Research Article

On the Frame Properties of Degenerate System of Sines

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Systems of sines with degenerate coefficients are considered in this paper. Frame properties of these systems in Lebesgue spaces are studied.

1. Introduction

Basis properties of classical system of exponents $\{e^{int}\}_{n \in \mathbb{Z}}$ ($\mathbb{Z}$ is the set of all integers) in Lebesgue spaces $L_p(-\pi, \pi)$, $1 \leq p < +\infty$, are well studied in the literature (see [1–4]). Bari in her fundamental work [5] raised the issue of the existence of normalized basis in L_2 which is not Riesz basis. The first example of this was given by Babenko [6]. He proved that the degenerate system of exponents $\{|t|^\alpha e^{int}\}_{n \in \mathbb{Z}}$ with $|\alpha| < (1/2)$ forms a basis for $L_2(-\pi, \pi)$ but is not Riesz basis when $\alpha \neq 0$. This result has been extended by Gaposkin [7]. In [8], the condition on the weight ρ was found which make the system $\{e^{int}\}_{n \in \mathbb{Z}}$ forms a basis for the weight space $L_{p,\rho}(-\pi, \pi)$ with a norm $\|f\|_{p,\rho} = (\int_{-\pi}^{\pi} |f(t)|^p \rho(t) dt)^{(1/p)}$. Basis properties of a degenerate system of exponents are closely related to the similar properties of an ordinary system of exponents in corresponding weight space. In all the mentioned works, the authors consider the cases when the weight or the degenerate coefficient satisfies the Muckenhoupt condition (see, e.g., [9]). It should be noted that the above stated is true for the systems of sines and cosines, too.

Basis properties of the system of exponents and sines with the linear phase in weighted Lebesgue spaces have been studied in [10–12]. Those of the systems of exponents with degenerate coefficients have been studied in [13, 14]. Similar questions have previously been considered in papers [15–18].

In this work, we study the frame properties of the system of sines with degenerate coefficient in Lebesgue spaces, when the degenerate coefficient, generally speaking, does not satisfy the Muckenhoupt condition.

2. Needful Information

To obtain our main results, we will use some concepts and facts from the theory of bases.

We will use the standard notation. $\mathbb{N}$ will be the set of all positive integers; $\exists$ will mean “there exist(s)”; $\Rightarrow$ will mean “it follows”; $\Leftrightarrow$ will mean “if and only if”; $\exists!$ will mean “there exists unique”; $K \equiv \mathbb{R}$ or $K \equiv \mathbb{C}$ will stand for the set of real or complex numbers, respectively; δ_{nk} is Kronecker symbol, $\delta_k = \{\delta_{kn}\}_{k \in \mathbb{N}}$.

Let X be some Banach space with a norm $\|\cdot\|_X$. Then X^* will denote its dual with a norm $\|\cdot\|_{X^*}$. By $L[M]$, we denote the linear span of the set $M \subset X$, and $\overline{M}$ will stand for the closure of M .

System $\{x_n\}_{n \in \mathbb{N}} \subset X$ is said to be uniformly minimal in X if $\exists \delta > 0$

$$\inf_{\forall u \in L[\{x_n\}_{n \neq k}]} \|x_k - u\|_X \geq \delta \|x_k\|_X, \quad \forall k \in \mathbb{N}. \quad (2.1)$$

System $\{x_n\}_{n \in \mathbb{N}} \subset X$ is said to be complete in X if $\overline{L[\{x_n\}_{n \in \mathbb{N}}]} = X$. It is called minimal in X if $x_k \notin \overline{L[\{x_n\}_{n \neq k}]}$, for all $k \in \mathbb{N}$.

The following criteria of completeness and minimality are available.

Criterion 1 (Hahn-Banach theorem). System $\{x_n\}_{n \in \mathbb{N}} \subset X$ is complete in X if $f(x_n) = 0$, for all $n \in \mathbb{N}$, $f \in X^* \Rightarrow f = 0$.

Criterion 2 (see [19]). System $\{x_n\}_{n \in \mathbb{N}} \subset X$ is minimal in $X \Leftrightarrow$ it has a biorthogonal system $\{f_n\}_{n \in \mathbb{N}} \subset X^*$, that is, $f_n(x_k) = \delta_{nk}$, for all $n, k \in \mathbb{N}$.

Criterion 3. Complete system $\{x_n\}_{n \in \mathbb{N}} \subset X$ is uniformly minimal in $X \Leftrightarrow \sup_n \|x_n\|_X \|y_n\|_{X^*} < +\infty$, where $\{y_n\}_{n \in \mathbb{N}} \subset X^*$ is a system biorthogonal to it.

System $\{x_n\}_{n \in \mathbb{N}} \subset X$ is said to be a basis for X if for all $x \in X$, $\exists! \{\lambda_n\}_{n \in \mathbb{N}} \subset K : x = \sum_{n=1}^{\infty} \lambda_n x_n$.

If system $\{x_n\}_{n \in \mathbb{N}} \subset X$ forms a basis for X , then it is uniformly minimal.

Definition 2.1 (see [20, 21]). Let X be a Banach space and $\mathcal{K}$ a Banach sequence space indexed by $\mathbb{N}$. Let $\{f_k\}_{k \in \mathbb{N}} \subset X$, $\{g_k\}_{k \in \mathbb{N}} \subset X^*$. Then $(\{g_k\}_{k \in \mathbb{N}}, \{f_k\}_{k \in \mathbb{N}})$ is an atomic decomposition of X with respect to $\mathcal{K}$ if

- (i) $\{g_k(f)\}_{k \in \mathbb{N}} \in \mathcal{K}$, for all $f \in X$;
- (ii) $\exists A, B > 0$:

$$A \|f\|_X \leq \|\{g_k(f)\}_{k \in \mathbb{N}}\|_{\mathcal{K}} \leq B \|f\|_X, \quad \forall f \in X; \quad (2.2)$$

- (iii) $f = \sum_{k=1}^{\infty} g_k(f) f_k$, for all $f \in X$.

Definition 2.2 (see [20, 21]). Let X be a Banach space and $\mathcal{K}$ a Banach sequence space indexed by $\mathbb{N}$. Let $\{g_k\}_{k \in \mathbb{N}} \subset X^*$ and $S : \mathcal{K} \rightarrow X$ be a bounded operator. Then $(\{g_k\}_{k \in \mathbb{N}}, S)$ is a Banach frame for X with respect to $\mathcal{K}$ if

- (i) $\{g_k(f)\}_{k \in \mathbb{N}} \in \mathcal{K}$, for all $f \in X$;
- (ii) $\exists A, B > 0$:

$$A\|f\|_X \leq \|\{g_k(f)\}_{k \in \mathbb{N}}\|_{\mathcal{K}} \leq B\|f\|_X, \quad \forall f \in X; \quad (2.3)$$

- (iii) $S[\{g_k(f)\}_{k \in \mathbb{N}}] = f$, for all $f \in X$.

It is true the following.

Proposition 2.3 (see [20, 21]). Let X be a Banach space and $\mathcal{K}$ a Banach sequence space indexed by $\mathbb{N}$. Assume that the canonical unit vectors $\{\delta_k\}_{k \in \mathbb{N}}$ constitute a basis for $\mathcal{K}$ and let $\{g_k\}_{k \in \mathbb{N}} \subset X^*$ and $S : \mathcal{K} \rightarrow X$ be a bounded operator. Then the following statements are equivalent:

- (i) $(\{g_k\}_{k \in \mathbb{N}}, S)$ is a Banach frame for X with respect to $\mathcal{K}$.
- (ii) $(\{g_k\}_{k \in \mathbb{N}}, \{S(\delta_k)\}_{k \in \mathbb{N}})$ is an atomic decomposition of X with respect to $\mathcal{K}$.

More details about these facts can be found in [20–23].

3. Completeness and Minimality

We consider a system of sines

$$\{S_n^v\}_{n \in \mathbb{N}} \equiv \{v(t) \sin nt\}_{n \in \mathbb{N}}, \quad (3.1)$$

with a degenerate coefficient v

$$v(t) = \prod_{k=0}^r |t - t_k|^{\alpha_k}, \quad (3.2)$$

where $0 = t_0 < t_1 < \dots < t_r = \pi$ are points of degeneration and $\{\alpha_k\}_0^r \subset \mathbb{R}$.

The notation $f \sim g$, $t \rightarrow a$, means that the inequality $0 < \delta \leq |f(t)/g(t)| \leq \delta^{-1} < +\infty$ holds in sufficiently small neighborhood of the point $t = a$ with respect to the functions f and g . Thus, it is clear that $\sin nt \sim t$, $t \rightarrow 0$ and $\sin nt \sim \pi - t$, $t \rightarrow \pi$ for all $n \in \mathbb{N}$. Proceeding from these relations, we immediately obtain that the inclusion $\{S_n^v\}_{n \in \mathbb{N}} \subset L_p(0, \pi)$, $1 \leq p < +\infty$, is true if and only if the following relations hold

$$\alpha_0; \alpha_r \in \left(-\frac{1}{p} - 1, +\infty\right), \quad \{\alpha_k\}_1^{r-1} \subset \left(-\frac{1}{p}, +\infty\right). \quad (3.3)$$

In what follows, we will always suppose that this condition is satisfied. Assume that the function $f \in L_q(0, \pi)((1/p) + (1/q) = 1)$ is orthogonal to the system $\{S_n^v\}_{n \in \mathbb{N}}$, that is

$$\int_0^\pi v(t) \sin nt \overline{f(t)} dt = 0, \quad \forall n \in \mathbb{N}, \quad (3.4)$$

where $(\overline{\cdot})$ is a complex conjugate. By $C_0[0, \pi]$, we denote the Banach space of functions which are continuous on $[0, \pi]$ with a sup-norm and vanish at the ends of the interval $[0, \pi]$. It is absolutely clear that $vf \in L_1(0, \pi) \subset C_0^*[0, \pi]$. As the system of sines $\{\sin nt\}_{n \in \mathbb{N}}$ is complete in $C_0[0, \pi]$, we obtain from the relations (3.4) that $v(t)f(t) = 0$ a.e. on $(0, \pi)$, and, consequently, $f(t) = 0$ a.e. on $(0, \pi)$. This proves the completeness of system (3.3) in $L_p(0, \pi)$.

Now consider the minimality of system (3.3) in $L_p(0, \pi)$. It is clear that $\{S_n^{v^{-1}}\}_{n \in \mathbb{N}} \subset L_q(0, \pi)$ if and only if

$$\alpha_0; \alpha_r \in \left(-\infty, \frac{1}{q} + 1\right), \quad \{\alpha_k\}_1^{r-1} \subset \left(-\infty, \frac{1}{q}\right). \quad (3.5)$$

It is easily seen that the system $\{\sqrt{(2/\pi)}S_n^{v^{-1}}\}_{n \in \mathbb{N}}$ is biorthogonal to $\{S_n^v\}_{n \in \mathbb{N}}$. So the following theorem is true.

Theorem 3.1. *System $\{S_n^v\}_{n \in \mathbb{N}}$ is complete in $L_p(0, \pi)$, $1 \leq p < +\infty$, if the relations (3.3) hold. Besides, it is minimal in $L_p(0, \pi)$ if both (3.3) and (3.5) hold. Consequently, system $\{S_n^v\}_{n \in \mathbb{N}}$ is complete and minimal in $L_p(0, \pi)$ if the following relations hold:*

$$\alpha_0; \alpha_r \in \left(-\frac{1}{p} - 1, \frac{1}{q} + 1\right), \quad \{\alpha_k\}_1^{r-1} \subset \left(-\frac{1}{p}, \frac{1}{q}\right). \quad (3.6)$$

It is known that (see, e.g., [10, 11]) if $\{\alpha_k\}_0^r \subset (-(1/p), (1/q))$, then system $\{S_n(v)\}_{n \in \mathbb{N}}$ forms a basis for $L_p(0, \pi)$, $1 < p < +\infty$. Let $\beta \in [(1/q), (1/q) + 1]$, where either $\beta = \alpha_0$ or $\beta = \alpha_r$. In the sequel, we will suppose that the condition (3.6) is satisfied for $\{\alpha_k\}_1^{r-1}$. We have

$$\|S_n^v\|_p^p = \int_0^\pi |v(t)|^p |\sin nt|^p dt \leq \int_0^\pi |v(t)|^p dt < +\infty, \quad \forall n \in \mathbb{N}. \quad (3.7)$$

On the other hand

$$\|S_n^v\|_p^p \geq \int_0^\delta |v(t)|^p |\sin nt|^p dt \geq c \int_0^\delta t^{\alpha_0 p} |\sin nt|^p dt = \frac{c}{n^{1+\alpha_0 p}} \int_0^{n\delta} t^{\alpha_0 p} |\sin t|^p dt, \quad (3.8)$$

where $c > 0$ is some constant (in what follows c will denote constants that may be different from each other), $\delta > 0$ is such that $[0, \delta]$ does not contain the points $\{\alpha_k\}_1^{r-1}$. Let us show that $\inf_{n \in \mathbb{N}} \|S_n^v\|_p > 0$. We have $(\alpha = \alpha_0 p + 1)$

$$\|S_n^v\|_p^p \geq \frac{c}{n^\alpha} \sum_{k \in M_n} \int_{k\pi+(\pi/4)}^{(k+1)\pi-(\pi/4)} t^{\alpha-1} |\sin t|^p dt \geq \frac{c}{n^\alpha} \sum_{k \in M_n} \int_{k\pi+(\pi/4)}^{(k+1)\pi-(\pi/4)} t^{\alpha-1} dt, \quad (3.9)$$

where $M_n \equiv \{k \geq 0 : (k+1)\pi - (\pi/4) \leq n\delta\}$. Thus

$$\|S_n^v\|_p^p \geq \frac{c}{n^\alpha} \sum_{k \in M_n} \int_{4k+1}^{4k+3} t^{\alpha-1} dt. \quad (3.10)$$

It is absolutely clear that

$$\int_{4k-1}^{4k+1} t^{\alpha-1} dt \leq \int_{4k+1}^{4k+3} t^{\alpha-1} dt, \quad \forall k \in M_n. \quad (3.11)$$

Taking into account this relation, we obtain

$$\|S_n^v\|_p^p \geq \frac{c}{n^\alpha} \sum_{k \in M_n} \left[\int_{4k+1}^{4k+3} t^{\alpha-1} dt + \int_{4k-1}^{4k+1} t^{\alpha-1} dt \right] \geq \frac{c}{n^\alpha} \int_0^{\lambda_n} t^{\alpha-1} dt = c \left(\frac{\lambda_n}{n} \right)^\alpha, \quad (3.12)$$

where $\lambda_n \geq (n\delta/\pi) - 2$. Consequently

$$\|S_n^v\|_p^p \geq c \left(\frac{\delta}{\pi} - \frac{2}{n} \right)^\alpha \longrightarrow c \left(\frac{\delta}{\pi} \right)^\alpha > 0, \quad n \longrightarrow \infty. \quad (3.13)$$

It follows immediately that $\inf_n \|S_n^v\|_p > 0$.

Regarding biorthogonal system we get

$$\|S_n^{-v}\|_q^q = \int_0^\pi v^{-q}(t) |\sin nt|^q dt. \quad (3.14)$$

Choose $\varepsilon > 0$ as small as the interval $[0, \varepsilon]$ does not contain the points $\{\alpha_k\}_1^{r-1}$. Consequently

$$\|S_n^{-v}\|_q^q = m \int_0^\varepsilon t^{-\alpha_0 q} |\sin nt|^q dt, \quad (3.15)$$

where $m > 0$ is some constant. We have

$$\|S_n^{-v}\|_q^q \geq mn^{\alpha_0 q - 1} \int_0^{n\varepsilon} \frac{|\sin t|^q}{t^{\alpha_0 q}} dt. \quad (3.16)$$

First we consider the case $\alpha_0 \in ((1/q), (1/q) + 1)$. In this case, for sufficiently great n , we have

$$\|S_n^{-v}\|_q^q \geq mn^{\alpha_0 q - 1} \int_0^1 \frac{|\sin t|^q}{t^{\alpha_0 q}} dt \longrightarrow \infty, \quad n \longrightarrow \infty. \quad (3.17)$$

Let $\alpha_0 = (1/q)$. Consequently

$$\|S_n^{-v}\|_q^q \geq m \int_0^{n\varepsilon} \frac{|\sin t|^q}{t} dt, \quad (3.18)$$

and, as a result

$$\begin{aligned} \sup_n \|S_n^{-\nu}\|_q^q &\geq m \int_1^{+\infty} \frac{|\sin t|^q}{t} dt \geq m \sum_{k=1}^{\infty} \int_{k\pi+(\pi/4)}^{(k+1)\pi-(\pi/4)} \frac{|\sin t|^q}{t} dt \\ &\geq mc_1 \sum_{k=1}^{\infty} \frac{1}{(k+1)\pi - (\pi/4)} \int_{k\pi+(\pi/4)}^{(k+1)\pi-(\pi/4)} 1 dt = c_2 \sum_{k=1}^{\infty} \frac{1}{(k+1)\pi - (\pi/4)} = +\infty, \end{aligned} \quad (3.19)$$

where c_i are some constants. So we obtain that for $\beta \in [(1/q), (1/q) + 1)$, $\sup_n \|S_n^{-\nu}\|_q = +\infty$. Consequently, in this case we have

$$\sup_n \|S_n^{\nu}\|_p \|S_n^{-\nu}\|_q = +\infty. \quad (3.20)$$

Then it is known that (see, e.g., [22]) the system $\{S_n(\nu)\}_{n \in \mathbb{N}}$ is not uniformly minimal and, besides, does not form a basis for L_p .

Consider the case $\beta \in (-(1/p) - 1, -(1/p)]$. Without limiting the generality, we will suppose that $\beta = \alpha_0$. In this case, with regard to the biorthogonal system we have

$$\|S_n^{-\nu}\|_q^q = \int_0^{\pi} \nu^{-q}(t) |\sin nt|^q dt \leq \int_0^{\pi} \nu^{-q}(t) dt < +\infty, \quad \forall n \in \mathbb{N}. \quad (3.21)$$

Taking sufficiently small $\delta > 0$, we obtain

$$\|S_n^{-\nu}\|_q^q \geq \int_0^{\delta} \nu^{-q}(t) |\sin nt|^q dt \geq c \int_0^{\delta} t^{-\alpha_0 q} |\sin nt|^q dt. \quad (3.22)$$

As $\alpha_0 q < 0$, then, in the absolutely same way as in the previous case, we get

$$\inf_n \|S_n^{-\nu}\|_q > 0. \quad (3.23)$$

On the other hand,

$$\|S_n^{\nu}\|_p^p = \int_0^{\pi} \nu^p(t) |\sin nt|^p dt \geq c \int_0^{\delta} \frac{|\sin nt|^p}{t^{-\alpha_0 p}} dt, \quad (3.24)$$

where $\alpha_0 p \leq 1$. Similarly to the previous case again, we get

$$\int_0^{\delta} t^{\alpha_0 p} |\sin nt|^p dt \longrightarrow +\infty, \quad n \longrightarrow \infty. \quad (3.25)$$

As a result we obtain

$$\sup_n \|S_n^{\nu}\|_p \|S_n^{-\nu}\|_q = +\infty, \quad \text{for } \beta \in \left(-\frac{1}{p} - 1, -\frac{1}{p}\right]. \quad (3.26)$$

Thus, the following theorem is true.

Theorem 3.2. Let $\{\alpha_0; \alpha_r\} \subset (-(1/p) - 1, -(1/p) + 2)$; $\{\alpha_0; \alpha_r\} \cap M_p^{(0)} \neq \emptyset$, and $A_r \subset (-(1/p), (1/q))$, where $M_p^{(0)} \equiv (-(1/p) - 1, -(1/p)] \cup [-(1/p) + 1, -(1/p) + 2)$, $A_r \equiv \{\alpha_k\}_1^{r-1}$. Then the system $\{S_n^v\}_{n \in \mathbb{N}}$ is complete and minimal in $L_p(0, \pi)$, $1 \leq p < +\infty$, but does not form a basis for it.

4. Defective Case

Here, we consider the defective system of sines $\{S_n^v\}_{n \in \mathbb{N}_{(k_0)}}$, where $\mathbb{N}_{(k_0)} \equiv \mathbb{N} \setminus \{k_0\}$, $k_0 \in \mathbb{N}$ is some number. It follows directly from Theorem 3.2 that if the condition

$$\{\alpha_0; \alpha_r\} \cap M_p^{(0)} \neq \emptyset; \quad A_r \subset \left(-\frac{1}{p}, \frac{1}{q}\right), \quad (4.1)$$

holds, then the system $\{S_n^v\}_{n \in \mathbb{N}_{(k_0)}}$ is minimal but not complete in $L_p(0, \pi)$, $1 \leq p < +\infty$. Assume $M_p^{(1)} \equiv [(1/q) + 1, (1/q) + 3)$. Let $\alpha_0 \in M_p^{(1)}$. Consider the completeness of system $\{S_n^v\}_{n \in \mathbb{N}_{(k_0)}}$ in $L_p(0, \pi)$. Suppose that $f \in L_q(0, \pi)$ is orthogonal to the system, that is,

$$\int_0^\pi v(t) \sin nt \overline{f(t)} dt = 0, \quad \forall n \in \mathbb{N}_{(k_0)}. \quad (4.2)$$

As $v f \in L_1 \subset C_0^*[0, \pi]$ and system $\{\sin nt\}_{n \in \mathbb{N}}$ is complete and minimal in $C_0[0, \pi]$, from (4.2) we get

$$v(t)f(t) = c \sin k_0 t \implies f(t) = c \frac{\sin k_0 t}{v(t)}. \quad (4.3)$$

It is clear that $v^{-1}(t) \sim t^{-\alpha_0}$, $\sin k_0 t \sim t$ as $t \rightarrow 0$. Consequently, $f \sim t^{-\alpha_0+1}$, $t \rightarrow 0$. As $q(-\alpha_0 + 1) \leq -1$, then $f \in L_q(0, \pi)$ if and only if $c = 0$, and, consequently, $f = 0$. The similar result is true for $\alpha_r \in M_p^{(1)}$. Thus, if $\alpha_0; \alpha_r > -(1/p) - 1$, and $\max\{\alpha_0; \alpha_r\} \in M_p^{(1)}$, then the system $\{S_n^v\}_{n \in \mathbb{N}_{(k_0)}}$ is complete in $L_p(0, \pi)$. Now we consider the minimality of this system. Let

$$\vartheta_n(t) = v^{-1}(t) \left[\frac{\sin nx}{n} - \frac{\sin k_0 x}{k_0} \right], \quad n \in \mathbb{N}_{(k_0)}. \quad (4.4)$$

We have

$$\langle S_k^v, \vartheta_n \rangle = \int_0^\pi S_k^v(t) \overline{\vartheta_n(t)} dt = \frac{\pi}{2n} \delta_{kn}, \quad \forall k, n \in \mathbb{N}_{(k_0)}. \quad (4.5)$$

Let us show that $\{\vartheta_n\}_{n \in \mathbb{N}_{(k_0)}} \subset L_q(0, \pi)$. In fact

$$\begin{aligned} \frac{\sin nt}{n} &= t - \frac{n^2 t^3}{6} + o(t^3), \quad t \longrightarrow 0, \\ \frac{\sin k_0 t}{k_0} &= t - \frac{k_0^2 t^3}{6} + o(t^3), \quad t \longrightarrow 0, \end{aligned} \quad (4.6)$$

and, consequently

$$\frac{\sin nt}{n} - \frac{\sin k_0 t}{k_0} = \frac{1}{6} (k_0^2 - n^2) t^3 + o(t^3), \quad t \longrightarrow 0. \quad (4.7)$$

From these relations, we immediately find that $\vartheta_n(t) \sim t^{3-\alpha_0}$, $t \rightarrow 0$. As a result, $\{\vartheta_n\}_{n \in \mathbb{N}_{(k_0)}} \subset L_q(0, \pi)$. Then the relations (4.5) imply the minimality of system $\{S_n^v\}_{n \in \mathbb{N}_{(k_0)}}$ in $L_p(0, \pi)$. Similar result is true for $\alpha_r \in M_p^{(1)}$. In the end, we obtain that if $\max\{\alpha_0; \alpha_r\} \in M_p^{(1)}$, then the system $\{S_n^v\}_{n \in \mathbb{N}}$ has a defect equal to 1.

Consider the case when $\max\{\alpha_0; \alpha_r\} = \alpha_0 \in M_p^{(2)}$, where $M_p^{(2)} \equiv [(1/q) + 3, (1/q) + 5)$. We look at the system $\{S_n^v\}_{n \in \mathbb{N}_{(k_1; k_2)}}$, where $\mathbb{N}_{(k_1; k_2)} \equiv \mathbb{N} \setminus \{k_1; k_2\}$, $k_1; k_2 \in \mathbb{N}$, $k_1 \neq k_2$, are some numbers. Let $f \in L_q(0, \pi)$ cancel this system out, that is,

$$\langle S_n^v, f \rangle = 0, \quad \forall n \in \mathbb{N}_{(k_1; k_2)}. \quad (4.8)$$

Using the previous reasoning, we find that for some constants c_1, c_2 , the following is true:

$$f(t) = c_1 v^{-1}(t) \sin k_1 t + c_2 v^{-1}(t) \sin k_2 t. \quad (4.9)$$

Using representations

$$\sin k_i t = k_i t - \frac{k_i^3}{6} t^3 + o(t^4), \quad t \longrightarrow 0, \quad i = 1, 2, \quad (4.10)$$

we obtain

$$f(t) \sim c t^{-\alpha_0} g_1(t) + g_2(t), \quad t \longrightarrow 0, \quad (4.11)$$

($c \neq 0$ is some constant), where it can be easily seen that $g_2 \in L_q$ and

$$g_1(t) = (c_1 k_1 + c_2 k_2) t - \frac{1}{6} (c_1 k_1^3 + c_2 k_2^3) t^3. \quad (4.12)$$

Thus, $f \in L_q$ if and only if $t^{-\alpha_0} g_1(t) \in L_q$. Assume $b_1 = c_1 k_1 + c_2 k_2$; $b_2 = -(1/6)(c_1 k_1^3 + c_2 k_2^3)$. As $(1 - \alpha_0)q \leq -1(3 - \alpha_0)q \leq -1$, it is clear that $t^{1-\alpha_0} \notin L_q$ and $t^{3-\alpha_0} \notin L_q$. Suppose $b_1 \neq 0$. We have

$$|t^{-\alpha_0} g_1(t)| = |b_1| t^{1-\alpha_0} |1 + b_1^{-1} b_2 t^2|. \quad (4.13)$$

It follows directly that for sufficiently small $\delta > 0$, we have

$$|t^{-\alpha_0} g_1(t)| \geq c_\delta |b_1| t^{1-\alpha_0}, \quad \forall t \in (0, \delta), \quad (4.14)$$

where $c_\delta > 0$ is some constant depending only on δ and b_2 . As a result, $f \notin L_q$.

Consequently, $b_1 = 0$. Moreover, it is not difficult to derive that $b_2 = 0$. Thus, we obtain the following system for c_1 and c_2 :

$$\begin{aligned} c_1 k_1 + c_2 k_2 &= 0, \\ c_1 k_1^3 + c_2 k_2^3 &= 0. \end{aligned} \quad (4.15)$$

It is clear that $\det \begin{vmatrix} k_1 & k_2 \\ k_1^3 & k_2^3 \end{vmatrix} \neq 0$. And, consequently, $c_1 = c_2 = 0$. As a result, $f = 0$, which, in turn, implies that the system $\{S_n^\nu\}_{n \in \mathbb{N}_{(k_1; k_2)}}$ is complete in L_p . Let us show that it is also minimal in L_p . Assume $\gamma_n^{(k)} = (1/6)(k^2 - n^2)$. Consider the system $\vartheta_n^{(k_1; k_2)}(t) \equiv [a_n^{(k_1; k_2)} \sin nt - (1/k_1 \gamma_n^{(k_1)}) \sin k_1 t + (1/k_2 \gamma_n^{(k_2)}) \sin k_2 t] \nu^{-1}(t)$, for all $n \in \mathbb{N}_{(k_1; k_2)}$, where

$$a_n^{(k_1; k_2)} = \frac{1}{n} \left(\frac{1}{\gamma_n^{(k_1)}} - \frac{1}{\gamma_n^{(k_2)}} \right) \neq 0. \quad (4.16)$$

Simple calculations give the following representation:

$$\vartheta_n^{(k_1; k_2)}(t) \sim c_n t^5, \quad t \rightarrow 0, \quad (4.17)$$

where $c_n \neq 0$, for all $n \in \mathbb{N}_{(k_1; k_2)}$ are some constants. We obtain directly from this representation that $\{\vartheta_n^{(k_1; k_2)}\}_{n \in \mathbb{N}_{(k_1; k_2)}} \subset L_q$. On the other hand,

$$\langle S_n^\nu, \vartheta_n^{(k_1; k_2)} \rangle = a_n^{(k_1; k_2)} \delta_{nk} = \begin{cases} \neq 0, & n \neq k, \\ = 0, & n = k, \end{cases} \quad \forall n, k \in \mathbb{N}_{(k_1; k_2)}. \quad (4.18)$$

Thus, if $\alpha_0 \in M_p^{(2)}$, then the system $\{S_n^\nu\}_{n \in \mathbb{N}_{(k_1; k_2)}}$ is complete and minimal in L_p , and, as a result, the system $\{S_n^\nu\}_{n \in \mathbb{N}}$ has a defect equal to 2. It is easy to see that the similar result is true if $\alpha_r \in M_p^{(2)}$ with $\alpha_0 \leq \alpha_r$. Continuing this way, we obtain that if $\beta = \max\{\alpha_0; \alpha_r\} = \alpha_0 \in M_p^{(k)}$, where $M_p^{(k)} \equiv [-(1/p) + 2k, -(1/p) + 2(k+1))$, then the system $\{S_n^\nu\}_{n \in \mathbb{N}_{(\bar{n}_k)}}$ is complete and minimal in L_p , where $\{\bar{n}_k\} = \{n_1; \dots; n_k\} \subset \mathbb{N}$, $n_i \neq n_j$ with $i \neq j$.

Consider the basicity of system $\{S_n^\nu\}_{n \in \mathbb{N}_{(\bar{n}_1)}}$ (i.e., the case of $k = 1$) in L_p . Similar to the case of $M_p^{(0)}$, it can be proved that

$$0 < \inf_{n \in \mathbb{N}_{(\bar{n}_1)}} \|S_n^\nu\|_p \leq \sup_{n \in \mathbb{N}_{(\bar{n}_1)}} \|S_n^\nu\|_p < +\infty. \quad (4.19)$$

Concerning biorthogonal system, we have

$$\int_0^\pi v^{-q}(x) \left| \frac{\sin nx}{n} - \frac{\sin k_1 x}{k_1} \right|^q dx \geq c \int_0^\varepsilon x^{-q\alpha_0} \left| \frac{\sin nx}{n} - \frac{\sin n_1 x}{n_1} \right|^q dx, \quad (4.20)$$

where the interval $[0, \varepsilon]$ ($\varepsilon > 0$) does not contain the points $\{t_k\}_1^{r-1}$. As $(\sin n_1 x / n_1) - x \sim x^3$, for $x \rightarrow 0$, it is clear that $(-q\alpha_0 + 3 > -1)$

$$\int_0^\varepsilon x^{-q\alpha_0} \left| \frac{\sin n_1 x}{n_1} - x \right|^q dx < +\infty. \quad (4.21)$$

Taking this circumstance into account, we have

$$\|S_n^{-\nu}\|_q^q \geq c \left[\int_0^\varepsilon x^{-q\alpha_0} \left| \frac{\sin nx}{n} - x \right|^q dx - \int_0^\varepsilon x^{-q\alpha_0} \left| \frac{\sin n_1 x}{n_1} - x \right|^q dx \right]. \quad (4.22)$$

Consider the case of $\alpha_0 \in ((1/q) + 1, (1/q) + 3)$:

$$\begin{aligned} \int_0^\varepsilon x^{-q\alpha_0} \left| \frac{\sin nx}{n} - x \right|^q dx &= n^{-q} \int_0^\varepsilon x^{-q\alpha_0} |\sin nx - nx|^q dx \\ &= n^{-\alpha} \int_0^{n\varepsilon} t^{-q\alpha_0} |\sin nt - t|^q dt \\ &\geq cn^{-\alpha} \int_0^{n\varepsilon} t^{-q\alpha_0+q} dt = \frac{c}{\alpha} (\varepsilon^\alpha - n^{-\alpha}), \end{aligned} \quad (4.23)$$

where $\alpha = -\alpha_0 q + q + 1 < 0$. Consequently, $\sup_n \|S_n^{-\nu}\|_q = +\infty$. Let $\alpha_0 = (1/q) + 1$. In this case we have

$$\int_0^\varepsilon x^{-q\alpha_0} \left| \frac{\sin nx}{n} - x \right|^q dx \geq c \int_1^{n\varepsilon} t^{-1} dt = c \ln n\varepsilon, \quad (4.24)$$

and, consequently $\sup_n \|S_n^{-\nu}\|_q = +\infty$. As a result, we get that for $\alpha_0 \in M_p^{(1)}$, the system $\{S_n^\nu\}_{n \in \mathbb{N}(\bar{n}_1)}$ does not form a basis for L_p . Assume that in this case, the system $\{S_n^\nu\}_{n \in \mathbb{N}}$ is a frame in L_p , that is any function from L_p can be expanded with respect to this system. As it does not form a basis for L_p , zero has a non trivial decomposition, that is,

$$0 = \sum_{n=1}^{\infty} a_n S_n^\nu, \quad (4.25)$$

where $\exists n_0 \in \mathbb{N} : a_{n_0} \neq 0$. As the system $\{S_n^\nu\}_{n \in \mathbb{N}(\bar{n}_1)}$ is complete and minimal in L_p , it is clear that $a_{n_1} \neq 0$. Consequently, $S_{n_1}^\nu = \sum_{n \neq n_1} (a_n / a_{n_1}) S_n^\nu$. It follows directly that the arbitrary element can be expanded with respect to the system $\{S_n^\nu\}_{n \in \mathbb{N}(\bar{n}_1)}$. But this is impossible. Similar result is true for $\max\{\alpha_0; \alpha_r\} \in M_p^{(1)}$.

Proceeding in an absolutely similar way as we did in the previous case, we can prove that for $\max\{\alpha_0; \alpha_r\} \in M_p^{(k)}$, the system $\{S_n^v\}_{n \in \mathbb{N}(\bar{n}_k)}$ is complete and minimal in L_p , but does not form a basis for it. Consequently, system $\{S_n^v\}_{n \in \mathbb{N}}$ has a defect equal to (k) . The fact that it is not a frame in L_p in this case too is proved as follows. Let $k = 2 : \{\bar{n}_k\} \equiv \{n_1; n_2\}$. Assume that the system $\{S_n\}_{n \in \mathbb{N}}$ is a frame in L_p . Then zero has a non trivial decomposition: $0 = \sum_{n=1}^{\infty} a_n S_n^v$. It is clear that $|a_{n_1}| + |a_{n_2}| > 0$, and let $a_{n_1} \neq 0$. It follows directly that the system $\{S_n\}_{n \neq n_1}$ is a frame in L_p . The further reasoning is absolutely similar to the case of $k = 1$. This scheme is applicable for for all $k \in \mathbb{N}$. Thus, we have proved the following main theorem.

Theorem 4.1. *Let the following necessary condition be satisfied*

$$\alpha_0; \alpha_r \in \left(-\frac{1}{p} - 1, +\infty\right), \quad \{\alpha_k\}_1^{r-1} \subset \left(-\frac{1}{p}; \frac{1}{q}\right). \quad (4.26)$$

Then the system $\{S_n^v\}_{n \in \mathbb{N}}$ is a frame (basis) in L_p if and only if $\alpha_0; \alpha_r \in (-(1/p), (1/q))$. Moreover, for $\max\{\alpha_0; \alpha_r\} \in M_p^{(k)}$, $k \in \mathbb{N}$, it has a defect equal to (k) , where $M_p^{(k)} \equiv [(1/q) + 2k, (1/q) + 2(k+1))$.

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