Research Article

Pointwise Multipliers of Triebel-Lizorkin Spaces on Carnot-Carathéodory Spaces

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Let $(\mathcal{X}, d, \mu)$ be a Carnot-Carathéodory space, namely, $\mathcal{X}$ is a smooth manifold, d is a control, or Carnot-Carathéodory, metric induced by a collection of vector fields of finite type. μ is a nonnegative Borel regular measure on $\mathcal{X}$ satisfying that there exists constant $C_0 \in [1, \infty)$ such that for all $x \in \mathcal{X}$ and $0 < r < \text{diam } \mathcal{X}$, $\mu(B(x, 2r)) := \mu(\{y \in \mathcal{X} : d(x, y) < 2r\}) \leq C_0 \mu(B(x, r)) < \infty$ (doubling property). Using the discrete Calderón reproducing formula and the Plancherel-Pólya characterization of the inhomogeneous Triebel-Lizorkin spaces developed in Han et al., in press and Han et al., 2008, pointwise multipliers of inhomogeneous Triebel-Lizorkin spaces are obtained.

1. Introduction

The multiplier theory of function spaces has been studied for a long time, and a lot of results have been obtained. As we know, the multiplier theory is one of the important parts in the studies of the Gleason problem, function space properties, and general operator theory. The pointwise multipliers on $\mathbb{R}^d$ are studied as a part of the researches of function spaces in several monographs, [1–8]. Pointwise multipliers have been found many important applications in partial differential equations.

However, it was not clear how to generalize the pointwise multipliers on $\mathbb{R}^n$ to spaces of homogeneous type introduced by Coifman and Weiss (see [9]) because the Fourier transform is no longer available. The main purpose of this paper is to establish pointwise multipliers on inhomogeneous Triebel-Lizorkin spaces in the setting of Carnot-Carathéodory spaces. To be more precisely, we first recall some necessary definitions. In this paper, we always assume that $(\mathcal{X}, d)$ is a metric space with a regular Borel measure μ such that all balls

defined by d have finite and positive measures. In what follows, set $\text{diam}(\mathcal{X}) \equiv \sup\{d(x, y) : x, y \in \mathcal{X}\}$, and for any $x \in \mathcal{X}$ and $r > 0$, set $B(x, r) \equiv \{y \in \mathcal{X} : d(x, y) < r\}$.

Definition 1.1 (see [10]). Let $(\mathcal{X}, d)$ be a metric space with a Borel regular measure μ such that all balls defined by d have finite and positive measures. The triple $(\mathcal{X}, d, \mu)$ is called a space of homogeneous type if there exists a constant $C_0 \in [1, \infty)$ such that for all $x \in \mathcal{X}$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C_0 \mu(B(x, r)) \quad (\text{doubling property}). \quad (1.1)$$

Remark 1.2. We point out that the doubling condition (1.1) implies that there exist positive constants C and n such that for all $x \in \mathcal{X}$ and $\lambda \geq 1$,

$$\mu(B(x, \lambda r)) \leq C \lambda^n \mu(B(x, r)), \quad (1.2)$$

where C is independent of x and r . Denote by n the homogeneous “dimension” of $\mathcal{X}$ as in [10].

A space of homogeneous type is called an RD space, if there exist constants $a_0, \tilde{C}_0 \in (1, \infty)$ such that for all $x \in \mathcal{X}$ and $0 < r < \text{diam}(\mathcal{X})/a_0$,

$$\tilde{C}_0 \mu(B(x, r)) \leq \mu(B(x, a_0 r)) \quad (1.3)$$

that is, some “reverse” doubling condition holds.

Clearly, any Ahlfors n -regular metric measure space $(\mathcal{X}, d, \mu)$ (which means that there exists some $n > 0$ such that $\mu(B(x, r)) \sim r^n$ for $x \in \mathcal{X}$ and $0 < r < \text{diam}(\mathcal{X})/2$) is a (n, n) -space (see [10]), also is an RD space and space of homogeneous type in the sense of Coifman in [10]. In other words, μ satisfies the doubling condition which is weaker than Ahlfors n -regular metric measure space and RD-spaces.

Another typical such a space is Carnot-Carathéodory space. One example with unbounded total measure studied in [11] is that $\mathcal{X}$ arises as the boundary of an unbounded model polynomial domain in $\mathbb{C}^2$. Let $\Omega = \{(z, w) \in \mathbb{C}^2 : \text{Im}(w) > P(z)\}$, where P is a real, subharmonic, non-harmonic polynomial of degree m . Then $\mathcal{X} = \partial\Omega$ can be identified with $\mathbb{C} \times \mathbb{R} = \{(z, t) : z \in \mathbb{C}, t \in \mathbb{R}\}$. The basic $(0, 1)$ Levi vector field is then $\bar{Z} = \partial/\partial\bar{z} - i(\partial P/\partial\bar{z})(\partial/\partial t)$, and we write $\bar{Z} = \mathbb{X}_1 + i\mathbb{X}_2$. The real vector fields $\{\mathbb{X}_1, \mathbb{X}_2\}$ and their commutators of order $\leq m$ span the tangent space to $\mathcal{X}$ at each point. See [10, 12] for more details and references therein.

We will also suppose that $\mu(\mathcal{X}) = \infty, \mu(\{x\}) = 0$ for all $x \in \mathcal{X}$. For any $x, y \in \mathcal{X}$ and $\delta > 0$, set $V_\delta(x) \equiv \mu(B(x, \delta))$ and $V(x, y) \equiv \mu(B(x, d(x, y)))$. It follows from (1.1) that $V(x, y) \sim V(y, x)$. The following notion of approximations of the identity on RD spaces was first introduced in [10]. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

We begin with recalling the definition of an approximation to the identity, which plays the same role as the heat kernel $H(s, x, y)$ does in Nagel-Stein’s theory [11].

Definition 1.3 (see [10, 12]). A sequence $\{S_k\}_{k \in \mathbb{Z}_+}$ of operators is said to be an approximation to the identity (in short, ATI) if there exists constant $C_1 > 0$ such that for all $k \in \mathbb{Z}_+$ and all x, x', y , and $y' \in \mathcal{X}$, $S_k(x, y)$, the kernel of S_k satisfies the following conditions:

$$\begin{aligned} S_k(x, y) &= 0 \quad \text{if } \rho(x, y) \geq C_1 2^{-k}, \quad |S_k(x, y)| \lesssim \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}, \\ |S_k(x, y) - S_k(x', y)| &\lesssim 2^k \rho(x, x') \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)} \end{aligned} \quad (1.4)$$

for $\rho(x, x') \leq \max\{C_1, 1\} 2^{-k}$,

$$|S_k(x, y) - S_k(x, y')| \lesssim 2^k \rho(y, y') \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)} \quad (1.5)$$

for $\rho(y, y') \leq \max\{C_1, 1\} 2^{-k}$,

$$|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]| \lesssim 2^{2k} \rho(x, x') \rho(y, y') \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)} \quad (1.6)$$

for $\rho(x, x') \leq \max\{C_1, 1\} 2^{-k}$ and $\rho(y, y') \leq \max\{C_1, 1\} 2^{-k}$,

$$\int_{\mathcal{X}} S_k(x, y) d\mu(y) = \int_{\mathcal{X}} S_k(x, y) d\mu(x) = 1. \quad (1.7)$$

The space of test functions plays a key role in this paper; see [10].

Definition 1.4. Fix two exponents $0 < \beta \leq 1$ and $\gamma > 0$. A function f defined on $\mathcal{X}$ is said to be a *test function* of type (β, γ) centered at $x_0 \in \mathcal{X}$ with width $r > 0$ if there exists a nonnegative constant C such that f satisfies the following conditions:

$$|f(x)| \leq C \frac{1}{(V_r(x_0) + V(x, x_0))} \frac{r^\gamma}{(r + d(x, x_0))^\gamma}, \quad (1.8)$$

$$|f(x) - f(x')| \leq C \left(\frac{d(x, x')}{r + d(x, x_0)} \right)^\beta \frac{1}{(V_r(x_0) + V(x, x_0))} \frac{r^\gamma}{(r + d(x, x_0))^\gamma} \quad (1.9)$$

for $d(x, x') \leq (1/2)(r + d(x, x_0))$.

If f is a test function of type (β, γ) centered at x_0 with width $r > 0$, we write $f \in \mathcal{M}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{M}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{M}(x_0, r, \beta, \gamma)} = \inf\{C \geq 0 : (1.8) \text{ and } (1.9) \text{ hold}\}. \quad (1.10)$$

We denote by $\mathcal{M}(\beta, \gamma)$ the class of all $f \in \mathcal{M}(x_0, 1, \beta, \gamma)$. It is easy to see that $\mathcal{M}(x_1, r, \beta, \gamma) = \mathcal{M}(\beta, \gamma)$ with the equivalent norms for all $x_1 \in \mathcal{X}$ and $r > 0$. Furthermore, it is also easy to check that $\mathcal{M}(\beta, \gamma)$ is a Banach space with respect to the norm in $\mathcal{M}(\beta, \gamma)$.

In what follows, for given $\epsilon \in (0, 1]$, we let $\widetilde{\mathcal{M}}(\beta, \gamma)$ be the completion of the space $\mathcal{M}(\epsilon, \epsilon)$ in $\mathcal{M}(\beta, \gamma)$ when $0 < \beta, \gamma \leq \epsilon$. Obviously $\widetilde{\mathcal{M}}(\epsilon, \epsilon) = \mathcal{M}(\epsilon, \epsilon)$. Moreover, $f \in \widetilde{\mathcal{M}}(\beta, \gamma)$ if and only if $f \in \mathcal{M}(\beta, \gamma)$ when $0 < \beta, \gamma \leq \epsilon$ and there exists $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}(\epsilon, \epsilon)$ such that $\|f - f_j\|_{\mathcal{M}(\beta, \gamma)} \rightarrow 0$ as $j \rightarrow \infty$. If $f \in \widetilde{\mathcal{M}}(\beta, \gamma)$, we then define $\|f\|_{\widetilde{\mathcal{M}}(\beta, \gamma)} = \|f\|_{\mathcal{M}(\beta, \gamma)}$. Obviously $\widetilde{\mathcal{M}}(\beta, \gamma)$ is a Banach space, and we also have $\|f\|_{\widetilde{\mathcal{M}}(\beta, \gamma)} = \lim_{j \rightarrow \infty} \|f_j\|_{\mathcal{M}(\beta, \gamma)}$ for the above chosen $\{f_j\}_{j \in \mathbb{N}}$.

We denote by $(\widetilde{\mathcal{M}}(\beta, \gamma))'$ the dual space of $\widetilde{\mathcal{M}}(\beta, \gamma)$ consisting of all linear functionals $\mathcal{L}$ from $\widetilde{\mathcal{M}}(\beta, \gamma)$ to $\mathbb{C}$ with the property that there exists a constant C such that for all $f \in \widetilde{\mathcal{M}}(\beta, \gamma)$,

$$|\mathcal{L}(f)| \leq C \|f\|_{\widetilde{\mathcal{M}}(\beta, \gamma)}. \quad (1.11)$$

We denote by $\langle h, f \rangle$ the natural pairing of elements $h \in (\widetilde{\mathcal{M}}(\beta, \gamma))'$ and $f \in \widetilde{\mathcal{M}}(\beta, \gamma)$. Since $\widetilde{\mathcal{M}}(x_1, r, \beta, \gamma) = \widetilde{\mathcal{M}}(\beta, \gamma)$ with the equivalent norms for all $x_1 \in \mathcal{X}$ and $r > 0$. Thus, for all $h \in (\widetilde{\mathcal{M}}(\beta, \gamma))'$, $\langle h, f \rangle$ is well defined for all $f \in \widetilde{\mathcal{M}}(x_0, r, \beta, \gamma)$ with $x_0 \in \mathcal{X}$ and $r > 0$.

The following constructions, which provide an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type, were given by Christ in [13].

Lemma 1.5. *Let $\mathcal{X}$ be a space of homogeneous type. Then there exist a collection $\{Q_\alpha^k \subset \mathcal{X} : k \in \mathbb{Z}_+, \alpha \in I_k\}$ of open subsets, where I_k is some (possible finite) index set, and constants $\delta \in (0, 1)$ and $C_5, C_6 > 0$ such that*

- (i) $\mu(\mathcal{X} \setminus \cup_\alpha Q_\alpha^k) = 0$ for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ if $\alpha \neq \beta$;
- (ii) for any α, β, k, l with $l \geq k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$;
- (iii) for each (k, α) and each $l < k$ there is a unique β such that $Q_\alpha^k \subset Q_\beta^l$;
- (iv) $\text{diam}(Q_\alpha^k) \leq C_2 \delta^k$;
- (v) each Q_α^k contains some ball $B(z_\alpha^k, C_3 \delta^k)$, where $z_\alpha^k \in \mathcal{X}$.

In fact, we can think of Q_α^k as being a dyadic cube with a diameter roughly δ^k and centered at z_α^k . In what follows, we always suppose $\delta = 1/2$. See [14] for how to remove this restriction. Also, in the following, for $k \in \mathbb{Z}_+, \tau \in I_k$, we will denote by $Q_\tau^{k, \nu}, \nu = 1, \dots, N(k, \tau, M)$, the set of all cubes $Q_\tau^{k+M} \subset Q_\tau^k$, where M is a fixed large positive integer.

Now, we can introduce the inhomogeneous Triebel-Lizorkin spaces $F_p^{\alpha, q}(\mathcal{X})$ via the approximation in Definition 1.3. Note that the Triebel-Lizorkin spaces have been already investigated for decades in the study of partial differential equations, interpolation theory, and approximation theory.

Definition 1.6. Suppose that $\{S_k\}_{k \in \mathbb{Z}_+}$ is an ATI and let $D_0 = S_0$, and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$. Let M be a fixed large positive integer, $Q_\tau^{0, \nu}$ be as above. Suppose that $-1 < s < 1$.

The inhomogeneous Triebel-Lizorkin space $F_p^{\alpha, q}(\mathcal{X})$ for $\max(n/(n+1), n/(n+1+s)) < p < \infty$ and $\max(n/(n+1), n/(n+1+s)) < q \leq \infty$ is the collection of $f \in (\widetilde{\mathcal{M}}(\beta, \gamma))'$, for some β and γ satisfying

$$\max\left(0, s, -s + n\left(\frac{1}{\min\{p, 1\}} - 1\right)\right) < \beta < 1, \quad \frac{n}{\min\{p, 1\}} - n < \gamma < 1 \quad (1.12)$$

such that

$$\begin{aligned} \|f\|_{F_p^{s,q}(\mathcal{X})} &= \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) \left[m_{Q_\tau^{0,\nu}}(|D_0(f)|) \right]^p \right\}^{1/p} \\ &\quad + \left\| \left\{ \sum_{k=1}^{\infty} [2^{ks} |D_k(f)|]^q \right\}^{1/q} \right\|_{L^p(\mathcal{X})} < \infty, \end{aligned} \quad (1.13)$$

where $m_{Q_\tau^{0,\nu}}(D_0(f))$ are averages of $D_0(f)$ over $Q_\tau^{0,\nu}$.

The restrictions (1.12) guarantee that the definitions of the *inhomogeneous Triebel-Lizorkin space* $F_p^{s,q}(\mathcal{X})$ for $\max(n/(n+1), n/(n+1+s)) < p < \infty$ and $\max(n/(n+1), n/(n+1+s)) < q \leq \infty$ are independent of the choices of β and γ satisfying these conditions and $F_p^{s,q}(\mathcal{X}) \subset (\widetilde{\mathcal{M}}(\beta, \gamma))'$, $\widetilde{\mathcal{M}}(\beta, \gamma) \subset \mathcal{M}(\beta, \gamma) \subset F_p^{s,q}(\mathcal{X})$ in [10].

The classical scale of inhomogeneous Triebel-Lizorkin spaces contains many well-known function spaces. For example, if $\alpha > 0$, $p = q = \infty$, one recovers the Hölder-Zygmund spaces $\mathcal{C}^\alpha(\mathcal{X})$, that is,

$$F_\infty^{\alpha,\infty}(\mathcal{X}) = \mathcal{C}^\alpha(\mathcal{X}), \quad \alpha > 0. \quad (1.14)$$

The space $\mathcal{C}^\alpha(\mathcal{X})$ is defined as the collection of f such that

$$\|f\|_{\mathcal{C}^\alpha(\mathcal{X})} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty. \quad (1.15)$$

If $-1 < s < 1$ and $1 < p < \infty$, then $H_p^s(\mathcal{X}) = F_p^{s,2}(\mathcal{X})$ are the Bessel potential spaces (Lebesgue spaces, Liouville spaces). If $m = 0, 1, 2, \dots$ and $1 < p < \infty$, then $W_p^m(\mathcal{X}) = H_p^m(\mathcal{X}) = F_p^{m,2}(\mathcal{X})$ are the usual Sobolev spaces. If $n/(n+1) < p \leq 1$, then $F_p^{0,2}(\mathcal{X}) = h^p(\mathcal{X})$ are the inhomogeneous Hardy spaces, which are closely related to the Hardy spaces $H^p(\mathcal{X})$ in [15, 16] (more precisely: the homogeneous spaces $\dot{F}_p^{0,2}(\mathcal{X})$ coincide with the usual Hardy spaces $H_p(\mathcal{X})$). We will use here the notation $F_p^{s,2}(\mathcal{X}) = H_p^s(\mathcal{X})$, $s \in (-1, 1)$, $n/(n+1) < p \leq \infty$. Then these spaces will be denoted as the (inhomogeneous) Hardy-Sobolev spaces, which include the above Lebesgue-Sobolev spaces for $1 < p < \infty$.

The inhomogeneous Triebel-Lizorkin spaces have the following Plancherel-Pölya characterizations in [10], which will be one of the the basic tools to prove the main results of this paper.

Lemma 1.7. *Let $\{D_k\}_{k \in \mathbb{Z}_+}$ be as in Definition 1.6, $-1 < s < 1$. Then if $\max(n/(n+1), n/(n+1+s)) < p < \infty$ and $\max(n/(n+1), n/(n+1+s)) < q \leq \infty$, for all $f \in (\widetilde{\mathcal{M}}(\beta, \gamma))'$ with β, γ satisfying (1.12), one has*

$$\begin{aligned} \|f\|_{F_p^{s,q}(\mathcal{X})} &\sim \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) [m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p \right\}^{1/p} \\ &\quad + \left\| \left\{ \sum_{k=1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(\tau,k,M)} \left[2^{ks} \inf_{z \in Q_\tau^{k,\nu}} |D_k(f)(z)| \chi_{Q_\tau^{k,\nu}}(\cdot) \right]^q \right\}^{1/q} \right\|_{L^p(\mathcal{X})} \\ &\sim \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) [m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p \right\}^{1/p} \\ &\quad + \left\| \left\{ \sum_{k=1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(\tau,k,M)} \left[2^{ks} \sup_{z \in Q_\tau^{k,\nu}} |D_k(f)(z)| \chi_{Q_\tau^{k,\nu}}(\cdot) \right]^q \right\}^{1/q} \right\|_{L^p(\mathcal{X})}. \end{aligned} \quad (1.16)$$

We now introduce the following definition of the pointwise multiplier.

Definition 1.8. Suppose that g is a given function on $\mathcal{X}$. Then g is called a pointwise multiplier for $F_p^{s,q}(\mathcal{X})$ if $f \rightarrow gf$ admits a bounded linear mapping from $F_p^{s,q}(\mathcal{X})$ into itself.

The main result in this paper is the following.

Theorem 1.9. *Let $-1 < s < 1$, $\max(n/(n+1), n/(n+1+s)) < p < \infty$ and $\max(n/(n+1), n/(n+1+s)) < q \leq \infty$ and $\alpha > \max(s, n/\min\{p, q, 1\} - n - s)$, then $g \in C^\alpha(\mathcal{X})$ with $0 < \alpha < 1$ is a multiplier for $F_p^{s,q}(\mathcal{X})$. In other words, $f \rightarrow gf$ yields a bounded linear mapping from $F_p^{s,q}(\mathcal{X})$ into itself and there is a positive constant C such that*

$$\|gf\|_{F_p^{s,q}(\mathcal{X})} \leq C \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})} \quad (1.17)$$

holds for all $g \in C^\alpha(\mathcal{X})$ and $f \in F_p^{s,q}(\mathcal{X})$.

We would like to point out that the study of pointwise multipliers is one of important problems in the theory of function spaces. It has attracted a lot of attention in the decades since starting with [7]. Pointwise multipliers in general spaces $F_p^{s,q}(\mathbb{R}^n)$, where $0 < p < \infty, 0 < q \leq \infty, s \in \mathbb{R}$ have been studied in great detail in [4, 5, 8].

Theorem 1.9 was proved in [8] for pointwise multipliers of inhomogeneous Triebel-Lizorkin spaces on $\mathbb{R}^n$ based on the Fourier transform. In the present setting, however, we do not have the Fourier transform at our disposal. Since the Fourier transform on Carnot-Carathéodory spaces is not available and hence the idea used in [8] does not work for this more general setting. A new idea to prove Theorem 1.9 is to use the discrete Calderón reproducing formula, which was developed in [10]. Therefore, this scheme easily extends to geometrical settings where the Fourier transform does not exist. The Fourier transform is missing but a version of pointwise multiplier is still present.

Throughout, we also denote by C a positive constant independent of main parameters involved, which may vary at different occurrences. Constants with subscripts do not change through the whole paper. We use $f \lesssim g$ and $f \gtrsim g$ to denote $f \leq Cg$ and $f \geq Cg$, respectively. If $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $a, b \in \mathbb{R}$, set $a \wedge b \doteq \min\{a, b\}$, $a \vee b \doteq \max\{a, b\}$. If $p > 1$, set $1/p + 1/p' = 1$.

2. Proof of Theorem 1.9

In this section, we will prove Theorem 1.9. Since there is no the Fourier transforms on spaces of homogeneous type, the proof of Theorem 1.9 is quite different from the proof of Theorem 2.8.2 in [8]. The key new ingredient in the proof of Theorem 1.9 is to apply the following discrete Calderón reproducing formulae established in [10, 12]. This formula can be stated as follows.

Lemma 2.1. *Suppose that $\{S_k\}_{k \in \mathbb{Z}_+}$ is an approximation to the identity as in Definition 1.3. Set $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. Then for any fixed $M \in \mathbb{N}$ large enough, there exists a family of functions $\{\tilde{D}_k(x, y)\}_{k \in \mathbb{Z}_+}$ and $\{\tilde{\tilde{D}}_k(x, y)\}_{k \in \mathbb{Z}_+}$ such that for any fixed $y_\tau^{k, \nu} \in Q_\tau^{k, \nu}$, $k \in \mathbb{N}$, $\tau \in I_k$ and $\nu \in \{1, \dots, N(k, \tau, M)\}$ and all $f \in (\tilde{M}(\beta, \gamma))'$ with $0 < \beta, \gamma < \epsilon$ and $x \in \mathcal{X}$*

$$\begin{aligned} f(x) &= \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0, \tau, M)} \int_{Q_\tau^{0, \nu}} \tilde{D}_0(x, y) d\mu(y) m_{Q_\tau^{0, \nu}}(D_0(f)) \\ &\quad + \sum_{k \in \mathbb{N}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau, M)} \mu(Q_\tau^{k, \nu}) \tilde{D}_k(x, y_\tau^{k, \nu}) D_k(f)(y_\tau^{k, \nu}) \\ &= \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0, \tau, M)} \int_{Q_\tau^{0, \nu}} D_0(x, y) d\mu(y) m_{Q_\tau^{0, \nu}}(\tilde{\tilde{D}}_0(f)) \\ &\quad + \sum_{k \in \mathbb{N}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau, M)} \mu(Q_\tau^{k, \nu}) D_k(x, y_\tau^{k, \nu}) \tilde{\tilde{D}}_k(f)(y_\tau^{k, \nu}), \end{aligned} \tag{2.1}$$

where the series converges in the norm of $F_p^{s, q}(\mathcal{X})$ with $\max(n/(n+1), n/(n+1+s)) < p < \infty$ and $\max(n/(n+1), n/(n+1+s)) < q \leq \infty$, $-1 < s < 1$, and $\tilde{M}(\beta', \gamma')$ for $f \in \tilde{M}(\beta, \gamma)$ with $\beta' < \beta$ and $\gamma' < \gamma$, and $(\tilde{M}(\beta', \gamma'))'$ for $f \in (\tilde{M}(\beta, \gamma))'$ with $\epsilon > \beta' > \beta$ and $\epsilon > \gamma' > \gamma$. Moreover, $\tilde{D}_k(x, y)$ and $\tilde{\tilde{D}}_k(x, y)$, the kernels of $\tilde{D}_k$ and $\tilde{\tilde{D}}_k$, satisfy the similar estimates but with x and y interchanged in (2.3): for $0 < \epsilon < 1$,

$$|\tilde{D}_k(x, y)| \leq C \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon}, \tag{2.2}$$

$$|\tilde{D}_k(x, y) - \tilde{D}_k(x', y)| \leq C \left(\frac{d(x, x')}{2^{-k} + d(x, y)} \right)^\epsilon \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon}, \tag{2.3}$$

for $d(x, x') \leq (2^{-k} + d(x, y))/2$,

$$\int_{\mathcal{X}} \tilde{D}_k(x, y) d\mu(y) = \int_{\mathcal{X}} \tilde{D}_k(x, y) d\mu(x) = 0, \quad (2.4)$$

when $k \in \mathbb{N}$; $\int_{\mathcal{X}} \tilde{D}_k(x, y) d\mu(y) = \int_{\mathcal{X}} \tilde{D}_k(x, y) d\mu(x) = 1$ when $k = 0$.

To prove Theorem 1.9, we first show the following lemma.

Lemma 2.2. Let $\{S_k(x, y)\}_{k \in \mathbb{Z}_+}$ and $\{G_k(x, y)\}_{k \in \mathbb{Z}_+}$ be two approximations to the identity as in Lemma 2.1 above and $D_k = S_k - S_{k-1}$, $E_k = G_k - G_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$, $E_0 = G_0$. For any given $\epsilon \in (0, 1)$ and $g \in C^\alpha(\mathcal{X})$ with $0 < \alpha < \epsilon$, then

$$\left| E_k g \tilde{D}_{k'}(x, y) \right| \lesssim \|g\|_{C^\alpha(\mathcal{X})} 2^{-|k-k'|\alpha} \frac{1}{V_{2^{-(k \wedge k')}}(x) + V_{2^{-(k \wedge k')}}(y) + V(x, y)} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(x, y))^\epsilon}, \quad (2.5)$$

where $k, k' \in \mathbb{Z}_+$.

Proof. We first show the inequality (2.5) of the case $k = k' = 0$ above. In fact, in this case, it follows that

$$\begin{aligned} \left| E_0 g \tilde{D}_0(x, y) \right| &\leq \left\| \|g\|_{C^\alpha(\mathcal{X})} \int E_0(x, z) \tilde{D}_0(z, y) d\mu(z) \right\| \\ &\lesssim \|g\|_{C^\alpha(\mathcal{X})} \frac{1}{V_1(x) + V_1(y) + V(x, y)} \frac{1}{(1 + d(x, y))^\epsilon}. \end{aligned} \quad (2.6)$$

We next consider the case $k' \geq k$ for $k > 0$ and $k' \geq 0$ or $k \geq 0$ and $k' > 0$. We write

$$\begin{aligned} &\left| E_k g \tilde{D}_{k'}(x, y) \right| \\ &= \left| \int [E_k(x, z)g(z) - E_k(x, y)g(y)] \tilde{D}_{k'}(z, y) d\mu(z) \right| \\ &\leq \int |E_k(x, z) - E_k(x, y)| |g(z)| |\tilde{D}_{k'}(z, y)| d\mu(z) \\ &\quad + \int |E_k(x, y)| |g(z) - g(y)| |\tilde{D}_{k'}(z, y)| d\mu(z) \\ &:= E + F. \end{aligned} \quad (2.7)$$

For E , we can obtain

$$E \lesssim \|g\|_{C^\alpha(\mathcal{X})} 2^{-(k'-k)\epsilon} \frac{1}{V_{2^{-(k \wedge k')}}(x) + V_{2^{-(k \wedge k')}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon}, \quad (2.8)$$

where $k, k' \in \mathbb{Z}_+$.

We may rewrite this integral as

$$\begin{aligned}
F &\lesssim \|g\|_{C^\alpha(\mathcal{X})} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon} \\
&\quad \times \left[\int_{W_1} d(z, y)^\alpha \frac{1}{V_{2^{-k'}}(z) + V_{2^{-k'}}(y) + V(z, y)} \frac{2^{-k'\epsilon}}{(2^{-k'} + d(z, y))^\epsilon} d\mu(z) \right. \\
&\quad \left. + \int_{W_2} d(z, y)^\alpha \frac{1}{V_{2^{-k'}}(z) + V_{2^{-k'}}(y) + V(z, y)} \frac{2^{-k'\epsilon}}{(2^{-k'} + d(z, y))^\epsilon} d\mu(z) \right] \\
&:= \|g\|_{C^\alpha(\mathcal{X})} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon} [F_1 + F_2],
\end{aligned} \tag{2.9}$$

where $W_1 = \{z : d(z, y) \geq 2^{-k'}\}$ and $W_2 = \{z : d(z, y) < 2^{-k'}\}$.

Observe further, a simple argument yields the same estimate for $k, k' \in \mathbb{Z}_+$ as follows:

$$\begin{aligned}
F_1 &\lesssim \sum_{i=-k'}^{\infty} \int_{\{z: 2^i \leq d(z, y) < 2^{i+1}\}} d(z, y)^\alpha \frac{1}{V_{2^{-k'}}(z) + V_{2^{-k'}}(y) + V(z, y)} \frac{2^{-k'\epsilon}}{(2^{-k'} + d(z, y))^\epsilon} d\mu(z) \\
&\lesssim \sum_{i=-k'}^{\infty} \frac{\mu(B(y, 2^{i+1}))}{\mu(B(y, 2^i))} 2^{(i+1)\alpha} 2^{-k'\epsilon} 2^{-i\epsilon} \lesssim 2^{-k'\alpha} \leq 2^{-(k'-k)\alpha},
\end{aligned} \tag{2.10}$$

where we used $0 < \alpha < \epsilon$.

For F_2 , we have

$$F_2 \lesssim 2^{-k'\alpha} \leq 2^{-(k'-k)\alpha}. \tag{2.11}$$

Combining the estimate for E, F , we conclude that if $k' \geq k$, then (2.5) holds. The case $k \geq k'$ is similar to above. This finishes the proof of Lemma 2.2. $\square$

Now we show the following technical version of Theorem 1.9.

Proposition 2.3. *For any $g \in C^\alpha(\mathcal{X})$, $f \in \widetilde{\mathcal{M}}(\beta, \gamma)$ with β and γ satisfying (1.12), then*

$$\|fg\|_{F_p^{s,q}(\mathcal{X})} \leq C \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})}, \tag{2.12}$$

where $-1 < s < 1$, $\max(n/(n+1), n/(n+1+s)) < p < \infty$ and $\max(n/(n+1), n/(n+1+s)) < q \leq \infty$ and $\alpha > \max(s, n/\min\{p, q, 1\} - n - s)$.

Proof. Using the Calderón reproducing formula, for any $g \in \mathcal{C}^\alpha(\mathcal{X})$, $f \in \widetilde{\mathcal{M}}(\beta, \gamma)$, We write

$$\begin{aligned}
& \|gf\|_{F_p^{s,q}(\mathcal{X})} \\
& \leq \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) \left[\sum_{\tau' \in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} m_{Q_\tau^{0,\nu}} \left(\left| E_0 g \tilde{D}_{Q_{\tau'}^{0,\nu'}} \right| \right) \mu(Q_{\tau'}^{0,\nu'}) \left| m_{Q_{\tau'}^{0,\nu'}}(D_0(f)) \right| \right]^p \right\}^{1/p} \\
& + \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) \left[\sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau',M)} \mu(Q_{\tau'}^{k',\nu'}) m_{Q_\tau^{0,\nu}} \right. \right. \\
& \quad \left. \left. \times \left(\left| E_0 g \tilde{D}_{k'}(\cdot, y_{\tau'}^{k',\nu'}) \right| \right) \left| D_{k'}(f)(y_{\tau'}^{k',\nu'}) \right| \right]^p \right\}^{1/p} \\
& + \left\| \left\{ \sum_{k=1}^{\infty} 2^{ksq} \left[\sum_{\tau' \in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'}) \left| E_k g \tilde{D}_{Q_{\tau'}^{0,\nu'}}(\cdot) \right| m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|) \right]^q \right\}^{1/q} \right\|_{L^p(\mathcal{X})} \\
& + \left\| \left\{ \sum_{k=1}^{\infty} 2^{ksq} \left[\sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau',M)} \mu(Q_{\tau'}^{k',\nu'}) \left| E_k g \tilde{D}_{k'}(\cdot, y_{\tau'}^{k',\nu'}) \right| \left| D_{k'}(f)(y_{\tau'}^{k',\nu'}) \right| \right]^q \right\}^{1/q} \right\|_{L^p(\mathcal{X})} \\
& := L_1 + L_2 + L_3 + L_4.
\end{aligned} \tag{2.13}$$

Applying the Hölder inequality for $p > 1$ and

$$\left(\sum_k |a_k| \right)^p \leq \sum_k |a_k|^p \tag{2.14}$$

for all $a_k \in \mathbb{C}$ and $p \leq 1$, for L_1 , it follows that

$$\begin{aligned}
L_1 & \lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})} \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) \sum_{\tau' \in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'}) \left| m_{Q_{\tau'}^{0,\nu'}}(D_0(f)) \right|^p \right. \\
& \quad \left. \times \mu(Q_{\tau'}^{0,\nu'})^{p\wedge 1-1} \inf_{x \in Q_\tau^{0,\nu}} \inf_{y \in Q_{\tau'}^{0,\nu'}} \left[\frac{1}{V_1(x) + V_1(y) + V(x,y)} \frac{1}{(1+d(x,y))^\epsilon} \right]^{p\wedge 1} \right\}^{1/p} \\
& \lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})} \left\{ \sum_{\tau \in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'}) \left| m_{Q_{\tau'}^{0,\nu'}}(D_0(f)) \right|^p \mu(Q_{\tau'}^{0,\nu'})^{p\wedge 1-1} [V_1(y_{\tau'}^{0,\nu'})]^{1-p\wedge 1} \right\}^{1/p} \\
& \lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})} \|f\|_{B_p^{s,q}(\mathcal{X})},
\end{aligned} \tag{2.15}$$

where we used the fact that for any $y_{\tau'}^{0,\nu'} \in Q_{\tau'}^{0,\nu'}$, $V_1(y_{\tau'}^{0,\nu'}) \mu(Q_{\tau'}^{0,\nu'})$.

For L_2 , in fact for all $k' \in \mathbb{N}$, $\tau' \in I_{k'}$ and $\nu' = 1, 2, \dots, N(k', \tau', M)$, and all $x \in \mathcal{X}$, $z \in Q_{\tau'}^{k', \nu'}$,

$$\mu(Q_{\tau'}^{k', \nu'}) \sim V_{2^{-k'}}(z), \quad V_1(y_{\tau'}^{k', \nu'}) + V(x, y_{\tau'}^{k', \nu'}) \sim V_1(z) + V(x, z), \quad (2.16)$$

$$V_1(z) \lesssim 2^{k'n} \mu(Q_{\tau'}^{k', \nu'}), \quad V_{2^{i+1}}(z) \lesssim 2^{\text{in}} V_1(z), 1 + d(x, y_{\tau'}^{k', \nu'}) \sim 1 + d(x, z). \quad (2.17)$$

From the inequality (2.5), the Hölder inequality for $q > 1$ and (2.14) for $q \leq 1$, the Fefferman-Stein vector-valued maximal function inequality in [17] and Lemma 1.7, it follows that

$$\begin{aligned} L_2 &\lesssim \|g\|_{C^\alpha(\mathcal{X})} \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0, \tau, M)} \mu(Q_\tau^{0, \nu}) \left[\sum_{k'=1}^{\infty} 2^{-k'(s+\alpha)} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k', \tau', M)} \mu(Q_{\tau'}^{k', \nu'}) 2^{k's} |D_{k'}(f)(y_{\tau'}^{k', \nu'})| \right. \right. \\ &\quad \left. \left. \times \inf_{x \in Q_\tau^{0, \nu}} \frac{1}{V_1(x) + V_1(y_{\tau'}^{k', \nu'}) + V(x, y_{\tau'}^{k', \nu'})} \frac{1}{(1 + d(x, y_{\tau'}^{k', \nu'}))^{\epsilon}} \right]^p \right\}^{1/p} \\ &\lesssim \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0, \tau, M)} \mu(Q_\tau^{0, \nu}) \left[\sum_{k'=1}^{\infty} 2^{-k'(s+\alpha)} \left(\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k', \tau', M)} \mu(Q_{\tau'}^{k', \nu'})^r |2^{k's} D_{k'}(f)(y_{\tau'}^{k', \nu'})|^r \right. \right. \\ &\quad \times \frac{1}{(V_1(x_\tau^{0, \nu}) + V_1(y_{\tau'}^{k', \nu'}) + V(x_\tau^{0, \nu}, y_{\tau'}^{k', \nu'}))^r} \\ &\quad \left. \left. \times \frac{1}{(1 + d(x_\tau^{0, \nu}, y_{\tau'}^{k', \nu'}))^{\epsilon r}} \right)^{1/r} \|g\|_{C^\alpha(\mathcal{X})} \right]^p \right\}^{1/p} \\ &\lesssim \|g\|_{C^\alpha(\mathcal{X})} \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0, \tau, M)} \mu(Q_\tau^{0, \nu}) \left[\sum_{k'=1}^{\infty} 2^{-k'(s+\alpha)} \left(\frac{1}{V_1(x_\tau^{0, \nu})} \int_{d(x_\tau^{0, \nu}, z) < 1} \sum_{\tau' \in I_{k'}} \right. \right. \right. \\ &\quad \left. \left. \times \sum_{\nu'=1}^{N(k', \tau', M)} \frac{V_1(x_\tau^{0, \nu})^{1-r}}{\mu(Q_{\tau'}^{k', \nu'})^{1-r}} |2^{k's} D_{k'}(f)(y_{\tau'}^{k', \nu'})|^r \chi_{Q_{\tau'}^{k', \nu'}}(z) d\mu(z) \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=0}^{\infty} \frac{1}{V_{2^{i+1}}(x_{\tau}^{0,\nu})} \int_{2^i \leq d(x_{\tau}^{0,\nu}, z) < 2^{i+1}} \frac{V_1(z)^{1-r}}{\mu(Q_{\tau'}^{k',\nu'})^{1-r}} \frac{V_{2^{i+1}}(z)^{1-r}}{V_1(z)^{1-r}} 2^{-i\epsilon r} \\
& \times \sum_{\tau' \in I_{k'}}^{N(k', \tau', M)} \sum_{\nu'=1} \left| 2^{k's} D_{k'}(f)(y_{\tau'}^{k',\nu'}) \right|^r \chi_{Q_{\tau'}^{k',\nu'}}(z) d\mu(z) \Bigg)^{1/r} \Bigg]^p \Bigg\}^{1/p} \\
& \lesssim \|g\|_{C^{\alpha}(\mathcal{X})} \\
& \times \left\| \sum_{k'=1}^{\infty} 2^{-k'(n+s+\alpha-n/r)} \left[\mathbf{M} \left(\sum_{\tau' \in I_{k'}}^{N(k', \tau', M)} \sum_{\nu'=1} 2^{k'sr} |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^r \chi_{Q_{\tau'}^{k',\nu'}} \right) \right]^{1/r} \right\|_{L^p(\mathcal{X})} \\
& \lesssim \|g\|_{C^{\alpha}(\mathcal{X})} \left\| \left\{ \sum_{k'=1}^{\infty} 2^{-k'(n+s+\alpha-n/r)(q \wedge 1)} \right. \right. \\
& \quad \times \left. \left[\mathbf{M} \left(\sum_{\tau' \in I_{k'}}^{N(k', \tau', M)} \sum_{\nu'=1} 2^{k'sr} |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^r \chi_{Q_{\tau'}^{k',\nu'}} \right) \right]^{q/r} \right\}^{1/q} \Bigg\|_{L^p(\mathcal{X})} \\
& \lesssim \|g\|_{C^{\alpha}(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})},
\end{aligned} \tag{2.18}$$

where we choose r satisfying $\max(n/(n+\epsilon), n/(n+s+\alpha)) < r < \min\{1, p, q\}$.

By the inequality (2.5), similarly,

$$\begin{aligned}
L_3 & \lesssim \|g\|_{C^{\alpha}(\mathcal{X})} \left\{ \int_{\mathcal{X}} \sum_{\tau' \in I_0} \sum_{\nu'=1}^{N(0, \tau', M)} \mu(Q_{\tau'}^{0,\nu'}) \left[m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|) \right]^p \right. \\
& \quad \left. \frac{1}{\left(V_1(x) + V_1(y_{\tau'}^{0,\nu'}) + V(x, y_{\tau'}^{0,\nu'}) \right)^{p \wedge 1}} \frac{1}{\left(1 + d(x, y_{\tau'}^{0,\nu'}) \right)^{\epsilon(p \wedge 1)}} d\mu(x) \right\}^{1/p} \tag{2.19} \\
& \lesssim \|g\|_{C^{\alpha}(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})},
\end{aligned}$$

where s can be any number in $(-\epsilon, \alpha) \subset (-1, 1)$.

We now consider the estimate of L_4 . Similar to the estimate of L_2 , using the equality (2.5), the Hölder inequality if $q > 1$ and (2.14) if $q \leq 1$, the Fefferman-Stein vector-valued maximal function inequality in [17], we obtain

$$\begin{aligned}
 L_4 &\lesssim \|g\|_{C^\alpha(\mathcal{X})} \left\| \left\{ \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \left[2^{(k-k')s} 2^{-|k'-k|\alpha} 2^{-k'n} 2^{(k \wedge k')n} 2^{[k'-(k \wedge k')]n/r} \right]^{q \wedge 1} \right. \right. \\
 &\quad \times \left. \left[\mathbf{M} \left(\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k', \tau', M)} 2^{k's} \left| D_{k'}(f) \left(y_{\tau'}^{k', \nu'} \right) \chi_{Q_{\tau'}^{k', \nu'}} \right| \right)^r \right]^{q/r} \right\}^{1/q} \Big\|_{L^p(\mathcal{X})} \\
 &\lesssim \|g\|_{C^\alpha(\mathcal{X})} \left\| \left\{ \sum_{k'=1}^{\infty} \left[\mathbf{M} \left(\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k', \tau', M)} 2^{k's} \left| D_{k'}(f) \left(y_{\tau'}^{k', \nu'} \right) \right|^r \chi_{Q_{\tau'}^{k', \nu'}} \right) \right]^{q/r} \right\}^{1/q} \right\|_{L^p(\mathcal{X})} \\
 &\lesssim \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{\alpha, q}(\mathcal{X})},
 \end{aligned} \tag{2.20}$$

where the first and the second inequalities follow from the estimates

$$\begin{aligned}
 \sup_k \sum_{k'=0}^{\infty} 2^{(k-k')s} 2^{-|k'-k|\alpha} 2^{-k'n} 2^{(k \wedge k')n} 2^{[k'-(k \wedge k')]n/r} &\leq C, \\
 \sup_{k'} \sum_{k=0}^{\infty} \left[2^{(k-k')s} 2^{-|k'-k|\alpha} 2^{-k'n} 2^{(k \wedge k')n} 2^{[k'-(k \wedge k')]n/r} \right]^{q \wedge 1} &\leq C
 \end{aligned} \tag{2.21}$$

with $\max(n/(n+1), n/(n+s+\alpha)) < r < \min\{1, p, q\}$, s can be any number in $(-\epsilon, \alpha)$, which verifies Proposition 2.3. $\square$

To introduce our definition of pointwise multiplication, the interesting estimate is needed.

Lemma 2.4. *Let $\{S_k(x, y)\}_{k \in \mathbb{Z}_+}$ be a approximation to the identity as in Lemma 2.1 above and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. For any $g \in C^\alpha(\mathcal{X})$ with $0 < \alpha < 1$, $h \in \widetilde{\mathcal{M}}(\beta, \gamma)$ with β and γ satisfying (1.12). Then*

$$|\langle D_k(\bullet, y)g, h \rangle| \leq C \|g\|_{C^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta, \gamma)} 2^{-k(\beta \wedge \alpha)} \frac{1}{V_1(x_0) + V(x_0, y)} \frac{1}{(1 + d(y, x_0))^r}, \tag{2.22}$$

where $k \in \mathbb{Z}_+$.

Proof. We first prove inequality (2.22) with $k = 0$. In fact, since $D_0(z, y) = 0$ if $d(z, y) \geq 2C_1$, then

$$\begin{aligned}
& |\langle D_0(\bullet, y)g, h \rangle| \\
& \lesssim \|g\|_{C^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta, \gamma)} \int_{d(y, z) < 2C_1} \frac{1}{V_1(z) + V_1(y)} \frac{1}{V_1(x_0) + V(x_0, z)} \frac{1}{(1 + d(z, x_0))^{\bar{\gamma}}} d\mu(z) \\
& = \|g\|_{C^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta, \gamma)} \left[\int_{d(y, z) < 2C_1, d(x_0, z) \geq d(x_0, y)/2} \cdots + \int_{d(y, z) < 2C_1, d(x_0, z) < d(x_0, y)/2} \cdots \right] \\
& := \|g\|_{C^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta, \gamma)} [B_1 + B_2].
\end{aligned} \tag{2.23}$$

For B_1 , since $V(x_0, y) = \mu(B(x_0, d(x_0, y))) \leq \mu(B(x_0, 2d(x_0, z)))\mu(B(x_0, d(x_0, z))) = V(x_0, z)$, it follows that

$$B_1 \lesssim \frac{1}{V_1(x_0) + V(x_0, y)} \frac{1}{(1 + d(y, x_0))^{\bar{\gamma}}}. \tag{2.24}$$

For B_2 , the fact $d(y, z) < 2C_1, d(x_0, z) < d(x_0, y)/2$ implies that $2C_1 > d(y, z) > d(x_0, y)/2$, then

$$B_2 \lesssim \frac{1}{V_1(x_0) + V(x_0, y)} \lesssim \frac{1}{V_1(x_0) + V(x_0, y)} \frac{1}{(1 + d(y, x_0))^{\bar{\gamma}}}. \tag{2.25}$$

That is, (2.22) with $k = 0$ holds.

To prove (2.22) with $k \in \mathbb{N}$, we write

$$\begin{aligned}
|\langle D_k(\bullet, y)g, h \rangle| & = \left| \int_{\mathcal{X}} D_k(x, y) [g(x)h(x) - g(y)h(y)] d\mu(x) \right| \\
& \leq \int_{\mathcal{X}} |D_k(x, y)| |g(x) - g(y)| |h(x)| d\mu(x) \\
& \quad + \int_{\mathcal{X}} |D_k(x, y)| |g(y)| |h(x) - h(y)| d\mu(x) \\
& := I + II.
\end{aligned} \tag{2.26}$$

For I , since $D_k(x, y) = 0$ if $d(x, y) \geq 2C_1 2^{-k}$, we obtain

$$\begin{aligned}
 I &\lesssim \|g\|_{C^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta, \gamma)} 2^{-k\alpha} \\
 &\quad \times \int_{d(x, y) < 2C_1 2^{-k}} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)} \frac{1}{V_1(x_0) + V(x_0, x)} \frac{1}{(1 + d(x, x_0))^{\bar{\gamma}}} d\mu(x) \\
 &\lesssim \|g\|_{C^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta, \gamma)} 2^{-k\alpha} \left[\int_{d(x, y) < 2C_1 2^{-k}, d(x, y) \leq (1/2)(1 + d(y, x_0))} \cdots + \int_{2C_1 2^{-k} > d(x, y) > (1/2)(1 + d(y, x_0))} \cdots \right] \\
 &\doteq \|g\|_{C^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta, \gamma)} 2^{-k\alpha} [I_1 + I_2].
 \end{aligned} \tag{2.27}$$

For I_1 , by $d(x, y) \leq (1/2)(1 + d(y, x_0))$ implies that $d(y, x_0) \leq 2(d(x, x_0) + 1)$ and

$$\frac{1}{V_1(x_0) + V(x_0, x)} \lesssim \frac{1}{V_1(x_0) + \mu(B(x_0, d(x_0, y)))} = \frac{1}{V_1(x_0) + V(x_0, y)}. \tag{2.28}$$

Thus

$$\begin{aligned}
 I_1 &\lesssim \frac{1}{V_1(x_0) + V(x_0, y)} \frac{1}{(1 + d(y, x_0))^{\bar{\gamma}}} \int_{d(x, y) < 2C_1 2^{-k}} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)} d\mu(x) \\
 &\lesssim \frac{1}{V_1(x_0) + V(x_0, y)} \frac{1}{(1 + d(y, x_0))^{\bar{\gamma}}}.
 \end{aligned} \tag{2.29}$$

For I_2 , by $2C_1 2^{-k} > d(x, y) > (1/2)(1 + d(y, x_0))$ implies that

$$1 + d(y, x_0) < C, \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)} \lesssim \frac{1}{V(y, x)} \lesssim \frac{1}{V_1(x_0) + V(x_0, y)}. \tag{2.30}$$

Thus

$$\begin{aligned}
 I_2 &\lesssim \frac{1}{V_1(x_0) + V(x_0, y)} \frac{1}{(1 + d(x_0, y))^{\bar{\gamma}}} \int_{\mathcal{X}} \frac{1}{V_1(x_0) + V(x_0, x)} \frac{1}{(1 + d(x, x_0))^{\bar{\gamma}}} d\mu(x) \\
 &\lesssim \frac{1}{V_1(x_0) + V(x_0, y)} \frac{1}{(1 + d(x_0, y))^{\bar{\gamma}}}.
 \end{aligned} \tag{2.31}$$

To obtain the estimate of the II , we write

$$\begin{aligned}
 II &\lesssim \|g\|_{C^a(\mathcal{X})} \|h\|_{\mathcal{M}(\beta, \gamma)} \left\{ \int_{\varpi_1} |D_k(x, y)| \frac{1}{V_1(x_0) + V(x_0, y)} \frac{d(x, y)^\beta}{(1 + d(y, x_0))^{\beta + \gamma}} d\mu(x) \right. \\
 &\quad + \int_{\varpi_2} |D_k(x, y)| \left[\frac{1}{V_1(x_0) + V(x_0, x)} \frac{1}{(1 + d(x, x_0))^\gamma} \right. \\
 &\quad \left. \left. + \frac{1}{V_1(x_0) + V(x_0, y)} \frac{1}{(1 + d(y, x_0))^\gamma} \right] d\mu(x) \right\} \\
 &:= \|g\|_{C^a(\mathcal{X})} \|h\|_{\mathcal{M}(\beta, \gamma)} [II_1 + II_2 + II_3],
 \end{aligned} \tag{2.32}$$

where $\varpi_1 = \{x \in \mathcal{X} : d(x, y) \leq (1/2)(1 + d(y, x_0))\}$ and $\varpi_2 = \{x \in \mathcal{X} : d(x, y) > (1/2)(1 + d(y, x_0))\}$.

For II_1 , we have

$$\begin{aligned}
 II_1 &\lesssim \frac{1}{V_1(x_0) + V(x_0, y)} \frac{2^{-k\beta}}{(1 + d(x_0, y))^\gamma} \int_{\varpi_1} |D_k(x, y)| d\mu(x) \\
 &\lesssim \frac{1}{V_1(x_0) + V(x_0, y)} \frac{2^{-k\beta}}{(1 + d(x_0, y))^\gamma}.
 \end{aligned} \tag{2.33}$$

For II_2 , similar to I_2 , we have

$$|D_k(x, y)| \lesssim \frac{1}{V_1(x_0) + V(x_0, y)} \frac{2^{-k\beta}}{(1 + d(x_0, y))^\gamma}, \tag{2.34}$$

thus

$$\begin{aligned}
 II_2 &\lesssim \frac{1}{V_1(x_0) + V(x_0, y)} \frac{2^{-k\beta}}{(1 + d(x_0, y))^\gamma} \int_{\varpi_2} \frac{1}{V_1(x_0) + V(x_0, x)} \frac{1}{(1 + d(x, x_0))^\gamma} d\mu(x) \\
 &\leq \frac{1}{V_1(x_0) + V(x_0, y)} \frac{2^{-k\beta}}{(1 + d(x_0, y))^\gamma}.
 \end{aligned} \tag{2.35}$$

For II_3 , obviously

$$\begin{aligned} II_3 &\lesssim \frac{2^{-k\beta}}{V_1(x_0) + V(x_0, y)} \frac{1}{(1 + d(y, x_0))^{\gamma}} \int_{\mathcal{X}} |D_k(x, y)| d\mu(x) \\ &\lesssim \frac{2^{-k\beta}}{V_1(x_0) + V(x_0, y)} \frac{1}{(1 + d(y, x_0))^{\gamma}}. \end{aligned} \quad (2.36)$$

Combining the estimate of I and II , we obtain that for $k \in \mathbb{N}$

$$|\langle D_k(\bullet, y)g, h \rangle| \leq C \|g\|_{C^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta, \gamma)} 2^{-k(\beta \wedge \alpha)} \frac{1}{V_1(x_0) + V(x_0, y)} \frac{1}{(1 + d(y, x_0))^{\gamma}}. \quad (2.37)$$

Thus, (2.22) also holds. This finishes the proof of Lemma 2.4. $\square$

If $g \in C^\alpha(X)$ and $f \in F_p^{s,q}(\mathcal{X})$ for $s \in (-1, 1)$, $\max(n/(n+1), d/(d+\epsilon+s)) < p < \infty$, $\max(n/(n+1), d/(d+\epsilon+s)) < q \leq \infty$, it is not clear in general what is meant by gf (pointwise multiplication). Our approach is the following.

Lemma 2.5. *For any $f \in F_p^{s,q}(\mathcal{X})$ with $\max(n/(n+1), n/(n+1+s)) < p < \infty$ and $\max(n/(n+1), n/(n+1+s)) < q \leq \infty$, $-1 < s < 1$, and $g \in C^\alpha(\mathcal{X})$ with $1 > \alpha > \max(s, (n/\min\{p, q, 1\}) - n - s)$. There exist a constant C and a sequence $\{f_j\}_{j \in \mathbb{N}}$ such that $f_j \in \widetilde{\mathcal{M}}(\epsilon, \epsilon)$, $\|f_j\|_{F_p^{s,q}(\mathcal{X})} \leq C \|f\|_{F_p^{s,q}(\mathcal{X})}$ and $\lim_{j \rightarrow \infty} \langle gf_j, h \rangle$ converges for any $h \in \widetilde{\mathcal{M}}(\beta, \gamma)$ with β and γ satisfying (1.12).*

Proof. For any $f \in F_p^{s,q}(\mathcal{X})$ with $\max(n/(n+1), n/(n+1+s)) < p < \infty$ and $\max(n/(n+1), n/(n+1+s)) < q \leq \infty$, $-1 < s < 1$, and $g \in C^\alpha(\mathcal{X})$ with $\alpha > \max(s, (n/\min\{p, q, 1\}) - n - s)$, denote

$$\begin{aligned} f_j &= \sum_{\tau=1}^j \sum_{\nu=1}^{N(0, \tau, M)} \mu(Q_\tau^{0, \nu}) m_{Q_\tau^{0, \nu}} \left(\widetilde{D}_0(f) \right) D_{Q_\tau^{0, \nu}}(x) \\ &\quad + \sum_{k=1}^j \sum_{\tau=1}^j \sum_{\nu=1}^{N(k, \tau, M)} \mu(Q_\tau^{k, \nu}) D_k(x, y_\tau^{k, \nu}) \widetilde{D}_k(f) (y_\tau^{k, \nu}), \end{aligned} \quad (2.38)$$

where $D_{Q_\tau^{0, \nu}}(x) = (1/\mu(Q_\tau^{0, \nu})) \int_{Q_\tau^{0, \nu}} D_0(x, y) d\mu(y)$.

It is easy to see that $f_j \in \widetilde{\mathcal{M}}(\epsilon, \epsilon)$. Applying a similar proof as in Proposition 2.3 with $g = 1$ and $f = f_j$ gives $\|f_j\|_{F_p^{s,q}(\mathcal{X})} \leq C\|f\|_{F_p^{s,q}(\mathcal{X})}$. We write

$$\begin{aligned}
 |\langle f_j - f_m, gh \rangle| &\leq \left| \sum_{\tau=m+1}^j \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) m_{Q_\tau^{0,\nu}}(\widetilde{D}_0(f)) \langle D_{Q_\tau^{0,\nu}}, gh \rangle \right| \\
 &\quad + \left| \sum_{k=m+1}^j \sum_{\tau=1}^j \sum_{\nu=1}^{N(k,\tau,M)} \mu(Q_\tau^{k,\nu}) \langle D_k(\bullet, y_\tau^{k,\nu}), gh \rangle \widetilde{D}_k(f)(y_\tau^{k,\nu}) \right| \\
 &\quad + \left| \sum_{k=1}^j \sum_{\tau=m+1}^j \sum_{\nu=1}^{N(k,\tau,M)} \mu(Q_\tau^{k,\nu}) \langle D_k(\bullet, y_\tau^{k,\nu}), gh \rangle \widetilde{D}_k(f)(y_\tau^{k,\nu}) \right| \\
 &:= R + T + Y.
 \end{aligned} \tag{2.39}$$

The fact (2.22) implies

$$\begin{aligned}
 &\left\{ \sum_{\tau=m+1}^j \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) |D_{Q_\tau^{0,\nu}}(gh)|^{p'} \right\}^{1/p'} \\
 &\lesssim \|g\|_{C^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} \left\{ \sum_{\tau \geq \log_2^m} \int_{\{y: 2^\tau < d(x_0, y) \leq 2^{\tau+1}\}} \left[\frac{1}{V_1(x_0) + V(x_0, y)} \right]^{p'} \right. \\
 &\quad \left. \times \frac{1}{(1 + d(y, x_0))^{p'}} d\mu(y) \right\}^{1/p'} \\
 &\lesssim \|g\|_{C^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} \left\{ \sum_{\tau \geq \log_2^m} 2^{-\tau\gamma p'} \right\}^{1/p'} \\
 &= m^{-\gamma} \|g\|_{C^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)}.
 \end{aligned} \tag{2.40}$$

Thus

$$\begin{aligned}
 R &\lesssim \begin{cases} \|f\|_{F_p^{s,q}(\mathcal{X})} \left\{ \sum_{\tau=m+1}^j \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) |D_{Q_\tau^{0,\nu}}^*(gh)|^{p'} \right\}^{1/p'} & : p > 1 \\ \|f\|_{F_p^{s,q}(\mathcal{X})} \sup_{y_\tau^{0,\nu}, m < d(x_0, y_\tau^{0,\nu}) \leq j} \frac{\mu(Q_\tau^{0,\nu})^{1-1/p}}{V_1(x_0) + V(x_0, y_\tau^{0,\nu})} \frac{1}{(1 + d(y_\tau^{0,\nu}, x_0))^\gamma} & : p \leq 1, \end{cases} \\
 &\lesssim \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} m^{-\gamma},
 \end{aligned} \tag{2.41}$$

this finishes the proof of $R \rightarrow 0$ as $j, m \rightarrow \infty$.

We now consider the estimate of T . By Hölder inequality if $p, q > 1$, and (2.14) if $p, q < 1$ and (2.22), it follows that

$$\begin{aligned}
T &\lesssim \begin{cases} \|f\|_{F_p^{s,q}(\mathcal{X})} \left\| \left\{ \sum_{k=m+1}^j |2^{-ks} D_k^*(gh)|^{q'} \right\}^{1/q'} \right\|_{L_{p'}(\mathcal{X})} & : p > 1, q > 1, \\ \|f\|_{B_p^{s-\varepsilon,\infty}(\mathcal{X})} \sum_{k=m+1}^j \sup_{x \in \mathcal{X}} 2^{k[-s+\varepsilon]} \mu(Q_{\tau}^{k,v})^{1-1/p} |D_k^*(gh)(x)| & : p \leq 1, q > 1, \\ \|f\|_{F_p^{s,q}(\mathcal{X})} \left\| \sup_{m+1 \leq k \leq j} |2^{-ks} D_k^*(gh)| \right\|_{L_{p'}(\mathcal{X})} & : p > 1, q \leq 1, \\ \|f\|_{B_p^{s,1}(\mathcal{X})} \sup_{x \in \mathcal{X}} \sup_{m+1 \leq k \leq j} 2^{-ks} \mu(Q_{\tau}^{k,v})^{1-1/p} |D_k^*(gh)(x)| & : p \leq 1, q \leq 1, \end{cases} \\
&\lesssim \begin{cases} \|f\|_{F_p^{s,q}(\mathcal{X})} \left\| \left\{ \sum_{k=m+1}^j |2^{-ks} D_k^*(gh)|^{q'} \right\}^{1/q'} \right\|_{L_{p'}(\mathcal{X})}, & : p > 1, q > 1, \\ \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} 2^{-m[s-\varepsilon+n(1-1/p)+\alpha\wedge\beta]}, & : p \leq 1, q > 1, \\ \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} 2^{-m[s+\alpha\wedge\beta]}, & : p > 1, q \leq 1, \\ \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} 2^{-m[s+n(1-1/p)+\alpha\wedge\beta]}, & : p \leq 1, q \leq 1, \end{cases} \quad (2.42)
\end{aligned}$$

where let ε be a positive number with $s - \varepsilon + n(1 - 1/p) + \alpha \wedge \beta > 0$ when $p \leq 1$ and $s + \alpha \wedge \beta > 0$ if $p > 1$, and the second inequality can be obtained by using the fact that

$$F_p^{s,q}(\mathcal{X}) \subset B_p^{s,\max(p,q)}(\mathcal{X}) \subset B_p^{s-\varepsilon,\infty}(\mathcal{X}), \quad (2.43)$$

when $-\varepsilon < s - \varepsilon$, see [10, Proposition 5.31].

By the Proposition 2.3, then $gh \in F_{p'}^{-s,q'}(\mathcal{X})$ if $s \in (-1, 1), p, q > 1$. Thus by the Calderón reproducing that the series converges in the norm of $F_{p'}^{-s,q'}(\mathcal{X})$ for $gh \in F_{p'}^{-s,q'}(\mathcal{X})$ with $s \in (-1, 1), p, q > 1$ in Lemma 2.1, which proves $T \rightarrow 0$ as $j, m \rightarrow \infty$ when $\max(n/(n+1), n/(n+1+s)) < p < \infty$ and $\max(n/(n+1), n/(n+1+s)) < q \leq \infty, -1 < s < 1$, and $g \in C^\alpha(\mathcal{X})$ with $\alpha > \max(s, (n/\min\{p, q, 1\}) - d - s)$.

For Y , applying the Hölder inequality for $p, q > 1$ and (2.14) for $p, q \leq 1$ and (2.22), we also have

$$Y \lesssim \begin{cases} \|f\|_{F_p^{s,q}(\mathcal{X})} \left\{ \int_{\{y|m < d(x_0,y) \leq j\}} \left\{ \sum_{k=1}^j |2^{-ks} D_k^*(gh)(y)|^{q'} \right\}^{p'/q'} d\mu(y) \right\}^{1/p'} & : p > 1, q > 1, \\ \|f\|_{B_p^{s-\varepsilon,\infty}(\mathcal{X})} \sup_{k \in \mathbb{N}} \sup_{y \in \{y|m < d(x_0,y) \leq j\}} 2^{-k[s-\varepsilon]} \mu(Q_{\tau}^{k,v})^{1-1/p} |D_k^*(gh)(y)| & : p \leq 1, q > 1, \\ \|f\|_{F_p^{s,q}(\mathcal{X})} \left\{ \int_{\{y|m < d(x_0,y) \leq j\}} \left\{ \sup_{1 \leq k \leq j} |2^{-ks} D_k^*(gh)(y)| \right\}^{p'} d\mu(y) \right\}^{1/p'} & : p > 1, q \leq 1, \\ \|f\|_{B_p^{s,1}(\mathcal{X})} \sup_{y \in \{y|m < d(x_0,y) \leq j\}} \sup_{1 \leq k \leq j} 2^{-ks} \mu(Q_{\tau}^{k,v})^{1-1/p} |D_k^*(gh)(y)| & : p \leq 1, q \leq 1, \end{cases}$$

$$\begin{aligned}
& \lesssim \begin{cases} \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} m^{-\gamma} \left\{ \sum_{k=1}^j 2^{-k(s+\alpha\wedge\beta)q'} \right\}^{1/q'} & : p > 1, q > 1, \\ \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} m^{-\gamma} \left\{ \sup_{k \in \mathbb{N}} 2^{-k[s-\varepsilon+n(1-1/p)+(\alpha\wedge\beta)]q'} \right\}^{1/q'} & : p \leq 1, q > 1, \\ \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} m^{-\gamma} \sup_{k \in \mathbb{N}} 2^{-k[s+\alpha\wedge\beta]} & : p > 1, q \leq 1, \\ \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} m^{-\gamma} \sup_{k \in \mathbb{N}} 2^{-k[s+n(1-1/p)+\alpha\wedge\beta]} & : p \leq 1, q \leq 1, \end{cases} \\
& \lesssim \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} m^{-\gamma},
\end{aligned} \tag{2.44}$$

where let ε be a positive number with $s-\varepsilon+n(1-1/p)+\alpha\wedge\beta > 0$ when $p \leq 1$ and $s+\alpha\wedge\beta > 0$ if $p > 1$, and we also used the fact that $F_p^{s,q}(\mathcal{X}) \subset B_p^{s,\max(p,q)}(\mathcal{X}) \subset B_p^{s-\varepsilon,\infty}(\mathcal{X})$, when $-\varepsilon < s-\varepsilon$. This finishes the proof of $Y \rightarrow 0$ as $j, m \rightarrow \infty$, and hence the proof of Lemma 2.5 is concluded. $\square$

The above estimate shows $\lim_{j \rightarrow \infty} \langle gf_j, h \rangle$ exists and the limit is independent of the choice of f_j . Therefore, for $g \in C^\alpha(\mathcal{X})$, $f \in F_p^{s,q}(\mathcal{X})$ with $\max(n/(n+1), n/(n+1+s)) < p < \infty$ and $\max(n/(n+1), n/(n+1+s)) < q \leq \infty$, $s \in (-1, 1)$, $\alpha > \max(s, (n/\min\{p, q, 1\}) - n - s)$. We define

$$\langle gf, h \rangle = \lim_{j \rightarrow \infty} \langle gf_j, h \rangle \tag{2.45}$$

for any $h \in \widetilde{\mathcal{M}}(\beta, \gamma)$ with β and γ satisfying (1.12), f_j is fundamental sequence defined in Lemma 2.5.

We now prove Theorem 1.9.

Proof of Theorem 1.9. By Proposition 2.3 and Lemma 2.5, for any $g \in C^\alpha(\mathcal{X})$, $f \in F_p^{s,q}(\mathcal{X})$, Fatou's lemma implies that

$$\begin{aligned}
& \|gf\|_{F_p^{s,q}(\mathcal{X})} \\
& \leq \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) \left[m_{Q_\tau^{0,\nu}} \left(\left| \lim_{j \rightarrow \infty} E_0(f_j g) \right| \right) \right]^p \right\}^{1/p} \\
& \quad + \left\| \left\{ \sum_{k=1}^{\infty} \left[2^{k\alpha} \lim_{j \rightarrow \infty} |E_k(f_j g)| \right]^q \right\}^{1/q} \right\|_{L^p(\mathcal{X})} \\
& \leq \liminf_{j \rightarrow \infty} \|gf_j\|_{F_p^{s,q}(\mathcal{X})} \lesssim \|g\|_{C^\alpha(\mathcal{X})} \|f\|_{F_p^{s,q}(\mathcal{X})}.
\end{aligned} \tag{2.46}$$

We complete the proof of Theorem 1.9. $\square$

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