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Research Article

On Convex Total Bounded Sets in the Space of Measurable Functions

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We estimate the measure of nonconvex total boundedness in terms of simpler quantitative characteristics in the space of *measurable functions* L_0. A Fréchet-Smulian type compactness criterion for convexly totally bounded subsets of L_0 is established.

1. Introduction

In 1988, Idzik [1] proved that the answer to the well-known Schauder's problem [2, Problem 54]: *does every continuous map $f : K \to K$ defined on a convex compact subset of a Hausdorff topological linear space have a fixed point?* is affirmative if K is convexly totally bounded.

This notion was introduced by Idzik [1]: a subset K of a topological linear space X is said to be *convexly totally bounded* (*ctb* for short), if for every 0-neighborhood U there are $x_1, \ldots, x_n \in K$ and convex subsets $C_1, \ldots, C_n$ of U such that $K \subseteq \bigcup_{i=1}^{n}(x_i + C_i)$. If X is locally convex, every convex compact subset of X is ctb. This is not true, in general, if X is nonlocally convex (see [3–5]).

In 1993, De Pascale et al. [3] defined the measure of nonconvex total boundedness, modelled on Idzik's concept, that may be regarded as the analogue of the well-known notion of Hausdorff measure of noncompactness in nonlocally convex linear spaces. The above notions of ctb set and of nonconvex total boundedness are especially useful working in the setting of nonlocally convex topological linear spaces (see, e.g., [6–8]).

Let $(E, \|\cdot\|_E)$ be a normed space, Ω a nonempty set, and $\mathcal{P}(\Omega)$ the power set of Ω. The space L_0 is an F-normed linear space of E-valued functions defined on Ω which depends on an algebra $\mathcal{A}$ in $\mathcal{P}(\Omega)$ and a submeasure $\eta : \mathcal{P}(\Omega) \to [0, +\infty]$.

We observe that the space L_0 defined above is a generalization of the space of measurable functions introduced in [9, Chapter III], in order to develop the integration theory

with respect to finitely additive measures. Recall that given $M \subseteq L_0$, the Hausdorff measure of noncompactness $\gamma(M)$ of M is defined by

$$\gamma(M) := \inf\left\{\varepsilon > 0 : \text{there exist functions } f_1, \ldots, f_n \in L_0 \text{ such that } M \subseteq \bigcup_{i=1}^{n}(f_i + B_\varepsilon(L_0))\right\}, \tag{1.1}$$

where $B_\varepsilon(L_0) := \{f \in L_0 : \|f\|_0 \leq \varepsilon\}$. In [10, 11], to estimate $\gamma(M)$ are used two quantitative characteristics $\lambda(M)$ and $\omega(M)$ which measure, respectively, the degree of non equi-quasiboundedness and the degree of non equi-measurability of M.

The main purpose of this note is to estimate the measure of nonconvex total boundedness in L_0 and to characterize the convexly totally bounded subsets of L_0. At this end we introduce two quantitative characteristics $\lambda_c(M)$ and $\omega_c(M)$ involving convex sets, which measure the *degree of nonconvex equi-quasiboundedness* and the degree of nonconvex equi-measurability of M, respectively. Then we establish some inequalities between $\gamma_c(M)$, $\lambda_c(M)$, $\omega_c(M)$, and $\lambda(M)$ that give, as a special case, a Fréchet-Smulian type convex total boundedness criterion in the space L_0. This generalizes previous results of Trombetta [12]. Finally, we point out that it is not so clear if the Schauder's problem has been solved in its generality. In particular, the proof given by Cauty in [13] contains some unsolved gaps (see [14, 15], Mathscinet review of [16], and Zentralblatt Math review of [17]). However, the results of this paper are meant to be independent from the Schauder's problem.

2. Definitions and Preliminaries

For the remainder of this section we present some definitions and known results which will be needed throughout this paper.

We use the convention $\inf \emptyset := +\infty$. Moreover, for two sets A and B, we denote by B^A the set of all maps from A to B. We recall the definition of the space L_0 (see [11, 18]). Let $(E, \|\cdot\|_E)$ be a normed space, Ω a nonempty set, $\mathcal{A}$ an algebra in the power set $\mathbb{P}(\Omega)$ of Ω, and $\eta : \mathbb{P}(\Omega) \to [0, \infty]$ a submeasure. Then

$$\|f\|_0 := \inf\{a > 0 : \eta(\{x \in \Omega : \|f(x)\|_E \geq a\}) \leq a\} \tag{2.1}$$

defines a group pseudonorm on E^Ω, that is, $\|0\|_0 = 0$, $\|-f\|_0 = \|f\|_0$, $\|f+g\|_0 \leq \|f\|_0 + \|g\|_0$, for all $f, g \in E^\Omega$. Let $L_0 := L_0(\Omega, \mathcal{A}, E, \eta)$ be the closure of the linear space $S := \operatorname{span}\{y\chi_A : y \in E \text{ and } A \in \mathcal{A}\}$ of E-valued $\mathcal{A}$-simple functions in $(E^\Omega, \|\cdot\|_0)$, where χ_A is the characteristics function of A. Identification of functions $f, g \in E^\Omega$ for which $\|f-g\|_0 = 0$ turns $(L_0, \|\cdot\|_0)$ (the space of measurable functions) into an F-normed linear space in the sense of [19, page 38], that is, $\|f+g\|_0 \leq \|f\|_0 + \|g\|_0$, $\lim_{n\to\infty}\|(1/n)f\|_0 = 0$, $\|\lambda f\|_0 \leq \|f\|_0$ for $f, g \in L_0$ and $|\lambda| \leq 1$.

Let $M \subseteq L_0$. The measure of nonconvex total boundedness $\gamma_c(M)$ of M is defined by

$$\gamma_c(M) := \inf\Big\{\varepsilon > 0 : \text{there exist functions } f_1, \ldots, f_n \in L_0 \text{ and convex subsets } C_1, \ldots, C_n \text{ of } B_\varepsilon(L_0) \text{ such that } M \subseteq \bigcup_{i=1}^{n}(f_i + C_i)\Big\}. \tag{2.2}$$

Clearly M is ctb if and only if $\gamma_c(M) = 0$.

Let $A_1, \ldots, A_m \in \mathcal{A}$ be a partition of Ω. We set

$$S(A_1, \ldots, A_m) := \left\{s \in S : s = \sum_{i=1}^{m} y_i \chi_{A_i}, \text{ where } y_i \in E \text{ for } i = 1, \ldots, m\right\}. \tag{2.3}$$

In [11] the following two quantitative characteristic λ and ω are used to estimate γ in L_0:

$$\begin{aligned} \lambda(M) &:= \inf\Big\{\varepsilon > 0 : \text{there exists a finite subset } G \text{ of } E \text{ such that } M \subseteq \left(G^{\Omega} \cap S\right) + B_\varepsilon(L_0)\Big\}, \\ \omega(M) &:= \inf\{\varepsilon > 0 : \text{there exists a partition } A_1, \ldots, A_m \in \mathcal{A} \text{ of } \Omega \\ &\qquad \text{such that } M \subseteq S(A_1, \ldots, A_m) + B_\varepsilon(L_0)\}. \end{aligned} \tag{2.4}$$

The set M is called *equi-quasibounded* if $\lambda(M) = 0$ and *equi-measurable* if $\omega(M) = 0$.

Theorem 2.1 (see [11, Theorem 2.2.2]). *Let $M \subseteq L_0$. Then*

$$\max\{\lambda(M), \omega(M)\} \le \gamma(M) \le \lambda(M) + 2\omega(M). \tag{2.5}$$

In particular, M is totally bounded if and only if $\lambda(M) = \omega(M) = 0$.

Moreover, in [18], it is defined the following quantitative characteristic $\sigma(M)$ which is useful for the calculation of $\lambda(M)$:

$$\begin{aligned} \sigma(M) := \inf\{&\varepsilon > 0 : \text{there exists a finite subset } G \text{ of } E \text{ such that for all } f \in M \\ &\text{there is } D_f \subseteq \Omega \text{ with } \eta(D_f) \le \varepsilon \text{ and } f(\Omega \setminus D_f) \subseteq G + B_\varepsilon(E)\}, \end{aligned} \tag{2.6}$$

where $B_\varepsilon(E) := \{y \in E : \|y\|_E \le \varepsilon\}$.

The following result was established in [18, Proposition 2.1].

Proposition 2.2. *Let $M \subseteq L_0$. Then $\lambda(M) = \sigma(M)$.*

We omit the proof of the following proposition which is similar to the proof of Proposition 2.6 of [12].

Proposition 2.3. *Let $M \subseteq L_0$. Then*

$$\gamma_c(M) = \inf\Bigg\{\varepsilon > 0 : \textit{there exist functions } s_1, \ldots, s_n \in S \textit{ and convex subsets} \\ C_1, \ldots, C_n \textit{ of } B_\varepsilon(L_0) \textit{ such that } M \subseteq \bigcup_{i=1}^{n}(s_i + C_i)\Bigg\}. \tag{2.7}$$

3. Inequalities in the Space L_0

In order to estimate the measure of nonconvex total boundedness in L_0, we introduce the following two quantitative characteristics.

Definition 3.1. Let $M \subseteq L_0$. We define the following:

$$\lambda_c(M) := \inf\Bigg\{\varepsilon > 0 : \text{there exist a finite subset } G \text{ of } E \text{ and convex subsets} \\ C_1, \ldots, C_n \text{ of } B_\varepsilon(L_0) \text{ such that } M \subseteq \left(G^\Omega \cap S\right) + \bigcup_{i=1}^{n} C_i\Bigg\},$$

$$\omega_c(M) := \inf\Bigg\{\varepsilon > 0 : \text{there exist a partition } A_1, \ldots, A_m \in \mathcal{A} \text{ of } \Omega \text{ and convex subsets} \\ C_1, \ldots, C_n \text{ of } B_\varepsilon(L_0) \text{ such that } M \subseteq S(A_1, \ldots, A_m) + \bigcup_{i=1}^{n} C_i\Bigg\}. \tag{3.1}$$

We call M *convexly equi-quasibounded* if $\lambda_c(M) = 0$ and *convexly equi-measurable* if $\omega_c(M) = 0$.

We observe that if $E = \mathbb{R}$, then the quantitative characteristics λ_c and ω_c coincide with those introduced in [12].

We point out that the request of convexity plays a crucial role in the definition of the parameters λ_c and ω_c, whereas it was not involved in the definition of λ and ω. We illustrate this with the following example.

Example 3.2. Let $L_0 := L_0([0, +\infty[, \mathcal{A}, E, \eta)$, where $\mathcal{A}$ is the algebra of all Lebesgue-measurable subsets of the interval $[0, +\infty[$ and $\eta|_{\mathcal{A}}$ the Lebesgue measure. Let $I_1 := [0, 1[$, $I_n := [\sum_{k=1}^{n-1}(1/k), \sum_{k=1}^{n}(1/k)[$ for $n \geq 2$, and $M_n := \{y\chi_{I_n} : y \in E\}$ for $n \geq 1$. If $M := \bigcup_{n=1}^{\infty} M_n$, then $\lambda(M) = 1$, $\omega(M) = 0$, $\lambda_c(M) = \omega_c(M) = +\infty$.

Proof. It is easy to check that $\lambda(M) = 1$ and $\omega(M) = 0$. We are going to prove that $\lambda_c(M) = +\infty$. On the contrary, suppose that $\lambda_c(M) < \alpha < +\infty$ then there exist a finite set $G = \{z_1, \ldots, z_p\} \subseteq E$ and convex sets $C_1, \ldots, C_m \subseteq B_\alpha(L_0)$ such that

$$M \subseteq \left(G^\Omega \cap S\right) + \bigcup_{j=1}^{m} C_j. \tag{3.2}$$

Set $c := \max\{\|z_l\|_E : l = 1, \ldots, p\}$. We have $\|s\|_0 \le c$ for all $s \in G^\Omega \cap S$. Fix $\overline{\alpha} > c + \alpha$, a natural number $\overline{n}$ such that $\sum_{n=1}^{\overline{n}}(1/n) > m\overline{\alpha}$, and $\overline{y} \in E$ with $\|\overline{y}\|_E > \overline{n} \cdot \overline{\alpha}$.

Put

$$S_{\overline{y}} := \left\{f = \overline{y}\chi_{I_n} : n = 1, 2, \ldots, \overline{n}\right\},$$
$$S_{\overline{y}}^j := \left\{f \in S_{\overline{y}} : \text{there is } s_f \in G^\Omega \cap S \text{ such that } f - s_f \in C_j\right\} \tag{3.3}$$

for $j = 1, \ldots, m$.

Then it is easy to see that there exist $\overline{j} \in \{1, \ldots, m\}$ and a subfamily $\{I_{n_1}, \ldots, I_{n_{\overline{k}}}\}$ of $\{I_1, \ldots, I_{\overline{n}}\}$ such that $f_{n_k} \in S_{\overline{y}}^{\overline{j}}$ for $k = 1, \ldots, \overline{k}$ and $\mu(\bigcup_{k=1}^{\overline{k}} I_{n_k}) \ge \overline{\alpha}$. Moreover, a straightforward computation shows that

$$\left\|\sum_{k=1}^{\overline{k}} \frac{1}{\overline{k}}(f_{n_k} - s_{f_{n_k}})\right\|_0 > \alpha, \tag{3.4}$$

which contradicts the convexity of the set $C_{\overline{j}}$. Since $\lambda_c(M) = +\infty$, the equality $\omega_c(M) = +\infty$ is a consequence of Theorem 3.4. □

The following lemma is crucial in the proof of Theorem 3.4.

Lemma 3.3. *Let $A_1, \ldots, A_n \in \mathcal{A}$ be a partition of Ω and $H \subseteq S(A_1, \ldots, A_n)$. Then $\lambda(H) = \gamma_c(H)$.*

Proof. Obviously, $\gamma(H) \le \gamma_c(H)$. Since $\omega(H) = 0$, it follows from Theorem 2.1 and Proposition 2.2 that $\lambda(H) = \gamma(H) = \sigma(H)$. Then it is sufficient to prove the inequality $\gamma_c(H) \le \sigma(H)$ which is trivial if $\sigma(H) = \eta(\Omega)$.

Assume that $\sigma(H) < \alpha < \eta(\Omega)$. By the definition of σ we can find a finite set $G \subseteq E$, containing the origin $0 \in E$ and sets $D_0 := \emptyset, D_1, \ldots, D_r \subseteq \Omega$, with $\eta(D_j) \le \alpha$ for $j = 1, \ldots, r$, such that

(i) each D_j for $j = 1, \ldots, r$ is the union of the members of a proper subfamily depending on j of the partition $\{A_1, \ldots, A_n\}$;

(ii) for each $s \in H$, there is $j \in \{0, \ldots, r\}$ such that $s(\Omega \setminus D_j) \subseteq G + B_\alpha(E)$.

For $j = 0, \ldots, r$ we consider the following convex subsets of $B_\alpha(L_0)$:

$$C_\alpha^j := \left\{s \in S(A_1, \ldots, A_n) : s(\Omega \setminus D_j) \subseteq B_\alpha(E)\right\}, \tag{3.5}$$

and we set

$$H_j := \left\{s \in H : s(\Omega \setminus D_j) \subseteq G + B_\alpha(E)\right\}. \tag{3.6}$$

We will prove that

$$H = \bigcup_{j=0}^{r} H_j \subseteq \left[G^\Omega \cap S(A_1, \ldots, A_n)\right] + \bigcup_{j=0}^{r} C_\alpha^j. \tag{3.7}$$

It follows that $\gamma_c(H) \le \alpha$, and therefore $\gamma_c(H) \le \sigma(H)$.

Let $s = \sum_{i=1}^{n} y_i \chi_{A_i} \in H$. Suppose $s \in H_j$, $j \geq 1$ and $D_j = \bigcup_{l=1}^{k} A_{i_l}$, where $\{A_{i_1}, \ldots, A_{i_k}\}$ is a proper subfamily of the partition $\{A_1, \ldots, A_n\}$.

Put $\{A_{i_{k+1}}, \ldots, A_{i_n}\} := \{A_1, \ldots, A_n\} \setminus \{A_{i_1}, \ldots, A_{i_k}\}$.

We have

$$s(\Omega \setminus D_j) = \{y_{i_{k+1}}, \ldots, y_{i_n}\} \subseteq G + B_\alpha(E). \tag{3.8}$$

Hence, for all $l \in \{k+1, \ldots, n\}$, there is $z_{i_l} \in G$ such that $y_{i_l} - z_{i_l} \in B_\alpha(E)$. Then $s = \varphi + h$, where

$$\begin{aligned} \varphi &:= \sum_{l=k+1}^{n} z_{i_l} \chi_{A_{i_l}} \in G^\Omega \cap S(A_1, \ldots, A_n), \\ h &:= \sum_{l=1}^{k} y_{i_l} \chi_{A_{i_l}} + \sum_{l=k+1}^{n} (y_{i_l} - z_{i_l}) \chi_{A_{i_l}} \in C_\alpha^j. \end{aligned} \tag{3.9}$$

Therefore,

$$s \in \left[G^\Omega \cap S(A_1, \ldots, A_n)\right] + C_\alpha^j. \tag{3.10}$$

Similarly, if $s \in H_0$, we can prove that

$$s \in \left[G^\Omega \cap S(A_1, \ldots, A_n)\right] + C_\alpha^0. \tag{3.11}$$

Thus, (3.7) immediately follows from (3.10) and (3.11). □

We are now in a position to prove the main result of this note.

Theorem 3.4. *Let $M \subseteq L_0$. Then,*

$$\max\{\lambda_c(M), \omega_c(M)\} \leq \gamma_c(M) \leq \lambda(M) + 2\omega_c(M). \tag{3.12}$$

Proof. We first prove the left inequality which is trivial if $\gamma_c(M) = +\infty$. Assume that $\gamma_c(M) < \alpha < +\infty$. By Proposition 2.3, there are functions $s_1, \ldots, s_n \in S$ and convex sets $C_1, \ldots, C_n$ in $B_\alpha(L_0)$ such that

$$M \subseteq \bigcup_{i=1}^{n} (s_i + C_i). \tag{3.13}$$

Put $F := \bigcup_{i=1}^{n} s_i(\Omega)$ and let $A_1, \ldots, A_m \in \mathcal{A}$ be a partition of Ω such that $s_i|_{A_j}$ is constant for $i = 1, \ldots, n$ and $j = i, \ldots, m$. Then,

$$M \subseteq \left(F^\Omega \cap S(A_1, \ldots, A_m)\right) + \bigcup_{i=1}^{n} C_i, \tag{3.14}$$

hence $\lambda_c(M) < \alpha$, and $\omega_c(M) < \alpha$.

Therefore,

$$\max\{\lambda_c(M), \omega_c(M)\} \leq \gamma_c(M). \tag{$*$}$$

We now prove the right inequality. Clearly, it is true if $\lambda(M) = +\infty$ or $\omega_c(M) = +\infty$. Assume that $\omega_c(M) < \beta < +\infty$. By the definition of ω_c, we can find a partition $A_1, \ldots, A_m \in \mathcal{A}$ and convex sets $K_1, \ldots, K_m$ of $B_\beta(L_0)$ such that

$$M \subseteq S(A_1, \ldots, A_m) + \bigcup_{j=1}^{m} K_j. \tag{3.15}$$

Set

$$H := \left(M - \bigcup_{j=1}^{m} K_j \right) \cap S(A_1, \ldots, A_m), \tag{3.16}$$

then we have $\lambda(H) \leq \lambda(M) + \beta$. It easy to see that

$$M \subseteq H + \bigcup_{j=1}^{m} K_j. \tag{3.17}$$

Therefore, by Lemma 3.3, we have that

$$\gamma_c(M) \leq \gamma_c(H) + \beta = \lambda(H) + \beta \leq \lambda(M) + \beta + \beta, \tag{3.18}$$

and so

$$\gamma_c(M) \leq \lambda(M) + 2\omega_c(M). \tag{3.19}$$

The proof is complete. □

As a corollary of Theorem 3.4, we obtain the following Fréchet-Smulian type convex total boundedness criterion.

Corollary 3.5. *A subset M of L_0 is ctb if and only if $\lambda(M) = \omega_c(M) = 0$.*

Remark 3.6. In [12], Trombetta proved that

$$\max\{\lambda_c(M), \omega_c(M)\} \leq \gamma_c(M) \leq \lambda_c(M) + 2\omega_c(M) \tag{3.20}$$

for a subset $M \subseteq L_0(\Omega, \mathcal{A}, \mathbb{R}, \eta)$. Since $\lambda(M) \leq \lambda_c(M)$, Theorem 3.4 improves and generalizes to E-valued case the above inequalities.

We point out that the approach used in the scalar case [12] in order to prove Theorem 3.7 cannot be used in our framework. The crucial difference is in the proof of Lemma 3.3. In fact, if $\dim(E) = +\infty$, it might exist some $i \in \{1, \ldots, n\}$ such that the set $H(A_i) := \{s(A_i) : s \in H\}$ is bounded but not necessarily totally bounded.

The following example shows how the value of the parameters λ and λ_c changes when passing from the scalar case to the E-valued case.

Example 3.7. Let $M := \{y\chi_\Omega : y \in B(E)\}$, where $B(E) = \{y \in E : \|y\|_E \leq 1\}$. Then $\lambda(M) = \lambda_c(M) = 0$ if E is finite dimensional, $\lambda(M) = \lambda_c(M) = 1$, otherwise. Moreover, since $\omega_c(M) = 0$, we have $\lambda(M) = \lambda_c(M) = \gamma_c(M)$.

Corollary 3.8. *Let $M \subseteq L_0$. Then,*

$$\begin{aligned} \max\{\lambda(M), \omega_c(M)\} &\leq \max\left\{\lambda\left(\overline{M}\right), \omega_c\left(\overline{M}\right)\right\} \\ &\leq \max\left\{\lambda_c\left(\overline{M}\right), \omega_c\left(\overline{M}\right)\right\} \\ &\leq \lambda(M) + 2\omega_c(M). \end{aligned} \tag{3.21}$$

Proof. It is sufficient to observe that $\gamma_c(M) = \gamma_c(\overline{M})$.

In particular, M ctb implies $\lambda(M) = \lambda(\overline{M}) = 0$, $\lambda_c(M) = \lambda_c(\overline{M}) = 0$, and $\omega_c(M) = \omega_c(\overline{M}) = 0$. □

The next two corollaries of Theorem 3.4 are useful in order to compute or to estimate γ_c in particular classes of subsets of L_0. Moreover, the second one generalizes [12, Proposition 3.10].

Corollary 3.9. *Let M be a convexly equi-measurable subset of L_0. Then, $\lambda(M) = \gamma_c(M)$.*

Corollary 3.10. *Let M be an equi-quasibounded subset of L_0. Then,*

$$\max\{\lambda_c(M), \omega_c(M)\} \leq \gamma_c(M) \leq 2\omega_c(M). \tag{$**$}$$

We observe that if K is a totally bounded subset of E and $M := \{f \in L_0 : f(\Omega) \subseteq K\}$, then, since $\lambda(M) = 0$, the inequalities (3.21) are true for M.

Example 3.11. Let L_0 be the space of Example 3.2 and assume that E is the Banach space l^∞ of all sequences $y = (\xi_1, \xi_2, \ldots)$ with finite norm $\|y\|_\infty := \sup\{|\xi_n| : n = 1, 2, \ldots\}$. If $K := \{y \in l_\infty : |\xi_n| \leq 1/n \text{ for } n = 1, 2, \ldots\}$ and $M := \{f \in L_0 : f(\Omega) \subseteq K\}$, it is easy to prove that $\lambda_c(M) = 0$ and $\omega_c(M) = \gamma_c(M) = 1$. If $B(l^\infty)$ is the closed unit ball of l^∞ and $M := \{f \in L_0 : f(\Omega) \subseteq B(l^\infty)\}$, the set M satisfies $\lambda(M) = \lambda_c(M) = \gamma_c(M) = \omega_c(M) = 1$.

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