l(k) \), and we identify the matrix \( B = (B^{(1)}_{l \in \mathbb{Z}}) \) with the sequence of associated distributions \( (f_l)_{l \in \mathbb{Z}} \).

Then for the sequence \( \alpha = (\alpha^1, \alpha^2, \ldots) \) and the matrix \( B = (f_l)_{l \in \mathbb{Z}} \), we denote by \( \alpha \circ B \) the matrix given by \( (\alpha^l f_l)_{l \in \mathbb{Z}} \).

In particular, if \( B \) is a Toeplitz matrix \( (B \in \mathbb{C}) \) and if \( \alpha \) is the constant sequence then \( \alpha \circ B \) coincides with the matrix \( \alpha B \).
Hence, if $[\alpha]$ is the matrix
\[
[\alpha] = \begin{pmatrix}
\alpha^1 & \alpha^1 & \alpha^1 & \cdots \\
\alpha^1 & \alpha^2 & \alpha^2 & \cdots \\
\alpha^1 & \alpha^2 & \alpha^3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
(9)
it is clear that $\alpha \odot B = [\alpha] \ast B$.

We define $ms$ to be the space of all sequences $\alpha$ such that $\alpha \odot B \in B(\ell_2)$ for all $B \in B(\ell_2)$, or equivalently $[\alpha] \in M(\ell_2)$.

On $ms$ we consider the norm $\|\alpha\|_{ms} = \|[\alpha]\|_{M(\ell_2)}$. Then $ms$ is a unital commutative Banach algebra with respect to the usual multiplication of sequences. As it was observed in [6], the multiplication of a function with a scalar corresponds to the multiplication $\odot$ of a sequence and an infinite matrix.

We call the matrices $[\alpha]$ scalar matrices. In this context, in [6] a theorem of Haar’s type for infinite matrices was proved. The product $\odot$ appeared also in [7] in other contexts.

An important role in applications is played by the upper triangular projection applied to the matrix $[\alpha]$. For an infinite matrix $A = (a_{ij})_{i \geq 1, j \geq 1}$, the upper triangular projection is
\[
P_T(A) = \begin{cases} 
a_{ij}, & \text{if } i \leq j, \\
0, & \text{otherwise.}
\end{cases}
\]
(10)

A sequence $b = (b_n)_{n \geq 1}$ belongs to $pms$ if and only if
\[
B = \{b\} = P_T([b]) \in M(\ell_2).
\]
(11)

The space $pms$ endowed with the norm $\|b\| = \|[b]\|_{M(\ell_2)}$ becomes a Banach algebra with respect to the usual product of sequences.

In [6] there were given sufficient and necessary conditions in order for matrices of the form $[\alpha]$ or $\{\alpha\}$, that is,
\[
\{\alpha\} = \begin{pmatrix}
\alpha^1 & \alpha^1 & \alpha^1 & \cdots \\
0 & \alpha^2 & \alpha^2 & \cdots \\
0 & 0 & \alpha^3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
(12)
to be Schur multipliers. The following result was proved.
Theorem BLP 1. Let $b = (b_n)_{n \geq 1}$ be a complex sequence.

1. If $(i_n)_{n \geq 1}$ is a strictly increasing sequence of natural numbers with $i_1 = 0$, and $z_{i_n} = \max_{i_n < k \leq i_{n+1}} |b_k|$, then there is a constant $R > 0$ such that

$$
\|b\|_{M(\ell^1)} \leq R \inf_{(i_n)_{n \geq 1}} \left\{ \| (z_{i_n})_{n \geq 1} \|_2 + \| (z_{i_n} \log(i_{n+1} - i_n))_{n \geq 1} \|_\infty \right\}.
$$

(13)

2. If $b \in pms$ then

$$
\sup_{n \geq 1; \ p \geq 1} \frac{(\log n)^2}{n} \sum_{k=p}^{n+p} |b_k|^2 < \infty.
$$

(14)

3. If $(|b_k|)_{k \geq 1}$ is a decreasing sequence, then $b \in pms$ if and only if $|b_k| = O(1/ \log k)$.

As an immediate consequence we have the following.

Corollary 1. (1) One has $\ell^2 \subset ms \subset \ell_\infty$.

(2) One has $\{(b_n)_{n \geq 1} \mid |b_n| = O(1/ \log n)\} \subset ms$.

A set of sufficient conditions in order for a matrix of the type $[\alpha]$ to be a Schur multiplier is given in [6], namely, the following theorem was proved.

Theorem BLP 2. Let $b = (b_n)_{n \geq 1}$ a complex sequence. Then,

1. if $\sup_{n \geq 1} \sum_{j=1}^{n} |b_j - b_n|^2 < \infty$, then $b \in ms$;

2. if $\|b\|_{BV} = |b_1| + \sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$ then $b \in ms$.

It is well known that $M(\ell^2)$ coincides with $(S_1, S_1)$, the space of all Schur multipliers from $S_1$ into $S_1$, see, for example, [4]. Using this fact we give a simpler proof of the first statement of Corollary 1.

Theorem 2. Let $c = (c_n)_{n \geq 1} \in \ell^2$. Then $c \in pms$.

Proof. By using the Schmidt decomposition of a matrix $A$, it is enough to show that $A^* \{c\} \in S_1$ for a matrix $A$ of rank 1. Let $A = \alpha \otimes \beta$ with $\alpha = (\alpha_n)_{n \geq 1} \in \ell^2$ and $\beta = (\beta_n)_{n \geq 1} \in \ell^2$. 

We have

\[
A * \{c\} = \begin{pmatrix}
\alpha_1 \beta_1 & \alpha_1 \beta_2 & \alpha_1 \beta_3 & \cdots \\
\alpha_2 \beta_1 & \alpha_2 \beta_2 & \alpha_2 \beta_3 & \cdots \\
\alpha_3 \beta_1 & \alpha_3 \beta_2 & \alpha_3 \beta_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} * \begin{pmatrix}
c_1 & c_1 & c_1 & \cdots \\
0 & c_2 & c_2 & \cdots \\
0 & 0 & c_3 & \cdots \\
0 & 0 & 0 & \ddots
\end{pmatrix}
\]

(15)

By the definition of \(S_1\) and Cauchy-Schwartz inequality we get

\[
\|A * \{c\}\|_{S_1} = \left(\sum_{j=1}^{\infty} \left(\sum_{|k|\leq j} |\alpha_j c_{j+k}| \right)^2 \right)^{1/2}
\]

(16)

\[
\leq \left(\sum_{j=1}^{\infty} |\alpha_j c_j|^2 \right)^{1/2} \leq \|\alpha\|_{\ell^2} \|\beta\|_{\ell^2} = \|c\|_{\ell^2} \|A\|_{S_1},
\]

that is, \(\|A\|_{M(\ell^2)} \leq \|c\|_{\ell^2}\) and the proof is complete.

We characterize now the upper triangular scalar matrices which are Schur multipliers, from the Hardy space \(H^2\), respectively, from the Schatten class \(S_2\) into \(B(\ell^2)\).

**Theorem 3.** (1) Let \(H^2\) be the Hardy space of Toeplitz matrices generated by the classical Hardy space of functions. Then an upper triangular matrix \(A = \{\alpha\}\) belongs to \((H^2, B(\ell^2))\) if and only if \(\alpha \in \ell^2\). Moreover, one has equality of the norms.

(2) Let \(T_2\) be the space of all upper triangular Hilbert-Schmidt matrices. Then \(\{\alpha\} \in (T_2, B(\ell^2))\) if and only if \(\alpha \in \ell^\infty\).

**Proof.** (1) We use the following identity proved in [6]:

\[
\|B\|_{B(\ell^2)} = \sup_{\|h\|_{L^1;H^2[0,1]}} \left(\sum_{k=1}^{\infty} \left(\int_0^1 \sum_{j=1}^{\infty} b_{kj} e^{2\pi i tj} h(-t) dt \right)^2 \right)^{1/2},
\]

(17)

where \(B\) is an upper triangular matrix \(B = (b_{kj})\).
Then, if \( f(t) = \sum_{k=0}^{\infty} c_k e^{2\pi i k t} \in H^2, \ t \in [0,1] \), \( F \) is the Toeplitz matrix associated to \( f \) (i.e., \( F \) is given by \( (c_k)_{k \geq 0} \)), and \( \alpha \in \ell_2 \), we have

\[
F \ast \{ \alpha \} = \begin{pmatrix}
\alpha_0 c_0 & \alpha_1 c_1 & \alpha_2 c_2 & \cdots \\
0 & \alpha_1 c_0 & \alpha_1 c_1 & \cdots \\
0 & 0 & \alpha_2 c_0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

(18)

\[
\|F \ast \{ \alpha \}\|_{B(\ell_2)} = \sup_{\|\alpha\|_2 \leq 1} \left( \sum_{k=1}^{\infty} \left( \sum_{j=0}^{\infty} c_j e^{2\pi i j t} \right) h(-t) dt \right)^2 \]

\[
= \sup_{\|\alpha\|_2 \leq 1} \left( \sum_{k=1}^{\infty} \left| \alpha_{k-1} \right|^2 \right)^{1/2} \left| \int_0^1 \left( \sum_{j=0}^{\infty} c_j e^{2\pi i j t} \right) h(-t) dt \right| = \| \alpha \|_{\ell_2} \| f \|_{H^2}.
\]

Hence \( \| \alpha \|_{(H^2, B(\ell_2))} = \| \alpha \|_{\ell_2} \), and this completes the proof.

(2) Let \( \alpha \in \ell_{\infty} \) and

\[
C = \begin{pmatrix}
c_{11} & c_{12} & c_{13} & \cdots \\
0 & c_{22} & c_{23} & \cdots \\
0 & 0 & c_{33} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \in T_2.
\]

(19)

Using formula (17) and Cauchy-Schwartz inequality we get

\[
\|\{ \alpha \} \ast C\|_{B(\ell_2)} \leq \left( \sum_{k=1}^{\infty} \left| \alpha_{k-1} \right|^2 \sup_{\|\alpha\|_2 \leq 1} \left| \int_0^1 \left( \sum_{j=0}^{\infty} c_j e^{2\pi i j t} \right) h(-t) dt \right|^2 \right)^{1/2}
\]

\[
\leq \left( \sum_{k=1}^{\infty} \left| \alpha_{k-1} \right|^2 \left( \sum_{j=0}^{\infty} |c_j|^2 \right) \right)^{1/2}
\]

(20)

\[
\leq \left( \sup_{k \geq 1} |\alpha_k| \right) \left( \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} |c_j|^2 \right)^{1/2}
\]

\[
= \| \alpha \|_{\ell_{\infty}} \| C \|_{T_2}.
\]

Hence \( \{ \alpha \} \in (T_2, B(\ell_2)) \), and this proves the first part of the theorem.

Conversely, let \( \{ \alpha \} \in (T_2, B(\ell_2)) \), that is, \( C \ast \{ \alpha \} \in B(\ell_2) \) for all \( C \in T_2 \) and take \( C = C_0 \), that is, the matrix \( C \) is reduced to main diagonal. It is clear that the sequence of entries of
Proof. Let first and to applications and this diagonal belongs to $\ell_2$. Consequently the sequence $(a_{k-1}c_{kk})_{k\geq 1}$ belongs to $\ell_\infty$ for every sequence $(c_{kk})_{k\geq 1} \in \ell_2$. Hence $(a_k)_{k=0}^\infty \in \ell_\infty$, and the proof is complete. □

Next we use the important results of Bennett proved in [8], in order to characterize the Schur multipliers of scalar type for some spaces of lower triangular infinite matrices contained in the Schatten classes $S_p$, $0 < p < \infty$. We denote these spaces by $\mathcal{LS}_p$.

Next we get a general description of upper triangular Schur multipliers of scalar type for different quasi-Banach spaces.

In order to state the following result we need to recall some definitions (see [9]).

Let $f$ be the space of all sequences with a finite number of nonzero elements. A norm $\Phi$ on $f$ is called symmetric if $\Phi(a) = \Phi(a^*)$, for all $a \in f$, that is, if $\Phi$ is invariant to permutations and to applications $a_n \rightarrow e^{i\theta_n}a_n$, where $\theta_n$ is a sequence of real numbers. Here $a^* = (a_n^*)_{n=1}^\infty$ is the decreasing rearrangement of the sequence $(a_n)$ which converges to 0.

We say that the sequence $(a_n)$ belongs to the space $s_{\Phi}$, if and only if $\lim_{n \rightarrow \infty} \Phi(a_1, \ldots, a_n, 0, 0, \ldots) = \Phi(a)$ exists.

We denote by $S_\Phi$ the space of all compact operators $A$ on $\ell_2$ with the sequence of their singular numbers $(\mu_n(A))$ belonging to $s_{\Phi}$. For $A \in S_\Phi$ we put $\Phi(A) = \Phi((\mu_n(A)))$.

Then the following noncommutative Hölder type inequality proved in [9] holds.

**Theorem AH.** Let $\Phi_1$, $\Phi_2$, $\Phi_3$ be symmetric norms such that if $a \in s_{\Phi_2}$, $b \in s_{\Phi_3}$ then $ab \in s_{\Phi_1}$ and

$$\Phi_1(ab) \leq \Phi_2(a)\Phi_3(b).$$

If $A \in S_{\Phi_2}$, $B \in S_{\Phi_3}$, then $AB \in S_{\Phi_1}$ and $\Phi_1(AB) \leq \Phi_2(A)\Phi_3(B)$.

Using this inequality we can state the following interesting result.

**Theorem 4.** Let $s_{\Phi_1} = s_{\Phi_2} s_{\Phi_3}$ (i.e., for each $\alpha \in s_{\Phi_1}$, there exist $\beta \in s_{\Phi_2}$, $\gamma \in s_{\Phi_3}$ such that $\alpha = \beta \gamma$, and $D_1(\alpha) \approx \inf_{\beta,\gamma} \Phi_2(\beta)\Phi_3(\gamma)$). Then a scalar matrix $[\alpha] \in (\mathcal{LS}_2, \mathcal{LS}_3)$ if and only if $\alpha \in s_{\Phi_1}$.

**Proof.** Let first $A \in \mathcal{LS}_2$, and $\alpha \in s_{\Phi_1}$. Then it is clear that

$$A \ast [\alpha] = A \cdot D_\alpha,$$

where

$$D_\alpha = \begin{pmatrix}
\alpha_1 & 0 & 0 & \cdots \\
0 & \alpha_2 & 0 & \cdots \\
0 & 0 & \alpha_3 & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}$$

By Theorem AH it follows that

$$\|A \ast [\alpha]\|_{s_{\Phi_1}} \leq \|A \cdot D_\alpha\|_{s_{\Phi_1}} \leq \|A\|_{s_{\Phi_2}} \|D_\alpha\|_{s_{\Phi_3}} \leq \|A\|_{s_{\Phi_2}} \|\alpha\|_{s_{\Phi_3}}.$$

Hence $[\alpha] \in (\mathcal{LS}_2, \mathcal{LS}_3)$, and this completes the first part of the proof.
For the reverse implication, take $A$ to be the main diagonal with the entries $(a_{jj})_{j=1}^{\infty} \in s_{\Phi}$ and $[\alpha] \in (L\mathcal{S}_{\Phi}, L\mathcal{S}_{\Phi})$.

Then

$$A \ast [\alpha] = A \cdot D_\alpha = \begin{pmatrix}
  a_{11} \alpha_1 & 0 & 0 & \cdots \\
  0 & a_{22} \alpha_2 & 0 & \cdots \\
  0 & 0 & a_{33} \alpha_3 & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \in S_{\Phi_1}$$

(25)

and we get that $(a_{jj})_{j=1}^{\infty} \in s_{\Phi_1}$ for all sequences $(a_{jj})_{j=1}^{\infty} \in s_{\Phi}$. Since $s_{\Phi_1} = s_{\Phi}, s_{\Phi_1}$, it follows that $\alpha \in s_{\Phi_1}$, and this completes the proof of the theorem.

Let $w = (w_n)$ be a positive decreasing sequence of numbers. Of course the Lorentz space of sequences $\ell_{p,w}$, $0 < p \leq \infty$, is a space of the previous type $s_{\Phi}$, see, for example, [9]. By the well-known fact that $\ell_{p,w} \cdot \ell_{q,w} = \ell_{r,w}$, for $1/p + 1/q = 1/r$, $0 < p, q, r < \infty$ we get the following result.

**Corollary 2.** (1) Let $1/p + 1/q = 1/r$, $0 < p, q, r < \infty$. Then $[\alpha] \in (S_p, S_r)$ if and only if $\alpha \in \ell_q$.

(2) Let $w_n$ be a decreasing positive sequence, and let $0 < p, q < \infty$, be such that $1/p + 1/q = 1$. Then $[\alpha] \in (S_{p,w}, S_{1,w})$ if and only if $\alpha \in \ell_{q,w}$, where $\ell_{p,w}$ is the weighted Lorentz space of sequences.

We call the Bergman-Schatten space of order $p$, $0 < p < \infty$, and we denote by $L_a^p(\ell_2)$ the space of all upper triangular matrices $A$ such that $\|A\|_{L_a^p(\ell_2)} = (\int_0^1 \|\sum_{k=0}^\infty A_k k^p \|_{S_p}^{p} 2rdr)^{1/p} < \infty$.

See, for example, [10] for further notations and details.

By Holder’s inequality we get the following result.

**Theorem 6.** Let $1 \leq p < \infty$. Then $[\alpha] \in (L_a^p(\ell_2), L_a^p(\ell_2))$ if and only if $\alpha \in \ell_q$, where $1/p + 1/q = 1$.

**Proof.** Let $A \in L_a^p(\ell_2)$ and $\alpha \in \ell_q$. We clearly have that $A \ast [\alpha] = D_\alpha \cdot A$. By Theorem AH we get

$$\|A \ast [\alpha]\|_{L_a^p(\ell_2)} = \int_0^1 \|D_\alpha \cdot A(r)\|_{S_p} 2rdr$$

$$\leq \left( \int_0^1 \|A(r)\|_{S_p}^{p} 2rdr \right)^{1/p} \|\alpha\|_{\ell_q} = \|A\|_{L_a^p(\ell_2)} \|\alpha\|_{\ell_q}$$

(26)

that is, $[\alpha] \in (L_a^p(\ell_2), L_a^p(\ell_2))$, and this completes the first part of the proof.
Conversely, let \([a] \in (L^p_d(e_2), L^1_d(e_2))\). By taking \(A = A_0 = (a_{ij})_{i,j=1}^{\infty} \in L^p_d(e_2)\), that is, for \((a_{ij})_j \in \ell_p\), we get

\[
A * [a] = \begin{pmatrix}
\alpha_1 a_{11} & 0 & \cdots \\
0 & \alpha_2 a_{22} & \cdots \\
& \ddots & \ddots \\
& & \ddots & \ddots
\end{pmatrix} \in L^1_d(e_2),
\] (27)

or, equivalently, \((a, a_{ij})_j \in \ell_1\). Hence by Hölder’s inequality it follows that \((a_j)_j \in \ell_q\), and the proof is complete. \(\square\)

Using the results of Bennett, proved in [8] we can also describe the Schur multipliers of scalar type also for others quasi-Banach spaces of matrices. The spaces of sequences \(d(a, p), g(a, p), \) and \(ces(p)\) were defined in [8].

We denote now by \(d^1_M(a, p), g^1_M(a, p), ces^1_M(p), \) and \(\ell^1_M(p)\) the spaces of upper triangular infinite matrices \(A = \sum_{k=0}^{\infty} A_k\), with all the sequences on the diagonals belonging to \(d(a, p)\) (resp., \(g(a, p), ces(p), \ell_p\)), and such that \(\|A\| = \left(\sum_k \|A_k\|^{q}_{d(a,p)}\right)^{1/q} < \infty\) (resp., \(\|A\| = \left(\sum_k \|A_k\|_{g(a,p)}^{q}\right)^{1/q} < \infty\) and so on) with the usual modification for \(q = \infty\).

Using Theorems 4.5 and 3.8 in [8], we have the following.

**Theorem 7.** (1) Let \(1 < p < \infty\). Then \(a \in g(p^*)\), where \(1/p + 1/p^* = 1\), if and only if \([a] \in (\ell^\infty_M(p), ces^\infty_M(p))\), where \(g(p^*) = g(a, p^*)\), with \(a = (1, 1, \ldots)\).

(2) Let \(0 < p < \infty\). Then \([a] \in (d^1_M(a, p), \ell^1_M(p))\) if and only if \(a \in g(a, p)\).

**Theorem 8.** (1) For \(1 < p < \infty\), \([a] \in (\ell^1_M(p), ces^1_M(p))\) if and only if \(a \in g(p^*)\).

(2) For \(0 < p < \infty\), \([a] \in (d^1_M(a, p), \ell^1_M(p))\) if and only if \(a \in g(a, p)\).

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**References**


