Research Article

Ideal Convergence of k -Positive Linear Operators

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We study some ideal convergence results of k -positive linear operators defined on an appropriate subspace of the space of all analytic functions on a bounded simply connected domain in the complex plane. We also show that our approximation results with respect to ideal convergence are more general than the classical ones.

1. Introduction

The classical Korovkin theory (see [1]) is mainly based on the “positivity” of real-valued linear operators. In [2], this theory was improved by Gadjiev and Orhan via the concept of statistical convergence (see, also, [3–6]). However, in order to obtain the Korovkin-type results for complex-valued linear operators, the concept of “ k -positivity” introduced by Gadžiev [7] is used instead of the classical positivity. Such results may be found in the papers [7–10]. In the recent works [11, 12], with the help of some convergence methods, such as A -statistical convergence and ideal convergence, various approximation theorems have been obtained for k -positive linear operators defined on some appropriate subspaces of all analytic functions on the unit disk. In the present paper, we study some ideal convergence results of a sequence of k -positive linear operators on a bounded simply connected domain that does not need to be the unit disk. Furthermore, we present a general family of k -positive linear operators which satisfy all conditions of our results but not the classical ones.

We first recall the concept of “ideal convergence.”

Let X be a nonempty set. A class $\mathcal{O}$ of subsets of X is said to be an *ideal* in X provided that

- (i) ϕ (the empty set) $\in \mathcal{O}$,
- (ii) if $A, B \in \mathcal{O}$, then $A \cup B \in \mathcal{O}$,
- (iii) if $A \in \mathcal{O}$ and $B \subseteq A$, then $B \in \mathcal{O}$.

An ideal is called *nontrivial* if $X \notin \mathcal{O}$. Also, a nontrivial ideal in X is called *admissible* if $\{x\} \in \mathcal{O}$ for each $x \in X$ (see [13] for details). In [14, 15], using the above definition of ideal, a new convergence method which is more general than the usual convergence has been introduced as follows.

Let $\mathcal{O}$ be a nontrivial ideal in $\mathbb{N}$, the set of all positive integers. A sequence $x = \{x_n\}_{n \in \mathbb{N}}$ is *ideal convergent* (or $\mathcal{O}$ -convergent) to a number L if, for every $\varepsilon > 0$, $\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{O}$, which is denoted by $\mathcal{O} - \lim_n x_n = L$. Notice that the method of ideal convergence includes many convergence methods such as A -statistical convergence, statistical convergence, lacunary statistical convergence, usual convergence, and so forth. For example, if $\mathcal{O}$ is the class of all finite subsets of $\mathbb{N}$, then $\mathcal{O}$ -convergence reduces to the usual. Furthermore, $\mathcal{O}$ -convergence coincides with the concept of A -statistical convergence (see [16]) by taking $\mathcal{O} = \{K \subseteq \mathbb{N} : \delta_A(K) = 0\}$, where A is a nonnegative regular summability matrix and $\delta_A(K)$ denotes the A density of K . Besides, if we choose $A = C_1$, the Cesàro matrix of order one, then we immediately obtain the statistical convergence (see [17]). With these properties, using the ideal convergence in the approximation theory provides us many advantages.

Now, we also recall some basic definitions and notations used in the paper.

Let D be a bounded simply connected domain in $\mathbb{C}$, the set of all complex numbers. By $A(D)$ we denote the space of all analytic functions on D . Let $\phi(z)$ be any analytic function mapping D conformally and one to one on the unit disk. Then, for every $f \in A(D)$, we have the Taylor expansion of f given by

$$f(z) = \sum_{k=0}^{\infty} f_k (\phi(z))^k, \quad (1.1)$$

where f_k , $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is the Taylor coefficient of f satisfying

$$\limsup_k \sqrt[k]{|f_k|} \leq 1. \quad (1.2)$$

It is known that Taylor's coefficients are calculated by the following formula (see [18]):

$$f_k = \frac{1}{2\pi i} \oint_C \frac{f(z)\phi'(z)}{(\phi(z))^{k+1}} dz, \quad (1.3)$$

where C is any closed contour lying in the interior of D . It is not hard to see that the series (1.1) under the condition (1.2) is uniformly convergent if $|\phi(z)| = r < 1$. It is well known that

$A(D)$ is Fréchet's space with topology of compact convergence in any closed subset of D . In this paper, we use the norm $\|\cdot\|_r$ on the space $A(D)$ defined by

$$\|f\|_r := \max_{|\phi(z)|=r<1} |f(z)|, \quad \text{for } f \in A(D), \quad (1.4)$$

and therefore the convergence in $A(D)$ is the convergence in norm $\|\cdot\|_r$ for any $0 < r < 1$.

Now let $A(D^+) := \{f \in A(D) : f_k \geq 0, k = 0, 1, 2, \dots\}$. Following [7] (see also [9]), if a linear operator T mapping $A(D)$ into itself satisfies the condition

$$T(A(D^+)) \subseteq A(D^+), \quad (1.5)$$

then we say that T is a "*k-positive linear operator*". As usual, for a function $f \in A(D)$, the value of $T(f)$ at a point z is denoted by $T(f; z)$ and also the Taylor expansion of $T(f)$ is

$$T(f; z) = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} T_{k,m} f_m \right) (\phi(z))^k, \quad (1.6)$$

where $f_m (m \in \mathbb{N}_0)$ is the Taylor coefficient of f and $T_{k,m} (k, m \in \mathbb{N}_0)$ is the Taylor coefficient of $T([\phi(z)]^k)$. If $\{T_n\}_{n \in \mathbb{N}}$ is a sequence of k -positive linear operators from $A(D)$ into itself, then, as in (1.6), we may write that

$$T_n(f; z) = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} T_{k,m}^{(n)} f_m \right) (\phi(z))^k, \quad \text{for each } n \in \mathbb{N}, \quad (1.7)$$

where, of course, $\sum_{m=0}^{\infty} T_{k,m}^{(n)} f_m$ is the Taylor coefficient of $T_n(f)$ for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Observe that T_n is k -positive if and only if $T_{k,m}^{(n)} \geq 0$ for every $n \in \mathbb{N}$ and $k, m \in \mathbb{N}_0$.

Throughout the paper we use the following three test functions:

$$e_i(z) := \sum_{k=0}^{\infty} k^i (\phi(z))^k, \quad i = 0, 1, 2. \quad (1.8)$$

We also consider the following subspace of $A(D)$:

$$A^*(D) := \left\{ f \in A(D) : |f_k| \leq M(1 + k^2) \text{ for every } k \in \mathbb{N}_0 \text{ and for some } M > 0 \right\}. \quad (1.9)$$

2. Ideal Convergence of k -Positive Linear Operators

In order to compute the degree of ideal convergence of sequences we introduce the following definition (see also [2]).

Definition 2.1. Let $\mathcal{O}$ be an admissible ideal in $\mathbb{N}$. Then, one says that a sequence $\{x_n\}_{n \in \mathbb{N}}$ is ideal convergent to a number L with degree $0 < \beta \leq 1$ if, for every $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{|x_n - L|}{n^{1-\beta}} \geq \varepsilon \right\} \in \mathcal{O}. \quad (2.1)$$

In this case, we write

$$x_n - L = \mathcal{O} - o(n^{-\beta}), \quad \text{as } n \rightarrow \infty. \quad (2.2)$$

Notice that if we choose, in particular, $\beta = 1$ in Definition 2.1, we immediately get the ideal convergence of (x_n) to L . Observe that, according to Definition 2.1, the degree β is controlled by the entries of the sequence $\{x_n\}_{n \in \mathbb{N}}$.

We first need the following two lemmas.

Lemma 2.2. Let $\mathcal{O}$ be an admissible ideal in $\mathbb{N}$. Assume further that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of analytic functions on D with the Taylor coefficients $f_{n,k}$ for each $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then, for $0 < \beta \leq 1$,

$$\|f_n\|_r = \mathcal{O} - o(n^{-\beta}), \quad \text{as } n \rightarrow \infty, \quad (2.3)$$

if and only if there exists a sequence $\{t_{n,k}\}_{n \in \mathbb{N}, k \in \mathbb{N}_0}$ for which the following conditions hold:

$$|f_{n,k}| \leq t_{n,k}, \quad \text{for every } n \in \mathbb{N}, k \in \mathbb{N}_0, \quad (2.4)$$

$$\limsup_k \sqrt[k]{t_{n,k}} \leq 1, \quad \text{for each fixed } n \in \mathbb{N}, \quad (2.5)$$

$$t_{n,k} = \mathcal{O} - o(n^{-\beta}), \quad \text{as } n \rightarrow \infty, \quad \text{for each fixed } k \in \mathbb{N}_0. \quad (2.6)$$

Proof.

Necessity. Assume that (2.3) holds. Then, choosing $t_{n,k} = |f_{n,k}|$, the conditions (2.4) and (2.5) can be obtained immediately. Now we prove (2.6). By (2.3), we may write that, for each fixed $k \in \mathbb{N}_0$,

$$\varepsilon_n := \|f_n\|_r = \max_{|\phi(z)|=r < 1} |f_n(z)| = \mathcal{O} - o(n^{-\beta}), \quad \text{as } n \rightarrow \infty. \quad (2.7)$$

By (1.3), for any $r < 1$, we obtain that

$$t_{n,k} = |f_{n,k}| \leq \frac{1}{2\pi} \int_{|\phi(z)|=r} \frac{|f_n(z)| |\phi'(z)|}{|\phi(z)|^{k+1}} |dz| \leq \frac{\varepsilon_n}{r^k}, \quad \text{for } n \in \mathbb{N}, k \in \mathbb{N}_0. \quad (2.8)$$

Hence, it follows from (2.8) that, for a given $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{t_{n,k}}{n^{1-\beta}} \geq \varepsilon \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{\varepsilon_n}{n^{1-\beta}} \geq r^k \varepsilon \right\}. \quad (2.9)$$

By (2.7), we get

$$\left\{ n \in \mathbb{N} : \frac{\varepsilon_n}{n^{1-\beta}} \geq \varepsilon' \right\} \in \mathcal{O}, \quad (2.10)$$

where $\varepsilon' = r^k \varepsilon$. Therefore, from (2.9), we conclude that

$$\left\{ n \in \mathbb{N} : \frac{t_{n,k}}{n^{1-\beta}} \geq \varepsilon \right\} \in \mathcal{O}, \quad (2.11)$$

which gives (2.6).

Sufficiency. Assume now that conditions (2.4)–(2.6) hold. It follows from (2.5) that the series $\sum_{k=0}^{\infty} t_{n,k} r^k$ is convergent for any $r < 1$. Then, for every $\varepsilon > 0$, there exists a positive natural number $N = N(\varepsilon)$ such that

$$\sum_{k=N+1}^{\infty} t_{n,k} r^k \leq \varepsilon \quad (2.12)$$

holds. Using (2.4) and (2.12) and also considering the fact that

$$\|f_n\|_r \leq \sum_{k=0}^{\infty} |f_{n,k}| r^k \leq \sum_{k=0}^N t_{n,k} r^k + \sum_{k=N+1}^{\infty} t_{n,k} r^k, \quad (2.13)$$

we get

$$\|f_n\|_r \leq \varepsilon + \sum_{k=0}^N t_{n,k} r^k, \quad (2.14)$$

which yields that

$$\frac{\|f_n\|_r}{n^{1-\beta}} \leq \frac{\varepsilon}{n^{1-\beta}} + \sum_{k=0}^N \frac{t_{n,k} r^k}{n^{1-\beta}}. \quad (2.15)$$

Since $r^k \leq 1$ for every $k = 0, 1, \dots, N$ and $1/n^{1-\beta} \leq 1$ for every $n \in \mathbb{N}$ and $0 < \beta \leq 1$, we obtain that

$$\frac{\|f_n\|_r}{n^{1-\beta}} \leq \varepsilon + \sum_{k=0}^N \frac{t_{n,k}}{n^{1-\beta}}. \quad (2.16)$$

Now, for a given $\gamma > 0$, choose an $\varepsilon > 0$ such that $\varepsilon < \gamma$. Then, define the following sets:

$$E := \left\{ n \in \mathbb{N} : \frac{\|f_n\|_r}{n^{1-\beta}} \geq \gamma \right\}, \quad (2.17)$$

$$E_k := \left\{ n \in \mathbb{N} : \frac{t_{n,k}}{n^{1-\beta}} \geq \frac{\gamma - \varepsilon}{N+1} \right\}, \quad k = 0, 1, \dots, N.$$

From (2.16), it is clear that

$$E \subseteq \bigcup_{k=0}^N E_k. \quad (2.18)$$

By hypothesis (2.6), we know $E_k \in \mathcal{O}$ for each $k = 0, 1, \dots, N$. Hence, by the definition of ideal, we immediately obtain that the set E belongs to $\mathcal{O}$ for every $\gamma > 0$, which implies (2.3). So, the proof is completed. $\square$

Lemma 2.3. *Let $\mathcal{O}$ be an admissible ideal in $\mathbb{N}$, and let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of k -positive linear operators from $A(D)$ into itself. If, for some β_i with $0 < \beta_i \leq 1$ ($i = 0, 1, 2$),*

$$\|T_n(e_i) - e_i\|_r = \mathcal{O} - o(n^{-\beta_i}), \quad \text{as } n \rightarrow \infty, \quad (2.19)$$

holds, then there exists a sequence $\{t_{n,k}\}_{n \in \mathbb{N}, k \in \mathbb{N}_0}$ satisfying (2.5), (2.6) with $\beta := \min\{\beta_0, \beta_1, \beta_2\}$ such that the following inequality

$$\sum_{m=0}^{\infty} T_{k,m}^{(n)} (m-k)^2 \leq (k+1)^2 t_{n,k} \quad (2.20)$$

holds, where $T_{k,m}^{(n)}$ is the same as (1.7).

Proof. We first observe that

$$\begin{aligned} \sum_{m=0}^{\infty} T_{k,m}^{(n)} (m-k)^2 &= \left(\sum_{m=0}^{\infty} T_{k,m}^{(n)} m^2 - k^2 \right) - 2k \left(\sum_{m=0}^{\infty} T_{k,m}^{(n)} m - k \right) + k^2 \left(\sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1 \right) \\ &\leq \left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} m^2 - k^2 \right| + 2k \left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} m - k \right| + k^2 \left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1 \right|. \end{aligned} \quad (2.21)$$

By (2.19) and Lemma 2.2, there exist sequences $\{t_{i,n,k}\}_{n \in \mathbb{N}, k \in \mathbb{N}_0}$ and numbers β_i with $0 < \beta_i \leq 1$ ($i = 0, 1, 2$) satisfying (2.5), (2.6) such that the following conditions

$$\begin{aligned} \left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1 \right| &\leq t_{0,n,k}, \\ \left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} m - k \right| &\leq t_{1,n,k}, \\ \left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} m^2 - k^2 \right| &\leq t_{2,n,k} \end{aligned} \quad (2.22)$$

hold. Hence, we easily get

$$\sum_{m=0}^{\infty} T_{k,m}^{(n)} (m - k)^2 \leq (k + 1)^2 t_{n,k}, \quad (2.23)$$

where $t_{n,k} := \max\{t_{0,n,k}, t_{1,n,k}, t_{2,n,k}\}$. Since $\beta := \min\{\beta_0, \beta_1, \beta_2\}$, we can see that

$$t_{n,k} = o(n^{-\beta}), \quad \text{as } n \rightarrow \infty, \quad (2.24)$$

whence the result. $\square$

Then, we are ready to give our main result.

Theorem 2.4. *Let $\mathcal{O}$ be an admissible ideal in $\mathbb{N}$, and let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of k -positive linear operators from $A(D)$ into itself. If, for some β_i with $0 < \beta_i \leq 1$ ($i = 0, 1, 2$), (2.19) holds, then, for every $f \in A^*(D)$, one has*

$$\|T_n(f) - f\|_r = \mathcal{O} - o(n^{1-\beta}), \quad \text{as } n \rightarrow \infty, \quad (2.25)$$

where $\beta := \min\{\beta_0, \beta_1, \beta_2\}$.

Proof. Assume that (2.19) holds for some β_i with $0 < \beta_i \leq 1$ ($i = 0, 1, 2$). Let $f \in A^*(D)$ and $z \in D$ be fixed. By (1.1) and (1.7), we may write that

$$T_n(f; z) - f(z) = \sum_{k=0}^{\infty} (\phi(z))^k \sum_{m=0}^{\infty} T_{k,m}^{(n)} (f_m - f_k) + \sum_{k=0}^{\infty} (\phi(z))^k f_k \left(\sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1 \right). \quad (2.26)$$

The last inequality gives that

$$\|T_n(f) - f\|_r \leq \sum_{k=0}^{\infty} r^k \sum_{m=0}^{\infty} T_{k,m}^{(n)} |f_m - f_k| + \sum_{k=0}^{\infty} r^k |f_k| \left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1 \right|. \quad (2.27)$$

By Lemma 2.2(ii) of [11], we know the fact that $|f_m - f_k| \leq M(3+k)^4(m-k)^2$. Also, since $f \in A^*(D)$, we see that $|f_k| \leq M(1+k^2)$. Hence, combining these inequalities, we have

$$\|T_n(f) - f\|_r \leq M \sum_{k=0}^{\infty} r^k (3+k)^4 \sum_{m=0}^{\infty} T_{k,m}^{(n)} (m-k)^2 + M \sum_{k=0}^{\infty} r^k (1+k^2) \left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1 \right|. \quad (2.28)$$

By Lemma 2.3, there exist a sequence $\{t_{n,k}\}_{n \in \mathbb{N}, k \in \mathbb{N}_0}$ satisfying (2.5), (2.6) with $\beta := \min\{\beta_0, \beta_1, \beta_2\}$ such that the inequality

$$\|T_n(f) - f\|_r \leq M \left(\sum_{k=0}^{\infty} r^k (3+k)^4 (k+1)^2 t_{n,k} + \sum_{k=0}^{\infty} r^k (1+k^2) (k+1)^2 t_{n,k} \right) \quad (2.29)$$

holds. Hence, we get, for every $n \in \mathbb{N}$, that

$$\|T_n(f) - f\|_r \leq M \sum_{k=0}^{\infty} r^k (3+k)^6 t_{n,k}. \quad (2.30)$$

Since, for every $k, n \in \mathbb{N}$,

$$r^k (3+k)^6 t_{n,k} \leq r^k 4^6 k^6 t_{n,k}, \quad (2.31)$$

we easily get that

$$\left\{ r^k (3+k)^6 t_{n,k} \right\}^{1/k} \leq r 4^{6/k} k^{6/k} t_{n,k}^{1/k}. \quad (2.32)$$

Now taking limsup as $k \rightarrow \infty$ in both sides of the last inequality and also using (2.5), we observe that the series in the right-hand side of (2.30) converges for any $0 < r < 1$. Therefore, the remain of the proof is very similar to the proof of the sufficiency part of Lemma 2.2. $\square$

3. Concluding Remarks

In this section, we give some useful consequences of our Theorem 2.4.

We first observe that, for $|\phi(z)| = r < 1$ with $\phi'(z) \neq 0$,

$$\begin{aligned}
 \frac{1}{1-\phi(z)} &= \sum_{k=0}^{\infty} (\phi(z))^k = e_0(z), \\
 \frac{\phi(z)}{(1-\phi(z))^2} &= \frac{\phi(z)}{\phi'(z)} \frac{d}{dz} \left(\frac{1}{1-\phi(z)} \right) = \phi(z) \sum_{k=0}^{\infty} k (\phi(z))^{k-1} \\
 &= \sum_{k=0}^{\infty} k (\phi(z))^k = e_1(z), \\
 \frac{\phi^2(z)}{(1-\phi(z))^3} &= \frac{\phi^2(z)}{2\phi'(z)} \frac{d}{dz} \left(\frac{1}{(1-\phi(z))^2} \right) = \frac{\phi^2(z)}{2} \sum_{k=0}^{\infty} k(k-1) (\phi(z))^{k-2} \\
 &= \frac{1}{2} \sum_{k=0}^{\infty} k(k-1) (\phi(z))^k = \frac{e_2(z) - e_1(z)}{2}.
 \end{aligned} \tag{3.1}$$

Then, defining

$$f_i(z) := \frac{(\phi(z))^i}{(1-\phi(z))^{i+1}}, \quad i = 0, 1, 2, \tag{3.2}$$

the next result is equivalent to Theorem 2.4.

Theorem 3.1. *Let $\mathcal{O}$ be an admissible ideal in $\mathbb{N}$, and let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of k -positive linear operators from $A(D)$ into itself. If, for some β_i with $0 < \beta_i \leq 1$ ($i = 0, 1, 2$),*

$$\|T_n(f_i) - f_i\|_r = \mathcal{O} - o\left(n^{-\beta_i}\right), \quad \text{as } n \longrightarrow \infty, \tag{3.3}$$

then, for every $f \in A^*(D)$, (2.25) holds for the same β as in Theorem 2.4.

If we take $\beta = 1$ in Theorems 2.4 and 3.1, then we immediately get the following characterization for ideal approximation by k -positive linear operators.

Corollary 3.2. *Let $\mathcal{O}$ be an admissible ideal in $\mathbb{N}$, and let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of k -positive linear operators from $A(D)$ into itself. Then, for every $f \in A^*(D)$,*

$$\mathcal{O} - \lim_n \|T_n(f) - f\|_r = 0, \tag{3.4}$$

if and only if, for each $i = 0, 1, 2$,

$$\mathcal{O} - \lim_n \|T_n(e_i) - e_i\|_r = 0, \tag{3.5}$$

or equivalently,

$$\mathcal{O} - \lim_n \|T_n(f_i) - f_i\|_r = 0, \quad (3.6)$$

Proof. Since each $e_i, f_i \in A^*(D)$, the implications (3.4) $\Rightarrow$ (3.5) and (3.4) $\Rightarrow$ (3.6) are obvious. The sufficiency immediately follows from Definition 2.1 and Theorems 2.4 and 3.1. $\square$

Finally, if we choose $\mathcal{O} = \{K \subseteq \mathbb{N} : \delta_A(K) = 0\}$, where A is a nonnegative regular summability matrix and $\delta_A(K)$ denotes the A density of K , and also if we take $D = \{z \in \mathbb{C} : |z| < 1\}$, then from Theorem 2.4 we obtain a slight modification of the result proved in [11].

It is known from [14] that if we choose $\mathcal{O} = \{K \subseteq \mathbb{N} : \delta(K) = 0\}$, where $\delta(K)$ denotes the asymptotic density of K given by

$$\delta(K) := \lim_n \frac{\#\{k \leq n : k \in K\}}{n} \quad (\text{provided the limit exists}), \quad (3.7)$$

then $\mathcal{O}$ -convergence reduces to the concept of statistical convergence which was first introduced by Fast [17]. In the last equality, by $\# \{B\}$ we denote the cardinality of the set B . Hence, let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence whose terms are defined by

$$u_n := \begin{cases} \frac{n}{n+1}, & \text{if } n = m^2, m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases} \quad (3.8)$$

Then, we easily observe that

$$\mathcal{O} - \lim_n u_n = \text{st} - \lim_n u_n = 0 \quad (3.9)$$

holds. Assume now that $\{T_n\}_{n \in \mathbb{N}}$ is any sequence of k -positive linear operators from $A(D)$ into itself, such that, for every $f \in A^*(D)$, the sequence $\{T_n\}_{n \in \mathbb{N}}$ is uniformly convergent to f on a bounded simply domain D with respect to any norm $\|\cdot\|_r$ ($0 < r < 1$). Then, consider the following operators:

$$L_n(f; z) := (1 + u_n)T_n(f, z). \quad (3.10)$$

Therefore, observe that $\{L_n\}_{n \in \mathbb{N}}$ is a sequence of k -positive linear operators from $A(D)$ into itself. By (3.10), we can write, for each $i = 0, 1, 2$, that

$$\begin{aligned} \|L_n(f_i) - f_i\|_r &\leq \|T_n(f_i) - f_i\|_r + u_n \|T_n(f_i)\|_r \\ &\leq (1 + u_n) \|T_n(f_i) - f_i\|_r + u_n \|f_i\|_r \\ &\leq (1 + u_n) \|T_n(f_i) - f_i\|_r + \frac{u_n r^i}{(1 - r)^{i+1}}. \end{aligned} \quad (3.11)$$

So, the last inequality gives

$$\text{st} - \lim_n \|L_n(f_i) - f_i\| \leq \left\{ \text{st} - \lim_n (1 + u_n) \right\} \left\{ \lim_n \|T_n(e_i) - e_i\| \right\} + \text{st} - \lim_n \frac{u_n r^i}{(1-r)^{i+1}} \quad (3.12)$$

by our conditions on $\{T_n\}_{n \in \mathbb{N}}$. Then, it follows from (3.9) that, for each $i = 0, 1, 2$,

$$\text{st} - \lim_n \|L_n(f_i) - f_i\| = 0. \quad (3.13)$$

Hence, by Corollary 3.2, we obtain, for every $f \in A^*(D)$, that

$$\mathcal{O} - \lim_n \|L_n(f) - f\| = \text{st} - \lim_n \|L_n(f) - f\| = 0. \quad (3.14)$$

However, by the definition (3.8), we see that, for every $f \in A^*(D)$,

$$u_n \|T_n(f)\|_r = \begin{cases} \frac{n}{n+1} \|T_n(f)\|_r, & \text{if } n = m^2, m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases} \quad (3.15)$$

Now since $\lim_n \|T_n(f)\|_r = \|f\|_r$, we immediately obtain that the subsequence $\{u_n \|T_n(f)\|_r\}_{n=m^2}$ converges to $\|f\|_r$ while the subsequence $\{u_n \|T_n(f)\|_r\}_{n \neq m^2}$ converges to zero. Hence, the sequence $\{u_n \|T_n(f)\|_r\}_{n \in \mathbb{N}}$ is nonconvergent. Therefore, we see that the sequence $\{L_n(f)\}_{n \in \mathbb{N}}$ cannot be uniformly convergent to f on D . Therefore, we can say that our ideal approximations by k -positive linear operators presented in this paper are more general and applicable than the classical ones.

Finally, for a given $m \in \mathbb{N}$, we consider the following subspace of $A(D)$:

$$A_m^*(D) := \left\{ f \in A(D) : |f_k| \leq M(1 + k^{2m}) \text{ for every } k \in \mathbb{N}_0 \text{ and for some } M > 0 \right\}. \quad (3.16)$$

In this case, we consider the following test functions:

$$g_i(z) = \sum_{k=0}^{\infty} k^{mi} (\phi(z))^k, \quad i = 0, 1, 2. \quad (3.17)$$

Thus, considering the same methods used in this paper, one can immediately get the following ideal approximation result on the subspace $A_m^*(D)$, $m \in \mathbb{N}$.

Corollary 3.3. *Let $\mathcal{O}$ be an admissible ideal in $\mathbb{N}$, and let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of k -positive linear operators from $A(D)$ into itself. Then, for every $f \in A_m^*(D)$, $m \in \mathbb{N}$,*

$$\mathcal{O} - \lim_n \|T_n(f) - f\|_r = 0, \quad (3.18)$$

if and only if, for each $i = 0, 1, 2$,

$$\mathcal{O} - \lim_n \|T_n(g_i) - g_i\|_r = 0. \quad (3.19)$$

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