Research Article

Derivatives of the Berezin Transform

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For a rotation invariant domain Ω , we consider $A^2(\Omega, \mu)$ the Bergman space and we investigate some properties of the rank one projection $A(z) := \langle \cdot, k_z \rangle k_z$. We prove that the trace of all the strong derivatives of $A(z)$ is zero. We also focus on the generalized Fock space $A^2(\mu_m)$, where μ_m is the measure with weight $e^{-|z|^m}$, $m > 0$, with respect to the Lebesgue measure on $\mathbb{C}^n$ and establish estimations of derivatives of the Berezin transform of a bounded operator T on $A^2(\mu_m)$.

1. Introduction and Statement of the Main Results

We consider a rotation invariant open set Ω in $\mathbb{C}^n$ and μ a positive rotation invariant measure on Ω ; we suppose that μ has moments of every order. Let $L^2(\Omega, \mu)$ be the Hilbert space of square integrable complex-valued functions on Ω and $\mathcal{A}^2(\Omega, \mu)$ its subspace consisting of holomorphic elements. We assume that for each compact set $K \subset \Omega$ there exists $C = C(K)$ such that for all $f \in \mathcal{A}^2(\Omega, \mu)$

$$\sup_{z \in K} |f(z)| \leq C \|f\|_{L^2(\Omega, \mu)}. \quad (1.1)$$

It is known that $\mathcal{A}^2(\Omega, \mu)$ is a closed space of $L^2(\Omega, \mu)$ and possesses a reproducing kernel $K(w, z) := K_z(w)$: we have

$$f(z) = \int_{\Omega} f(w) K(z, w) d\mu(w) = \langle f, K_z \rangle, \quad (1.2)$$

for all $f \in \mathcal{A}^2(\Omega, \mu)$ and $z \in \Omega$.

For a bounded linear operator T on $\mathcal{A}^2(\Omega, \mu)$, the Berezin transform of T is the function $\tilde{T}$ defined on Ω by

$$\tilde{T}(z) := \langle Tk_z, k_z \rangle, \quad (1.3)$$

where k_z is the normalized reproducing kernel

$$k_z(\omega) := \frac{K(\omega, z)}{K(z, z)^{1/2}}. \quad (1.4)$$

The case of the Fock space, where $\Omega = \mathbb{C}^n$ and μ_2 is the Gaussian measure, was considered by Coburn, Englis, and Zhang. Coburn [1] has shown that $\tilde{T}$ is a Lipschitz function. Namely, for $x, y \in \mathbb{C}^n$,

$$|\tilde{T}(x) - \tilde{T}(y)| \leq 2\|T\||x - y|, \quad (1.5)$$

the constant 2 being sharp (see [2]).

Englis and Zhang [3] have shown that $\tilde{T}$ has bounded derivatives of all orders. Namely, for any multi-indices β, γ , there exists a constant $c_{\beta, \gamma}$, depending on β, γ , and n only, such that

$$\|\partial^\beta \bar{\partial}^\gamma \tilde{T}\|_\infty \leq c_{\beta, \gamma} \|T\|, \quad (1.6)$$

where ∂^β will stand for

$$\frac{\partial^{|\beta|}}{\partial z_1^{\beta_1} \cdots \partial z_n^{\beta_n}}, \quad \text{where } \frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \text{ for } z_j = x_j + iy_j \quad (1.7)$$

and similarly for $\bar{\partial}$. Recently, the author extended Coburn's result to weighted Fock spaces, corresponding to $\Omega = \mathbb{C}^n$ endowed with the measure

$$d\mu_m(z) = e^{-|z|^m} dV(z), \quad (1.8)$$

where m is a positive parameter and dV is the normalized Lebesgue measure on $\mathbb{C}^n$, such that the volume of the unit ball in $\mathbb{C}^n$ is equal to $2n$. We have showed that $\tilde{T}$ satisfies a local Lipschitz condition. There exist positive constants A, C , depending on n and m only, such that, for any $a \in \mathbb{C}^n$ with $|a| > A$, there is a neighbourhood U of a that satisfies

$$|\tilde{T}(a) - \tilde{T}(x)| \leq C\|T\||a|^{(m/2)-1}|x - a|, \quad \forall x \text{ in } U. \quad (1.9)$$

In this paper we will investigate some properties of the derivatives of the Berezin transform on $\mathcal{A}^2(\Omega, \mu)$. For z in Ω , if $A(z)$ is the rank one projection

$$A(z) := \langle \cdot, k_z \rangle k_z, \quad (1.10)$$

we have

$$\tilde{T}(z) = \text{tr}(TA(z)). \quad (1.11)$$

We first fix some notations. Let $\mathbb{N}_0^n$ denote the set of all n -tuples with components in the set $\mathbb{N}_0$ of all nonnegative integers. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, we let $|\beta| := \beta_1 + \dots + \beta_n$ denote the length of β and $\beta!$ stands for $\prod_{j=1}^n \beta_j!$. If $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}_0^n$ satisfies $\gamma_j \leq \beta_j$ for all $j = 1, \dots, n$, then we write $\gamma \leq \beta$. Our first main result is about the strong derivatives of $A(z)$.

Theorem 1.1. *Let Ω be a rotation invariant open set in $\mathbb{C}^n$ and μ a rotation invariant positive measure on Ω that satisfies (1.1) and has moments of every order. Moreover one assumes that, for any multi-index α and any compact set K of Ω , there exists $G_{\alpha,K} \in L^2(\Omega, \mu)$ such that*

$$|\overline{w}^\alpha \partial^\alpha K(z, w)| \leq G_{\alpha,K}(w), \quad \forall z \in K. \quad (1.12)$$

Then for all β, γ multi-indices, the operators $\partial^\beta \overline{\partial}^\gamma A(z)$ and $\overline{\partial}^\beta \partial^\gamma A(z)$ are adjoint to each other; their rank is smaller or equal to the infimum of $\#\{\delta \in \mathbb{N}_0^n, \delta \leq \gamma\}$ and $\#\{\delta \in \mathbb{N}_0^n, \delta \leq \beta\}$. Moreover, if at least β or γ is different from 0, one has

$$\text{tr}[\partial^\beta \overline{\partial}^\gamma A(z)] = 0. \quad (1.13)$$

Our second main result generalizes the estimates of Engliš and Zhang for the strong derivatives of the Berezin transform on weighted Fock spaces.

Theorem 1.2. *For a bounded linear operator T on $\mathcal{A}^2(\mathbb{C}^n, \mu_m)$, the Berezin transform $\tilde{T}$ has derivatives of all orders. In addition to any multi-indices β, γ , there exist positive constants $c_{\beta,\gamma}$ and A , depending on β, γ , and n only such that*

$$|\partial^\beta \overline{\partial}^\gamma \tilde{T}(z)| \leq c_{\beta,\gamma} |z|^{((m/2)-1)(|\beta|+|\gamma|)} \quad \text{for } |z| > A. \quad (1.14)$$

2. Preliminaries

We recall some properties of the Bergman kernel K , when μ is a positive rotation invariant measure on Ω . The kernel K is given by

$$K(z, w) = \sum_{\beta \in \mathbb{N}_0^n} \frac{1}{c_\beta} z^\beta \overline{w}^\beta, \quad (2.1)$$

for z, w in Ω , with the usual convention that, for $z = (z_1, \dots, z_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, z^β stands for $z_1^{\beta_1} \cdots z_n^{\beta_n}$ and where

$$c_\beta := \int_{\Omega} x^\beta \overline{x}^\beta d\mu(x). \quad (2.2)$$

By Lemma 2.1 in [4], we know that

$$c_\beta = \frac{(n-1)! \beta! m_{|\beta|}}{(n-1+|\beta|)!}, \quad \text{where } m_k = \int_{\Omega} |z|^{2k} d\mu(z) \text{ for } k \in \mathbb{N}_0. \quad (2.3)$$

Thus we can write

$$K(z, \omega) = F(\langle z, \omega \rangle), \quad (2.4)$$

where

$$F(t) := \frac{1}{(n-1)!} \sum_{d \in \mathbb{N}_0} \frac{(n-1+d)!}{m_d} t^d \quad (2.5)$$

is a holomorphic function of one complex variable.

For a bounded operator T on $\mathcal{A}^2(\Omega, \mu)$, the Berezin transform can be written in the form

$$\tilde{T}(z) = \text{tr}(TA(z)), \quad \text{for } z \in \Omega, \quad (2.6)$$

where $A(z)$ is the rank one projection $A(z) = \langle \cdot, k_z \rangle k_z$.

Recall [3] that a mapping h from a domain in $\mathbb{C}^n$ into a Banach space possesses a strong holomorphic derivative $\partial h / \partial z_1(z)$ at a point $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ if

$$\lim_{t \rightarrow 0} \left\| \frac{h(z_1 + t, z_2, \dots, z_n) - h(z_1, \dots, z_n)}{t} - \frac{\partial h}{\partial z_1}(z_1, \dots, z_n) \right\| = 0; \quad (2.7)$$

similarly one can define the antiholomorphic derivative $\bar{\partial}$. Englis and Zhang showed that the mapping $z \mapsto A(z)$ has strong derivatives.

Lemma 2.1. *Let Ω be a domain in $\mathbb{C}^n$, μ a measure on Ω that satisfies (1.1). Then the function $z \mapsto k_z$ has strong derivatives of all orders.*

Lemma 2.2. *Under the same hypothesis, the trace-class-operator-valued function*

$$z \mapsto \langle \cdot, k_z \rangle k_z \quad (2.8)$$

from Ω into the space of trace-class operators on $\mathcal{A}^2(\Omega, \mu)$ has strong derivatives of all orders.

The mapping $z \mapsto A(z)$ and its derivatives can be expressed in terms of the function F . It is easy to see that, for z, u in Ω ,

$$A(z) \cdot f(u) = \left\langle f, F(\langle \cdot, z \rangle) \frac{F(\langle u, z \rangle)}{F(\langle z, z \rangle)} \right\rangle. \quad (2.9)$$

3. Proof of Theorem 1.1

We write, for $f \in \mathcal{A}^2(\Omega, \mu)$ and z, u in Ω ,

$$A(z) \cdot f(u) = \left[\int_{\Omega} f(\omega) F(\langle z, \omega \rangle) d\mu(\omega) \right] \frac{F(\langle u, z \rangle)}{F(\langle z, z \rangle)}. \quad (3.1)$$

Since it is possible to differentiate under the integral sign at any order with respect to z , for β in $\mathbb{N}_0^n$, we have

$$(\partial^\beta A(z) \cdot f)(u) = \int_{\Omega} f(\omega) \partial^\beta \left[\frac{F(\langle z, \omega \rangle)}{F(\langle z, z \rangle)} \right] d\mu(\omega) F(\langle u, z \rangle). \quad (3.2)$$

The Leibnitz rule leads to

$$(\partial^\beta A(z) \cdot f)(u) = \sum_{v \leq \beta} \binom{\beta}{v} \int_{\Omega} f(\omega) \bar{\omega}^v F^{(|v|)}(\langle z, \omega \rangle) \bar{z}^{\beta-v} \left(\frac{1}{F} \right)^{(|\beta-v|)}(\langle z, z \rangle) d\mu(\omega) F(\langle u, z \rangle). \quad (3.3)$$

Setting $\dot{K}_z^{(\beta)}(\omega) = F(\langle z, z \rangle)^{1/2} \sum_{v \leq \beta} \binom{\beta}{v} \omega^v \bar{z}^{\beta-v} (1/F)^{(|\beta-v|)}(\langle z, z \rangle) F^{(|v|)}(\langle \omega, z \rangle)$, we get

$$\partial^\beta A(z) \cdot f = \left\langle f, \dot{K}_z^{(\beta)} \right\rangle_{k_z} \quad (3.4)$$

and then

$$\bar{\partial}^\gamma \partial^\beta A(z) \cdot f = \sum_{\delta \leq \gamma} \langle f, L_{z,\delta} \rangle M_{z,\delta}, \quad (3.5)$$

where $L_{z,\delta}(\omega) = (\partial^\delta / \partial z^\delta)(F(\langle z, z \rangle)^{-1/2} \dot{K}_z^{(\beta)}(\omega))$ and $M_{z,\delta}(u) = \binom{\gamma}{\delta} u^{\gamma-\delta} F^{(|\gamma-\delta|)}(\langle u, z \rangle)$.

Notice that the rank of the operator $\bar{\partial}^\gamma \partial^\beta A(z)$ is smaller than or equal to $\#\{\delta \in \mathbb{N}_0^n, \delta \leq \gamma\}$.

Setting now $S := \bar{\partial}^\gamma \partial^\beta A(z)$, $L_\delta := L_{z,\delta}$, and $M_\delta := M_{z,\delta}$ for a fixed element z in Ω , we obtain, for f, g in $\mathcal{A}^2(\Omega, \mu)$,

$$\begin{aligned} \langle Sf, g \rangle &= \left\langle \sum_{\delta \leq \gamma} \langle f, L_\delta \rangle M_\delta, g \right\rangle \\ &= \left\langle f, \sum_{\delta \leq \gamma} \langle g, M_\delta \rangle L_\delta \right\rangle. \end{aligned} \quad (3.6)$$

Consequently,

$$S^* g = \sum_{\delta \leq \gamma} \langle g, M_\delta \rangle L_\delta, \quad (3.7)$$

that is,

$$\begin{aligned}
(S^*g)(u) &= \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} \int_{\Omega} g(\omega) \bar{\omega}^{\gamma-\delta} F^{(|\gamma-\delta|)}(\langle z, \omega \rangle) d\mu(\omega) \frac{\partial^\delta}{\partial z^\delta} \left(\frac{1}{\sqrt{F(\langle z, z \rangle)}} \dot{K}_z^{(\beta)}(u) \right) \\
&= \frac{\partial^\gamma}{\partial z^\gamma} \left[\int_{\Omega} g(\omega) F(\langle z, \omega \rangle) d\mu(\omega) \left(\frac{1}{\sqrt{F(\langle z, z \rangle)}} \dot{K}_z^{(\beta)}(u) \right) \right] \\
&= \frac{\partial^\gamma}{\partial z^\gamma} [\langle g, k_z \rangle \dot{K}_z^{(\beta)}(u)] \\
&= \frac{\partial^\gamma}{\partial z^\gamma} \frac{\partial^\beta}{\partial \bar{z}^\beta} [\langle g, k_z \rangle k_z(u)] \\
&= (\partial^\gamma \partial^{\bar{\beta}} A(z) \cdot g)(u).
\end{aligned} \tag{3.8}$$

It follows that the rank of the operator $\partial^{\bar{\gamma}} \partial^\beta A(z)$ is also smaller than or equal to $\#\{\delta \in \mathbb{N}_0^n, \delta \leq \beta\}$. Thus

$$(\partial^{\bar{\gamma}} \partial^\beta A(z))^* = \partial^\gamma \partial^{\bar{\beta}} A(z), \quad \text{for } z \in \Omega. \tag{3.9}$$

Now let β and γ be multi-indices. Due to Lemma 2.2 and the continuity of the linear form $\text{Tr}(X)$, we have

$$\text{Tr}(\partial^{\bar{\gamma}} \partial^\beta A(z)) = \partial^{\bar{\gamma}} \partial^\beta \text{Tr} A(z) = \partial^{\bar{\gamma}} \partial^\beta 1 = \delta_{\beta 0} \delta_{\gamma 0}. \tag{3.10}$$

4. Proof of Theorem 1.2

When $\Omega = \mathbb{C}^n$ and $d\mu_m(z) = e^{-|z|^m} dv(z)$, for brevity we set $\mathcal{A}^2(\mu_m)$ for $\mathcal{A}^2(\mathbb{C}^n, \mu_m)$. The Bergman kernel K_m of $\mathcal{A}^2(\mu_m)$ can be expressed in terms of the Mittag-Leffler function (see [5]). Putting $\alpha = 2/m$, we have

$$K_m(z, \omega) = F(\langle z, \omega \rangle), \quad \text{where } F(t) = C E_{\alpha, \alpha}^{(n-1)}(t), \quad C = \frac{m}{(n-1)!}, \tag{4.1}$$

$E_{\alpha, \alpha}^{(n-1)}$ being the $n-1$ -th derivative of the Mittag-Leffler function $E_{\alpha, \alpha}$, entire on $\mathbb{C}$ and defined by

$$E_{\alpha, \alpha}(t) := \sum_{d=0}^{+\infty} \frac{t^d}{\Gamma(\alpha d + \alpha)}, \quad \text{for } t \in \mathbb{C}. \tag{4.2}$$

In what follows, we will use some asymptotic properties of this function (see [6]) near the positive real axis.

Lemma 4.1. *Let p a nonnegative integer. There exists a constant $\epsilon > 0$, such that, for any complex number z with $|\arg z| < \epsilon$, one has*

$$F^{(p)}(z) = F_0^{(p)}(z) + \epsilon_p(z), \quad (4.3)$$

where

$$F_0(z) = Ce^{Z_0(z)} Z_0(z)^{-n\alpha} P_n(Z_0(z)) \quad (4.4)$$

and $Z_s(z) = |z|^{1/\alpha} \exp[(i/\alpha)(\arg z + 2\pi s)]$, for s integer.

The polynomials $(P_p)_{p \in \mathbb{N}}$ are defined by $(d^p/dz^p)[e^{u^{1/\alpha}}] = (e^{u^{1/\alpha}}/u^p)P_p(u^{1/\alpha})$, P_p is of degree p , and when $|z| \rightarrow +\infty$, $\epsilon_p^{(r)}(z) = o(F_0^{(q)}(Z_0))$ for any nonnegative integers r and q .

We also need asymptotic estimates for some auxiliaries functions.

Lemma 4.2. *Let p and d be nonnegative integers. When t is real and tends to $+\infty$,*

$$\begin{aligned} \left(\frac{1}{F}\right)^{(d)}(t) &\sim \left(\frac{1}{F_0}\right)^{(d)}(t), \\ \left(\frac{F^{(p)}}{F}\right)^{(d)}(t) &\sim \left(\frac{F_0^{(p)}}{F_0}\right)^{(d)}(t). \end{aligned} \quad (4.5)$$

For T a bounded linear operator on $\mathcal{A}^2(\mu_m)$, the Berezin transform is given by (see [3])

$$\tilde{T}(z) = \text{tr}(TA(z)). \quad (4.6)$$

Let β and γ be some multi-indices. By differentiation, Lemma 2.2 gives

$$\partial^\beta \bar{\partial}^\gamma \tilde{T}(z) = \text{tr}\left(T \partial^\beta \bar{\partial}^\gamma A(z)\right). \quad (4.7)$$

Due to the continuity of the bilinear form $(X, Y) \mapsto \text{tr}(XY)$, we have

$$|\text{tr}(XY)| \leq \|X\| \|Y\|_{\text{tr}}, \quad (4.8)$$

for all bounded operators X and trace class operators Y . Therefore, we obtain

$$\left| \partial^\beta \bar{\partial}^\gamma \tilde{T}(z) \right| \leq \|T\| \left\| \partial^\beta \bar{\partial}^\gamma A(z) \right\|_{\text{tr}}. \quad (4.9)$$

Like in Section 3, we set $S := \partial^\beta \bar{\partial}^\gamma A(z)$. Next we shall estimate $\text{tr}((SS^*)^{1/2})$.

Fix f in $\mathcal{A}^2(\mu_m)$ and z in $\mathbb{C}^n$. We recall that

$$Sf = \sum_{\delta \leq \gamma} \langle f, L_{z,\delta} \rangle M_{z,\delta}, \quad (4.10)$$

where $L_{z,\delta}(\omega) = (\partial^\delta / \partial z^\delta)(F(\langle z, z \rangle))^{-1/2} K_z^{(\beta)}(\omega)$ and $M_{z,\delta}(u) = \binom{\gamma}{\delta} u^{\gamma-\delta} F(|\gamma-\delta|)(\langle u, z \rangle)$. Then

$$SS^*f = \sum_{\delta \leq \gamma} \left[\sum_{\delta' \leq \gamma} \langle f, M_{z,\delta'} \rangle \langle L_{z,\delta'}, L_{z,\delta} \rangle \right] M_{z,\delta}. \quad (4.11)$$

To estimate the eigenvalues of the finite rank positive operator SS^* , we compute its trace:

$$\text{tr}(SS^*) = \sum_{\delta \leq \gamma} a_\delta, \quad \text{with } a_\delta = \sum_{\delta' \leq \gamma} \langle M_{z,\delta}, M_{z,\delta'} \rangle \langle L_{z,\delta'}, L_{z,\delta} \rangle. \quad (4.12)$$

With the operator being of finite rank, we observe that $\text{Tr}(SS^*) = \text{Tr}(S^*S)$. We now compute and then estimate each diagonal term a_δ . Fix some multi-indices δ and δ' such that $\delta \leq \gamma$ and $\delta' \leq \gamma$. To simplify the notation we set L_δ and M_δ instead of $L_{z,\delta}$ and $M_{z,\delta}$. We obtain

$$\begin{aligned} \langle M_\delta, M_{\delta'} \rangle &= C^2 \binom{\gamma}{\delta} \binom{\gamma}{\delta'} \int_{\mathbb{C}^n} \omega^{\gamma-\delta} F(|\gamma-\delta|)(\langle \omega, z \rangle) \overline{\omega}^{\gamma-\delta'} F(|\gamma-\delta'|)(\langle z, \omega \rangle) d\mu(\omega) \\ &= C^2 \binom{\gamma}{\delta} \binom{\gamma}{\delta'} \frac{1}{C} \frac{\partial^{\gamma-\delta}}{\partial \bar{z}^{\gamma-\delta}} \frac{\partial^{\gamma-\delta'}}{\partial z^{\gamma-\delta'}} F(\langle z, z \rangle) \\ &= C \binom{\gamma}{\delta} \binom{\gamma}{\delta'} \frac{\partial^{\gamma-\delta}}{\partial \bar{z}^{\gamma-\delta}} \left[\bar{z}^{\gamma-\delta'} F(|\gamma-\delta'|)(\langle z, z \rangle) \right] \\ &= C \binom{\gamma}{\delta} \binom{\gamma}{\delta'} \left\{ \sum_{\kappa \leq \gamma-\delta, \kappa \leq \gamma-\delta'} \binom{\gamma-\delta}{\kappa} \binom{\gamma-\delta'}{\kappa} \kappa! \right. \\ &\quad \left. \times \bar{z}^{\gamma-\delta'-\kappa} z^{\gamma-\delta-\kappa} F(|\gamma-\delta'+\gamma-\delta-\kappa|)(\langle z, z \rangle) \right\}. \end{aligned} \quad (4.13)$$

For the second term, since

$$\begin{aligned} L_\delta(\omega) &= \frac{\partial^\delta}{\partial z^\delta} \left\{ \sum_{\nu \leq \beta} \binom{\beta}{\nu} \omega^{\beta-\nu} F(|\beta-\nu|)(\langle \omega, z \rangle) z^\nu \left(\frac{1}{F} \right)^{(|\nu|)}(\langle z, z \rangle) \right\} \\ &= \sum_{\nu \leq \beta} \binom{\beta}{\nu} \omega^{\beta-\nu} F(|\beta-\nu|)(\langle \omega, z \rangle) \frac{\partial^\delta}{\partial z^\delta} \left[z^\nu \left(\frac{1}{F} \right)^{(|\nu|)}(\langle z, z \rangle) \right], \end{aligned} \quad (4.14)$$

we get

$$\begin{aligned} \langle L_{\delta'}, L_{\delta} \rangle &= \sum_{\nu, \nu' \leq \beta} \binom{\beta}{\nu} \binom{\beta}{\nu'} \frac{\partial^{\delta'}}{\partial z^{\delta'}} \left[z^{\nu'} \left(\frac{1}{F} \right)^{(|\nu'|)} (\langle z, z \rangle) \right] \frac{\partial^{\delta}}{\partial \bar{z}^{\delta}} \\ &\quad \times \left[\bar{z}^{\nu} \left(\frac{1}{F} \right)^{(|\nu|)} (\langle z, z \rangle) \right] \frac{\partial^{\beta-\nu'}}{\partial \bar{z}^{\beta-\nu'}} \frac{\partial^{\beta-\nu}}{\partial z^{\beta-\nu}} \left[\frac{F(\langle z, z \rangle)}{C} \right], \end{aligned} \quad (4.15)$$

$$\begin{aligned} \langle L_{\delta'}, L_{\delta} \rangle &= \frac{1}{C} \sum_{\nu, \nu' \leq \beta} \binom{\beta}{\nu} \binom{\beta}{\nu'} \frac{\partial^{\delta'}}{\partial z^{\delta'}} \left[z^{\beta-\nu'} \left(\frac{1}{F} \right)^{(|\beta-\nu'|)} (\langle z, z \rangle) \right] \frac{\partial^{\delta}}{\partial \bar{z}^{\delta}} \\ &\quad \times \left[\bar{z}^{\beta-\nu} \left(\frac{1}{F} \right)^{(|\beta-\nu|)} (\langle z, z \rangle) \right] \frac{\partial^{\nu'}}{\partial \bar{z}^{\nu'}} \left[\bar{z}^{\nu} F^{(|\nu|)} (\langle z, z \rangle) \right]. \end{aligned} \quad (4.16)$$

By the Leibnitz rule, we see that

$$\begin{aligned} \frac{\partial^{\delta}}{\partial \bar{z}^{\delta}} \left[\bar{z}^{\beta-\nu} \left(\frac{1}{F} \right)^{(|\beta-\nu|)} (\langle z, z \rangle) \right] &= \sum_{\lambda \leq \delta, \lambda \leq \beta-\nu} \binom{\delta}{\lambda} \binom{\beta-\nu}{\lambda} \lambda! \bar{z}^{\beta-\nu-\lambda} z^{\delta-\lambda} \left(\frac{1}{F} \right)^{(|\beta-\nu+\delta-\lambda|)} (\langle z, z \rangle), \\ \frac{\partial^{\delta'}}{\partial z^{\delta'}} \left[z^{\beta-\nu'} \left(\frac{1}{F} \right)^{(|\beta-\nu'|)} (\langle z, z \rangle) \right] &= \sum_{\lambda' \leq \delta', \lambda' \leq \beta-\nu'} \binom{\delta'}{\lambda'} \binom{\beta-\nu'}{\lambda'} \lambda'! z^{\beta-\nu'-\lambda'} \bar{z}^{\delta'-\lambda'} \\ &\quad \times \left(\frac{1}{F} \right)^{(|\beta-\nu'+\delta'-\lambda'|)} (\langle z, z \rangle), \\ \frac{\partial^{\nu'}}{\partial \bar{z}^{\nu'}} \left[\bar{z}^{\nu} F^{(|\nu|)} (\langle z, z \rangle) \right] &= \sum_{\sigma \leq \nu, \sigma \leq \nu'} \binom{\nu'}{\sigma} \binom{\nu}{\sigma} \sigma! \bar{z}^{\nu-\sigma} z^{\nu'-\sigma} F^{(|\nu+\nu'-\sigma|)} (\langle z, z \rangle). \end{aligned} \quad (4.17)$$

Then

$$\langle M_{\delta}, M_{\delta'} \rangle \langle L_{\delta'}, L_{\delta} \rangle = \binom{\gamma}{\delta} \binom{\gamma}{\delta'} \sum_{\nu \leq \beta, \nu' \leq \beta} \binom{\beta}{\nu} \binom{\beta}{\nu'} \sum_{\substack{\lambda \leq \delta, \beta-\nu \\ \lambda' \leq \delta', \beta-\nu' \\ \sigma \leq \nu, \nu' \\ \kappa \leq \gamma-\delta, \gamma-\delta'}} \binom{\nu'}{\sigma} \binom{\nu}{\sigma} \sigma! \binom{\delta'}{\lambda'}$$

$$\begin{aligned}
& \times \binom{\beta - \nu'}{\lambda'} \lambda'! \binom{\delta}{\lambda} \binom{\beta - \nu}{\lambda} \lambda! \binom{\gamma - \delta}{\kappa} \\
& \times \binom{\gamma - \delta'}{\kappa} \kappa! z^{\beta + \gamma - \lambda - \lambda' - \sigma - \kappa} \bar{z}^{\beta + \gamma - \lambda - \lambda' - \sigma - \kappa} \\
& \times F^{(|\gamma - \delta + \gamma - \delta' - \kappa|)}(\langle z, z \rangle) F^{(|\nu + \nu' - \sigma|)}(\langle z, z \rangle) \left(\frac{1}{F} \right)^{(|\beta - \nu' + \delta' - \lambda'|)} \\
& \times (\langle z, z \rangle) \left(\frac{1}{F} \right)^{(|\beta - \nu + \delta - \lambda|)} (\langle z, z \rangle).
\end{aligned} \tag{4.18}$$

Thus

$$\begin{aligned}
a_\delta &= \binom{\gamma}{\delta} \sum_{\delta' \leq \gamma} \binom{\gamma}{\delta'} \sum_{\substack{\nu \leq \beta \\ \nu' \leq \beta}} \binom{\beta}{\nu} \binom{\beta}{\nu'} \sum_{\substack{\lambda \leq \delta, \beta - \nu \\ \lambda' \leq \delta', \beta - \nu' \\ \beta - \nu, \beta - \nu' \leq \rho \\ \kappa \leq \gamma - \delta, \gamma - \delta'}} \binom{\nu'}{\beta - \rho} \binom{\nu}{\beta - \rho} (\beta - \rho)! \binom{\delta'}{\lambda'} \binom{\beta - \nu'}{\lambda'} \\
& \times \lambda'! \binom{\delta}{\lambda} \binom{\beta - \nu}{\lambda} \lambda! \binom{\gamma - \delta}{\kappa} \binom{\gamma - \delta'}{\kappa} \kappa! z^{\gamma - \lambda - \lambda' - \kappa + \rho} \bar{z}^{\gamma - \lambda - \lambda' - \kappa + \rho} \\
& \times F^{(|\gamma - \delta + \gamma - \delta' - \kappa|)}(\langle z, z \rangle) F^{(|\nu + \nu' - \beta + \rho|)}(\langle z, z \rangle) \left(\frac{1}{F} \right)^{(|\beta - \nu' + \delta' - \lambda'|)} (\langle z, z \rangle) \\
& \times \left(\frac{1}{F} \right)^{(|\beta - \nu + \delta - \lambda|)} (\langle z, z \rangle),
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
a_\delta &= \binom{\gamma}{\delta} \sum_{\delta' \leq \gamma} \binom{\gamma}{\delta'} \sum_{\kappa} \sum_{\lambda, \lambda'} \frac{\beta!}{(\beta - \lambda)!} \frac{\beta!}{(\beta - \lambda')!} \sum_{\nu, \nu'} \binom{\beta - \lambda}{\rho - \lambda} \binom{\beta - \lambda'}{\rho - \lambda'} (\beta - \rho)! \binom{\rho - \lambda}{\beta - \nu - \lambda} \\
& \times \binom{\rho - \lambda'}{\beta - \nu' - \lambda'} \binom{\delta}{\lambda} \binom{\delta'}{\lambda'} \binom{\gamma - \delta}{\kappa} \binom{\gamma - \delta'}{\kappa} \kappa! z^{\gamma - \lambda - \lambda' - \kappa + \rho} \bar{z}^{\gamma - \lambda - \lambda' - \kappa + \rho} \\
& \times F^{(|\gamma - \delta + \gamma - \delta' - \kappa|)}(\langle z, z \rangle) F^{(|\nu + \nu' - \beta + \rho|)}(\langle z, z \rangle) \\
& \times \left(\frac{1}{F} \right)^{(|\beta - \nu' + \delta' - \lambda'|)} (\langle z, z \rangle) \left(\frac{1}{F} \right)^{(|\beta - \nu + \delta - \lambda|)} (\langle z, z \rangle),
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
a_\delta &= \binom{\gamma}{\delta} \sum_{\delta' \leq \gamma} \binom{\gamma}{\delta'} \sum_{\kappa} \binom{\gamma - \delta}{\kappa} \binom{\gamma - \delta'}{\kappa} \kappa! \sum_{\nu, \nu', \lambda, \lambda', \rho} \frac{\beta!}{(\beta - \lambda)!} \frac{\beta!}{(\beta - \lambda')!} \binom{\beta - \lambda}{\rho - \lambda} \binom{\beta - \lambda'}{\rho - \lambda'} \\
&\quad \times (\beta - \rho)! \binom{\delta}{\lambda} \binom{\delta'}{\lambda'} \binom{\rho - \lambda}{\beta - \nu - \lambda} \binom{\rho - \lambda'}{\beta - \nu' - \lambda'} z^{\gamma - \lambda - \lambda' - \kappa + \rho} \bar{z}^{\gamma - \lambda - \lambda' - \kappa + \rho} \\
&\quad \times F^{(|\gamma - \delta + \gamma - \delta' - \kappa|)}(\langle z, z \rangle) F^{(|\nu + \nu' - \beta + \rho|)}(\langle z, z \rangle) \\
&\quad \times \left(\frac{1}{F} \right)^{(|\beta - \nu' + \delta' - \lambda'|)}(\langle z, z \rangle) \left(\frac{1}{F} \right)^{(|\beta - \nu + \delta - \lambda|)}(\langle z, z \rangle).
\end{aligned} \tag{4.21}$$

Setting $\sigma = \beta - \nu - \lambda$ and $\sigma' = \beta - \nu' - \lambda'$, we have

$$\begin{aligned}
a_\delta &= \binom{\gamma}{\delta} \sum_{\delta' \leq \gamma} \binom{\gamma}{\delta'} \sum_{\kappa} \binom{\gamma - \delta}{\kappa} \binom{\gamma - \delta'}{\kappa} \kappa! \sum_{\lambda \leq \delta, \lambda' \leq \delta'} \frac{\beta!}{(\beta - \lambda)!} \frac{\beta!}{(\beta - \lambda')!} \binom{\delta}{\lambda} \binom{\delta'}{\lambda'} \\
&\quad \times \sum_{\substack{\lambda \leq \rho, \lambda' \leq \rho}} (\beta - \rho)! \binom{\beta - \lambda}{\rho - \lambda} \binom{\beta - \lambda'}{\rho - \lambda'} \sum_{\substack{\sigma \leq \rho - \lambda, \\ \sigma' \leq \rho - \lambda'}} \binom{\rho - \lambda}{\sigma} \binom{\rho - \lambda'}{\sigma'} \\
&\quad \times F^{(|\gamma - \delta + \gamma - \delta' - \kappa|)}(\langle z, z \rangle) F^{(|\beta + \rho - \lambda - \lambda' - \sigma - \sigma'|)}(\langle z, z \rangle) \\
&\quad \times \left(\frac{1}{F} \right)^{(|\sigma + \delta|)}(\langle z, z \rangle) \left(\frac{1}{F} \right)^{(|\sigma' + \delta'|)}(\langle z, z \rangle) z^{\gamma - \lambda - \lambda' - \kappa + \rho} \bar{z}^{\gamma - \lambda - \lambda' - \kappa + \rho},
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
a_\delta &= \binom{\gamma}{\delta} \sum_{\delta' \leq \gamma} \binom{\gamma}{\delta'} \sum_{\kappa} \binom{\gamma - \delta}{\kappa} \binom{\gamma - \delta'}{\kappa} \kappa! \sum_{\lambda \leq \delta, \lambda' \leq \delta'} \frac{\beta!}{(\beta - \lambda)!} \frac{\beta!}{(\beta - \lambda')!} \binom{\delta}{\lambda} \binom{\delta'}{\lambda'} \\
&\quad \times \sum_{\substack{\lambda, \lambda' \leq \rho \leq \beta}} (\beta - \rho)! \binom{\beta - \lambda}{\rho - \lambda} \binom{\beta - \lambda'}{\rho - \lambda'} z^{\gamma - \lambda - \lambda' - \kappa + \rho} \bar{z}^{\gamma - \lambda - \lambda' - \kappa + \rho} S_\rho(\langle z, z \rangle),
\end{aligned} \tag{4.23}$$

where

$$S_\rho = \sum_{\substack{\sigma \leq \rho - \lambda \\ \sigma' \leq \rho - \lambda'}} \binom{\rho - \lambda}{\sigma} \binom{\rho - \lambda'}{\sigma'} F^{(|\gamma - \delta + \gamma - \delta' - \kappa|)} F^{(|\beta + \rho - \lambda - \lambda' - \sigma - \sigma'|)} \left(\frac{1}{F} \right)^{(|\sigma + \delta|)} \left(\frac{1}{F} \right)^{(|\sigma' + \delta'|)}. \tag{4.24}$$

Setting $\check{\tau} = (\tau_1, \dots, \tau_{n-1})$ for a multiindex $\tau = (\tau_1, \dots, \tau_n)$, we see that

$$S_\rho = F^{(|\gamma-\delta+\gamma-\delta'-\kappa|)} \sum_{\substack{\check{\sigma} \leq \check{\rho}-\check{\lambda} \\ \check{\sigma}' \leq \check{\rho}-\check{\lambda}'}} \binom{\check{\rho}-\check{\lambda}}{\check{\sigma}} \binom{\check{\rho}-\check{\lambda}'}{\check{\sigma}'} \sum_{\sigma'_n \leq \rho_n - \lambda'_n} \binom{\rho_n - \lambda'_n}{\sigma'_n} \left(\frac{1}{F}\right)^{(|\sigma'+\delta'|)} \quad (4.25)$$

$$\times \sum_{\sigma_n \leq \rho_n - \lambda_n} \binom{\rho_n - \lambda_n}{\sigma_n} \left(\left(\frac{1}{F}\right)^{(|\delta|+\sigma_1+\dots+\sigma_{n-1})}\right)^{(\sigma_n)} \left((F)^{(|\beta-\lambda'-\sigma'|+|\check{\rho}-\check{\lambda}-\check{\sigma}|)}\right)^{(\rho_n-\lambda_n-\sigma_n)}.$$

Using the Leibnitz rule

$$S_\rho = F^{(|\gamma-\delta+\gamma-\delta'-\kappa|)} \sum_{\substack{\check{\sigma} \leq \check{\rho}-\check{\lambda} \\ \check{\sigma}' \leq \check{\rho}-\check{\lambda}'}} \binom{\check{\rho}-\check{\lambda}}{\check{\sigma}} \binom{\check{\rho}-\check{\lambda}'}{\check{\sigma}'} \sum_{\sigma'_n \leq \rho_n - \lambda'_n} \binom{\rho_n - \lambda'_n}{\sigma'_n} \left(\frac{1}{F}\right)^{(|\sigma'+\delta'|)} \quad (4.26)$$

$$\times \left[\left(\frac{1}{F}\right)^{(|\delta|+\sigma_1+\dots+\sigma_{n-1})} F^{(|\beta-\lambda'-\sigma'|+|\check{\rho}-\check{\lambda}-\check{\sigma}|)} \right]^{(\rho_n-\lambda_n)}.$$

An induction process shows that

$$S_\rho = F^{(|\gamma-\delta+\gamma-\delta'-\kappa|)} \sum_{\sigma' \leq \rho - \lambda'} \binom{\rho - \lambda'}{\sigma'} \left(\frac{1}{F}\right)^{(|\sigma'+\delta'|)} \left[F^{(|\beta-\lambda'-\sigma'|)} \left(\frac{1}{F}\right)^{(|\delta|)} \right]^{(|\rho-\lambda|)}. \quad (4.27)$$

Therefore

$$a_\delta = \binom{\gamma}{\delta} \sum_{\delta' \leq \gamma} \binom{\gamma}{\delta'} \sum_{\kappa} \binom{\gamma-\delta}{\kappa} \binom{\gamma-\delta'}{\kappa} \kappa! \sum_{\lambda \leq \delta, \lambda' \leq \delta'} \frac{\beta!}{(\beta-\lambda)!} \frac{\beta!}{(\beta-\lambda')!} \binom{\delta}{\lambda} \binom{\delta'}{\lambda'} \quad (4.28)$$

$$\times \sum_{\lambda, \lambda' \leq \rho \leq \beta} (\beta-\rho)! \binom{\beta-\lambda}{\rho-\lambda} \binom{\beta-\lambda'}{\rho-\lambda'} F^{(|\gamma-\delta+\gamma-\delta'-\kappa|)} (\langle z, z \rangle) z^{\gamma-\lambda-\lambda'-\kappa+\rho} \bar{z}^{\gamma-\lambda-\lambda'-\kappa+\rho}$$

$$\times \sum_{\sigma' \leq \rho - \lambda'} \binom{\rho - \lambda'}{\sigma'} \left(\frac{1}{F}\right)^{(|\sigma'+\delta'|)} (\langle z, z \rangle) \left[F^{(|\beta-\lambda'-\sigma'|)} \left(\frac{1}{F}\right)^{(|\delta|)} \right]^{(|\rho-\lambda|)} (\langle z, z \rangle).$$

Now let $\tau = \gamma - \lambda' - \kappa$. It follows that

$$\begin{aligned}
 a_\delta &= \binom{\gamma}{\delta} \sum_{\rho \leq \beta} \sum_{\tau \leq \gamma} \sum_{\lambda, \lambda' \leq \rho} \sum_{\lambda' \leq \delta' \leq \tau + \lambda'} \binom{\gamma}{\delta'} \binom{\gamma - \delta}{\gamma - \lambda' - \tau} \binom{\gamma - \delta'}{\gamma - \lambda' - \tau} (\gamma - \lambda' - \tau)! \frac{\beta!}{(\beta - \lambda)!} \frac{\beta!}{(\beta - \lambda')!} \\
 &\times \binom{\delta}{\lambda} \binom{\delta'}{\lambda'} (\beta - \rho)! \binom{\beta - \lambda}{\rho - \lambda} \binom{\beta - \lambda'}{\rho - \lambda'} F^{(|\gamma - \delta - \delta' + \lambda' + \tau|)}(\langle z, z \rangle) \sum_{\sigma' \leq \rho - \lambda'} \binom{\rho - \lambda'}{\sigma'} \\
 &\times \left(\frac{1}{F} \right)^{(|\sigma' + \delta'|)}(\langle z, z \rangle) \left[F^{(|\beta - \lambda' - \sigma'|)} \left(\frac{1}{F} \right)^{(|\delta|)} \right]^{(|\rho - \lambda|)}(\langle z, z \rangle) z^\tau \bar{z}^\tau z^{\rho - \lambda} \bar{z}^{\rho - \lambda}.
 \end{aligned} \tag{4.29}$$

On the other hand

$$\begin{aligned}
 \binom{\gamma}{\delta'} \binom{\gamma - \delta'}{\gamma - \lambda' - \tau} (\gamma - \lambda' - \tau)! \binom{\delta'}{\lambda'} &= \frac{\gamma!}{(-\delta' + \lambda' + \tau)! \lambda'! (\delta' - \lambda')!} \\
 &= \frac{\gamma!}{\tau! \lambda'!} \binom{\tau}{\delta' - \lambda'}.
 \end{aligned} \tag{4.30}$$

Thus, if we set $\sigma = \delta' - \lambda'$, then

$$\begin{aligned}
 a_\delta &= \binom{\gamma}{\delta} \sum_{\rho \leq \beta} \sum_{\tau \leq \gamma} \sum_{\lambda, \lambda' \leq \rho} \sum_{\sigma \leq \tau} \binom{\rho - \lambda'}{\sigma'} (\beta - \rho)! \binom{\beta - \lambda}{\rho - \lambda} \binom{\beta - \lambda'}{\rho - \lambda'} \binom{\delta}{\lambda} \gamma! z^{\tau + \rho - \lambda} \bar{z}^{\tau + \rho - \lambda} \\
 &\times \binom{\gamma - \delta}{\gamma - \lambda' - \tau} \frac{1}{\tau! \lambda'!} \binom{\tau}{\sigma} F^{(|\gamma - \delta + \tau - \sigma|)}(\langle z, z \rangle) \\
 &\times \left(\frac{1}{F} \right)^{(|\sigma + \sigma' + \lambda'|)}(\langle z, z \rangle) \left[F^{(|\beta - \lambda' - \sigma'|)} \left(\frac{1}{F} \right)^{(|\delta|)} \right]^{(|\rho - \lambda|)}(\langle z, z \rangle).
 \end{aligned} \tag{4.31}$$

Using the identity

$$\sum_{\sigma \leq \tau} \binom{\tau}{\sigma} \left(F^{(|\gamma - \delta|)} \right)^{(|\tau - \sigma|)} \left(\left(\frac{1}{F} \right)^{(|\sigma' + \lambda'|)} \right)^{(|\sigma|)} = \left[\left(\frac{1}{F} \right)^{(|\gamma - \delta|)} F^{(|\sigma' + \lambda'|)} \right]^{(|\tau|)}, \tag{4.32}$$

we see that

$$\begin{aligned}
 a_\delta &= \binom{\gamma}{\delta} \gamma! \sum_{\substack{\rho \leq \beta \\ \tau \leq \gamma}} \sum_{\substack{\lambda, \lambda' \leq \rho \\ \sigma' \leq \rho - \lambda'}} \binom{\rho - \lambda'}{\sigma'} (\beta - \rho)! \binom{\beta - \lambda}{\rho - \lambda} \binom{\beta - \lambda'}{\rho - \lambda'} \binom{\delta}{\lambda} \binom{\gamma - \delta}{\gamma - \lambda' - \tau} \frac{1}{\tau! \lambda'!} z^{\tau + \rho - \lambda} \bar{z}^{\tau + \rho - \lambda} \\
 &\quad \times \left[F^{(|\beta - \lambda' - \sigma'|)} \left(\frac{1}{F} \right)^{(|\delta|)} \right]^{(|\rho - \lambda|)} (\langle z, z \rangle) \left[\left(\frac{1}{F} \right)^{(|\gamma - \delta|)} F^{(|\sigma' + \lambda'|)} \right]^{(|\tau|)} (\langle z, z \rangle).
 \end{aligned} \tag{4.33}$$

Notice that, for p and q integers, we have, when t is real and $t \rightarrow +\infty$,

$$\begin{aligned}
 \frac{F^{(p)}(t)}{F(t)} &\sim t^{((1/\alpha)-1)p}, \\
 \left(\frac{1}{F} \right)^{(q)}(t) &\sim \frac{t^{((1/\alpha)-1)p}}{F(t)}.
 \end{aligned} \tag{4.34}$$

Hence by Lemmas 4.1 and 4.2 we have the following estimates:

$$\begin{aligned}
 \left| \left[F^{(|\beta - \lambda' - \sigma'|)} \left(\frac{1}{F} \right)^{(|\delta|)} \right]^{(|\rho - \lambda|)}(t) \right| &\leq C t^{((1/\alpha)-1)(|\beta + \delta - \lambda' - \sigma'| - |\rho - \lambda|)}, \\
 \left| \left[\left(\frac{1}{F} \right)^{(|\gamma - \delta|)} F^{(|\sigma' + \lambda'|)} \right]^{(|\tau|)}(t) \right| &\leq C t^{((1/\alpha)-1)(|\gamma - \delta + \sigma' + \lambda'| - |\tau|)}
 \end{aligned} \tag{4.35}$$

for some constant C when the real t tends to $+\infty$. Thus for any multiindex $\delta \leq \gamma$, there exists a constant C' such that $a_\delta \leq C' |z|^{2((1/\alpha)-1)(|\beta + \gamma|)}$. Then taking the sum over δ we get

$$\text{tr}(SS^*) \leq C'' |z|^{2((1/\alpha)-1)(|\beta + \gamma|)}. \tag{4.36}$$

Since $\text{Tr}(SS^*) = \text{Tr}(S^*S)$ and the operator S^*S is positive, its eigenvalues are bounded above by $C'' |z|^{2((1/\alpha)-1)(|\beta + \gamma|)}$. Thus the eigenvalues of $(SS^*)^{1/2}$ are bounded above by $C''' |z|^{((1/\alpha)-1)(|\beta + \gamma|)}$. Since $(S^*S)^{1/2}$ is of finite rank, the proof is complete.

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