Research Article

On Functions of Bounded (p, k) -Variation

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We introduce and study the concept of (p, k) -variation ($1 < p < \infty$, $k \in \mathbb{N}$) of a real function on a compact interval. In particular, we prove that a function $u : [a, b] \rightarrow \mathbb{R}$ has bounded (p, k) -variation if and only if $u^{(k-1)}$ is absolutely continuous on $[a, b]$ and $u^{(k)}$ belongs to $L_p[a, b]$. Moreover, an explicit connection between the (p, k) -variation of u and the L_p -norm of $u^{(k)}$ is given which is parallel to the classical Riesz formula characterizing functions in the spaces $RV_p[a, b]$ and $A_p[a, b]$. This may also be considered as an alternative characterization of the one variable Sobolev space $W_p^k[a, b]$.

1. Introduction

About 120 years ago, Jordan [1] introduced the notion of a function of bounded variation and the corresponding function space $BV[a, b]$. He also proved the important result that $u \in BV[a, b]$ if and only if $u = f - g$ with both f and g being monotonically increasing. Later the concept of bounded variation was generalized in various directions. In 1908, De la Vallée Poussin [2] introduced the space $BV^2[a, b]$ of functions with bounded second variation. It is known that $u \in BV^2[a, b]$ if and only if u can be represented in the form $u = f - g$, where both f and g are convex.

This was further generalized by Popoviciu [3] who introduced, for any $k \in \mathbb{N}$, the notion of k th variation and defined the corresponding space $BV^k[a, b]$ of functions $u : [a, b] \rightarrow \mathbb{R}$ of bounded k th variation on $[a, b]$. It is known that, for any $u \in BV^k[a, b]$, the derivative $u^{(k-2)}$ belongs to $BV^2[a, b]$; therefore, there exist the right- and left-hand derivatives $u_+^{(k-1)}$ and $u_-^{(k-1)}$ on $[a, b]$. Moreover, the set E of points x where $u^{(k-1)}(x)$ does not exist is at most countable, the derivative $u^{(k-1)}$ is continuous on $[a, b] \setminus E$, and the unilateral right derivative $u_+^{(k-1)}$ and the unilateral left derivative $u_-^{(k-1)}$ are right continuous and left continuous, respectively.

In [4], the first author defined and studied the notion of the so-called bounded $(p, 2)$ -variation ($1 < p < \infty$) and proved a generalization of the well known Riesz lemma. More precisely, a function $u : [a, b] \rightarrow \mathbb{R}$ has bounded $(p, 2)$ -variation if and only if $u' \in AC[a, b]$, where $AC[a, b]$ denotes the space of all absolutely continuous functions on $[a, b]$, and $u'' \in L_p[a, b]$. Moreover, the $(p, 2)$ -variation of u on $[a, b]$ is then given by the formula

$$V_{(p,2)}(u; [a, b]) = \|u''\|_{L_p}^p. \quad (1.1)$$

In this paper we will prove the following parallel result: given $p \in (1, \infty)$ and $k \in \mathbb{N}$, a function $u : [a, b] \rightarrow \mathbb{R}$ has bounded (p, k) -variation on $[a, b]$ if and only if $u^{(k-1)} \in AC[a, b]$ and $u^{(k)} \in L_p[a, b]$. Moreover, the (p, k) -variation of u on $[a, b]$ is then given by

$$V_{(p,k)}(u; [a, b]) = \frac{\|u^{(k)}\|_{L_p}^p}{((k-1)!)^p}. \quad (1.2)$$

This characterization can be considered as a natural generalization of the classical Riesz lemma [5] for the class $A_p[a, b]$ of all functions $u : [a, b] \rightarrow \mathbb{R}$ such that $u \in AC[a, b]$ and $u' \in L_p[a, b]$. We point out that the Riesz lemma provides a criterion for functions to belong to the Sobolev space $W_p^1[a, b]$ more than 20 years before this space has been introduced by Sobolev [6] in 1934.

2. Preliminaries

In this section, we recall some definitions and known results concerning the Riesz p -variation, the De la Vallée Poussin second variation, and the Popoviciu k th variation.

Given a function $u : [a, b] \rightarrow \mathbb{R}$ and a partition $\pi := \{a = t_0 < t_1 < \dots < t_m = b\}$ of $[a, b]$, consider the expression

$$\sigma_p(u; \pi) = \sum_{j=1}^m \frac{|u(t_j) - u(t_{j-1})|^p}{|t_j - t_{j-1}|^{p-1}}. \quad (2.1)$$

The number $V_p(u; [a, b]) := \sup_{\pi} \sigma_p(u; \pi)$, where the supremum is taken over all partitions π of $[a, b]$, is called the *Riesz p -variation* of u on $[a, b]$. In case $V_p(u; [a, b]) < \infty$, the function u is said to have *bounded p -variation*. In what follows, by $RV_p[a, b]$ we will denote the class of all functions $u : [a, b] \rightarrow \mathbb{R}$ of bounded p -variation on $[a, b]$ (in the Riesz sense).

The class $RV_p[a, b]$ is a Banach space with respect to the norm

$$\|u\|_p := |u(a)| + (V_p(u; [a, b]))^{1/p}. \quad (2.2)$$

The following characterization of functions $u \in RV_p[a, b]$ is known in the literature as the *Riesz lemma* [5].

Proposition 2.1. *A function u belongs to $RV_p[a, b]$ ($1 < p < \infty$) if and only if $u \in A_p[a, b]$, that is, $u \in AC[a, b]$ and $u' \in L_p[a, b]$. Moreover, the equality*

$$V_p(u; [a, b]) = \|u'\|_{L_p}^p \quad (2.3)$$

holds in this case.

In 1908, De la Vallée Poussin [2] introduced the class of functions of bounded second variation as follows. Given a function $u : [a, b] \rightarrow \mathbb{R}$ and a partition π of $[a, b]$ of the form

$$a = a_1 < c_1 \leq d_1 < b_1 \leq a_2 < \cdots < b_{m-1} \leq a_m < c_m \leq d_m < b_m = b, \quad (2.4)$$

consider the expression

$$\sigma^2(u; \pi) := \sum_{j=1}^m \left| \frac{u(b_j) - u(d_j)}{b_j - d_j} - \frac{u(c_j) - u(a_j)}{c_j - a_j} \right| \quad (2.5)$$

and let

$$V^2(u; [a, b]) = \sup_{\pi} \sigma^2(u; \pi), \quad (2.6)$$

where the supremum is taken over all partitions π of the form (2.4). The number $V^2(u; [a, b])$ is called the *De La Vallée Poussin second variation* of u on $[a, b]$. In case $V^2(u; [a, b]) < \infty$, the function u is said to have *bounded second variation*. In what follows, by $BV^2[a, b]$ we will denote the class of all functions $u : [a, b] \rightarrow \mathbb{R}$ of bounded second variation on $[a, b]$.

The following result has been proved in [7], see also [8].

Proposition 2.2. *Every function $u \in BV^2[a, b]$ is Lipschitz continuous on $[a, b]$ and can be expressed as a difference of two convex functions.*

If $u \in BV^2[a, b]$, then from standard properties of convex functions it follows that the unilateral right derivative u'_+ and the unilateral left derivative u'_- exist on $[a, b]$. Moreover, the set E of points x where $u'(x)$ fails to exist is countable, u' is continuous on $[a, b] \setminus E$, the right unilateral derivative u'_+ is right continuous, and the left unilateral derivative u'_- is left continuous on $[a, b]$.

Finally, let us recall Popoviciu's more general definition of bounded k th variation [3]. To this end, we need the concept of the k th *divided difference* of a function $u : [a, b] \rightarrow \mathbb{R}$ with respect to distinct points $t_0, t_1, \dots, t_k \in [a, b]$ (not necessarily in increasing order) defined by

$$u[t_0, t_1, \dots, t_k] := \sum_{j=0}^k \frac{u(t_j)}{(t_j - t_0) \cdots (t_j - t_{j-1})(t_j - t_{j+1}) \cdots (t_j - t_k)}. \quad (2.7)$$

By definition, the k th divided difference (2.7) is independent of the order in which the points $t_0, t_1, \dots, t_k$ appear. The following two results are well known (see, e.g., [3, 9]).

Proposition 2.3. Suppose that $u : [a, b] \rightarrow \mathbb{R}$ belongs to $C^k[a, b]$ for some $k \geq 1$. Then,

$$u[t_0, t_1, \dots, t_k] := \frac{u^{(k)}(\xi)}{k!} \quad (2.8)$$

for some $\xi \in \text{conv}\{t_0, t_1, \dots, t_k\}$, where $\text{conv}M$ denotes the smallest interval containing M .

Proposition 2.4. Suppose that $u : [a, b] \rightarrow \mathbb{R}$ is such that $u^{(k)}$ is Riemann integrable on $[a, b]$ for some $k \geq 1$. Then,

$$u[t_0, t_1, \dots, t_k] = \int_0^1 \int_0^{x_1} \dots \int_0^{x_{k-1}} u^{(k)}[x_k(t_k - t_{k-1}) + \dots + x_1(t_1 - t_0) + t_0] dx_k \dots dx_1. \quad (2.9)$$

If the function u has a Riemann integrable derivative of order $k \geq 1$, then the last result allows us to generalize the concept of the k th divided difference for points $t_0, t_1, \dots, t_k$ which are not necessarily distinct. Moreover, if the unilateral derivatives $u_+^{(k-1)}$ and $u_-^{(k-1)}$ are right and left continuous on $[a, b]$, respectively, then the function of $k+1$ variables $(t_0, t_1, \dots, t_k) \mapsto u[t_0, t_1, \dots, t_k]$ is continuous in each variable separately.

As a consequence of the source Propositions 2.3 and 2.4 we have that, if $u \in C^k[a, b]$ for some $k \geq 1$, then

$$\lim_{h \rightarrow 0+} u[t_0, \dots, t_0, t_0 + h] = \frac{u^{(k)}(t_0)}{k!} = \lim_{h \rightarrow 0+} u[t_0 - h, t_0, \dots, t_0]. \quad (2.10)$$

However, as a corollary to Proposition 2.4 we can obtain a weaker version of the preceding results.

Proposition 2.5. Suppose that $u : [a, b] \rightarrow \mathbb{R}$ is such that $u^{(k)}$ is Riemann integrable on $[a, b]$ for some $k \geq 1$. Assume, in addition, that the unilateral right derivative $u_+^{(k-1)}$ is right continuous and the unilateral left derivative $u_-^{(k-1)}$ is left continuous on $[a, b]$. Then,

$$\begin{aligned} \lim_{h \rightarrow 0+} u[t_0, \dots, t_0, t_0 + h] &= \frac{u_+^{(k)}(t_0)}{k!} (a < t_0 < b), \\ \lim_{h \rightarrow 0+} u[t_0 - h, t_0, \dots, t_0] &= \frac{u_-^{(k)}(t_0)}{k!} (a < t_0 < b). \end{aligned} \quad (2.11)$$

Let $u : [a, b] \rightarrow \mathbb{R}$ be a function and π a partition of the form

$$\begin{aligned} a &= t_{1,1} < t_{1,2} < \dots < t_{1,k} \leq t_{1,k+1} < \dots < t_{1,2k} \leq \dots < t_{2,k+1} < \dots < t_{2,2k} \\ &\leq \dots < t_{3,1} < \dots < t_{j,1} < \dots < \dots < t_{j,k} \leq t_{j,k+1} < \dots < t_{j,2k} \leq \dots < t_{m-1,2k} \\ &\leq t_{m-1} < \dots < t_{m,k} \leq t_{m,k+1} < \dots < t_{m,2k} = b. \end{aligned} \quad (2.12)$$

Moreover, let

$$\sigma^k(u; \pi) := \sum_{j=1}^m |u[t_{j,k+1}, \dots, t_{j,2k}] - u[t_{j,1}, \dots, t_{j,k}]|, \quad (2.13)$$

$$V^k(u; [a, b]) := \sup_{\pi} \sigma^k(u; \pi), \quad (2.14)$$

where the supremum is taken over all partitions π of the form (2.12). The number $V^k(u; [a, b])$ is called the *Popoviciu k th variation* of u on $[a, b]$. In case $V^k(u; [a, b]) < \infty$, the function u is said to have *bounded k th variation* on $[a, b]$ and the set of such functions is denoted by $BV^k[a, b]$.

The following result on the regularity of higher derivatives is also well known [7].

Proposition 2.6. *Suppose that $u : [a, b] \rightarrow \mathbb{R}$ belongs to $BV^k[a, b]$ for some $k \geq 2$. Then, the derivative $u^{(k-2)}$ belongs to $BV^2[a, b]$; therefore, the unilateral right derivative $u_+^{(k-1)}$ and the unilateral left derivative $u_-^{(k-1)}$ exist on $[a, b]$. Moreover, the set E of points x where the derivative $u^{(k-1)}(x)$ fails to exist is countable, $u^{(k-1)}$ is continuous on $[a, b] \setminus E$, the unilateral right derivative $u_+^{(k-1)}$ is right continuous, and the unilateral left derivative $u_-^{(k-1)}$ is left continuous on $[a, b]$.*

3. Main Result

In what follows, for $1 \leq p < \infty$ and $k \in \mathbb{N}$ we denote by $A_{(p,k)}[a, b]$ the class of all functions $u : [a, b] \rightarrow \mathbb{R}$ such that $u^{(k-1)} \in AC[a, b]$ and $u^{(k)} \in L_p[a, b]$. Thus, $A_{(p,1)}[a, b] = A_p[a, b]$.

In this section we introduce the notion of (p, k) -variation, where p always denotes a real number in $[1, \infty)$ and $k \geq 1$ denotes a natural number. For $p > 1$ we will prove that a function $u : [a, b] \rightarrow \mathbb{R}$ has bounded (p, k) -variation (for the definition see below) if and only if $u \in A_{(p,k)}[a, b]$. Moreover, we will show that equality (1.2) holds, as one should expect.

So given $u : [a, b] \rightarrow \mathbb{R}$, consider a partition π of the form

$$\begin{aligned} a = t_{1,1} &< t_{1,2} < \dots < t_{1,k} \leq t_{1,k+1} < \dots < t_{1,2k} \leq \dots < t_{2,k+1} < \dots < t_{2,2k} \\ &\leq \dots < t_{3,1} < \dots < t_{j,1} < \dots < t_{j,k} \leq t_{j,k+1} < \dots < t_{j,2k} \leq \dots < t_{m-1,2k} \\ &\leq t_{m-1} < \dots < t_{m,k} \leq t_{m,k+1} < \dots < t_{m,2k} = b. \end{aligned} \quad (3.1)$$

Moreover, define

$$\begin{aligned} \sigma_{(p,k)}(u; \pi) &:= \sum_{j=1}^m \frac{|u[t_{j,k+1}, \dots, t_{j,2k}] - u[t_{j,1}, \dots, t_{j,k}]|^p}{|t_{j,2k} - t_{j,1}|^{p-1}}, \\ V_{(p,k)}(u; [a, b]) &:= \sup_{\pi} \sigma_{(p,k)}(u; \pi), \end{aligned} \quad (3.2)$$

where the supremum is taken over all partitions π of the form (3.1). We call the number $V_{(p,k)}(u; [a, b])$ the (p, k) -variation of u on $[a, b]$. In case $V_{(p,k)}(u; [a, b]) < \infty$, we say that the function u has *bounded (p, k) -variation*, and we denote the space of such functions by $RV_{(p,k)}[a, b]$.

Proposition 3.1. For $k \geq 1$ and $p \in (1, \infty)$, the inclusion

$$RV_{(p,k)}[a, b] \subseteq BV^k[a, b] \quad (3.3)$$

and the inequality

$$V^k(u; [a, b]) \leq (b - a)^{1-1/p} V_{(p,k)}(u; [a, b])^{1/p} \quad (3.4)$$

hold true, where $V^k(u; [a, b])$ is given by (2.14).

Proof. Let π be a partition of $[a, b]$ of the form (3.1). By Hölder's inequality, we obtain

$$\begin{aligned} & \sum_{j=1}^m \frac{|u[t_{j,k+1}, \dots, t_{j,2k}] - u[t_{j,1}, \dots, t_{j,k}]|}{|t_{j,2k} - t_{j,1}|} |t_{j,2k} - t_{j,1}| \\ & \leq \left(\sum_{j=1}^m \frac{|u[t_{j,k+1}, \dots, t_{j,2k}] - u[t_{j,1}, \dots, t_{j,k}]|}{|t_{j,2k} - t_{j,1}|^{p-1}} \right)^{1/p} \left(\sum_{j=1}^m |t_{j,2k} - t_{j,1}| \right)^{1-1/p}. \end{aligned} \quad (3.5)$$

Passing on both sides of to the supremum over all partitions π of the form (3.1), we conclude that (3.4) is true, and so also (3.3), by definition of $BV^k[a, b]$. $\square$

From Proposition 2.6 and Proposition 3.1 we immediately deduce the following.

Corollary 3.2. Let $1 < p < \infty$, $k \geq 2$, and $u \in RV_{(p,k)}[a, b]$. Then, the derivative $u^{(k-2)}$ belongs to $BV^2[a, b]$; therefore, the unilateral right and the unilateral left derivatives $u_+^{(k-1)}$ and $u_-^{(k-1)}$ exist on $[a, b]$ and are right and left continuous, respectively.

Now we prove a statement on the existence of the derivative $u^{(k-1)}(t_0)$ for all $t_0 \in (a, b)$ in case $u \in RV_{(p,k)}[a, b]$.

Proposition 3.3. Let $1 < p < \infty$, $k \geq 2$, and $u \in RV_{(p,k)}[a, b]$. Then, the derivative $u^{(k-1)}(t)$ exists for all $t \in (a, b)$.

Proof. Suppose that there exists $x_0 \in (a, b)$ such that $\alpha_{x_0} := |u_+^{(k-1)}(x_0) - u_-^{(k-1)}(x_0)| > 0$. Let π be a partition of $[a, b]$ of the form

$$\begin{aligned} a &= t_{1,1} < t_{1,2} < \dots < t_{1,k} \leq t_{1,k+1} < \dots < t_{1,2k} \leq \dots < t_{j,1} = x_0 - h \dots \\ &< t_{j,2} < \dots < t_{j,k} < \dots < t_{j,2k-1} = x_0 \\ &< t_{j,2k} = x_0 + h < \dots < t_{m,1} \leq t_{m,2} < \dots < t_{m,k} \leq t_{m,k+1} < \dots < t_{m,2k} = b, \end{aligned} \quad (3.6)$$

where h satisfies

$$0 < h < \min \left\{ \frac{t_{j,2k} - t_{j,1}}{2} : j = 1, 2, \dots, m \right\}. \quad (3.7)$$

By definition of the (p, k) -variation of u , we have then

$$V_{(p,k)}(u; [a, b]) \geq \frac{|u[t_{j,k+1}, t_{j,k+2}, \dots, t_{j,2k-2}, x_0, x_0 + h] - u[x_0 - h, t_{j,2}, \dots, t_{j,k}]|^p}{|2h|^{p-1}}. \quad (3.8)$$

From Corollary 3.2 we know that the unilateral right derivative $u_+^{(k-1)}$ and the unilateral left derivative $u_-^{(k-1)}$ are right and left continuous, respectively, on $[a, b]$. Now, by Corollary 3.2, we further have

$$\begin{aligned} V_{(p,k)}(u; [a, b]) &\geq \lim_{h \rightarrow 0+} \lim_{t_{j,2} \rightarrow x_0} \frac{|u[t_{j,k+1}, t_{j,k+2}, \dots, t_{j,2k-2}, x_0, x_0 + h] - u[x_0 - h, t_{j,2}, \dots, t_{j,k}]|^p}{|2h|^{p-1}} \\ &= \frac{\alpha_{x_0}^p}{2^{p-1}((k-1)!)^p} \lim_{h \rightarrow 0+} \frac{1}{h^{p-1}} = \infty, \end{aligned} \quad (3.9)$$

contradicting our assumption $u \in RV_{(p,k)}$. Consequently, the function u has a derivative $u^{(k-1)}$ everywhere on (a, b) . $\square$

We are now in a position to formulate and prove our main result.

Theorem 3.4. *Let $1 < p < \infty$ and $k \geq 2$. Then, $u \in RV_{(p,k)}[a, b]$ if and only if $u \in A_{(p,k)}[a, b]$. Moreover, in this case the (p, k) -variation of u on $[a, b]$ is given by (1.2).*

Proof. Suppose first that $u : [a, b] \rightarrow \mathbb{R}$ belongs to $RV_{(p,k)}[a, b]$. By Proposition 3.3, we know then that the derivative $u^{(k-1)}$ exists on $[a, b]$.

Let $\pi = \{a = t_0 < t_1 < \dots < t_m = b\}$ be a partition of $[a, b]$. For any subinterval $[t_j, t_{j+1}]$, $j = 0, 1, \dots, m-1$, we define a partition π_j in the form

$$t_j = t_{j,1} < t_{j,2} < \dots < t_{j,k-1} < t_{j,k} = t_j + h < \dots < t_{j,k+1} = t_{j+1} - h < t_{j,k+2} < \dots < t_{j,2k} = t_{j+1}, \quad (3.10)$$

where

$$h := \min \left\{ \frac{t_{j+1} - t_j}{2} : j = 1, 2, \dots, m-1 \right\}. \quad (3.11)$$

Using the shortcut

$$U_{j,k}(\pi_j; h) := \frac{|u[t_{j+1} - h, t_{j,k+2}, \dots, t_{j,2k-1}] - u[t_j, t_{j,2}, \dots, t_{j,k-1}, t_j + h]|^p}{|t_{j+1} - t_j|^{p-1}}, \quad (3.12)$$

by Corollary 3.2, we have then

$$\frac{|u^{(k-1)}(t_{j+1}) - u^{(k-1)}(t_j)|^p}{|t_{j+1} - t_j|^{p-1}} = \lim_{h \rightarrow 0+} \lim_{t_{j,k-1} \rightarrow t_j} \lim_{t_{j,k+2} \rightarrow t_{j+1}} U_{j,k}(\pi_j; h). \quad (3.13)$$

Consequently,

$$\frac{1}{((k-1)!)^p} \sum_{j=1}^m \frac{|u^{(k-1)}(t_{j+1}) - u^{(k-1)}(t_j)|^p}{|t_{j+1} - t_j|^{p-1}} = \lim_{h \rightarrow 0+} \lim_{t_{j,k-1} \rightarrow t_j} \lim_{t_{j,k+2} \rightarrow t_{j+1}} U_{j,k}(\pi_j; h) \leq V_{(p,k)}(u; [a, b]). \quad (3.14)$$

Hence, by the Riesz lemma, we see that $u^{(k-1)} \in AC[a, b]$ and $u^{(k)} \in L_p[a, b]$, that is, $u \in A_{(p,k)}[a, b]$. Moreover,

$$\int_a^b \frac{|u^{(k)}(t)|^p}{((k-1)!)^p} dt \leq V_{(p,k)}(u; [a, b]). \quad (3.15)$$

Conversely, suppose now that $u : [a, b] \rightarrow \mathbb{R}$ belongs to $A_{(p,k)}[a, b]$, and let π be a partition of $[a, b]$ of the form (3.10). Since $u^{(k-1)} \in AC[a, b]$ and $u^{(k)} \in L_p[a, b]$, by Proposition 2.4, we have

$$\begin{aligned} \sum_{j=1}^m \frac{|u[t_{j,k+1}, \dots, t_{j,2k}] - u[t_{j,1}, \dots, t_{j,k}]|^p}{|t_{j,2k} - t_{j,1}|^{p-1}} &= \sum_{j=1}^m \frac{|u^{(k-1)}(\xi_{j,2k}) - u^{(k-1)}(\xi_{j-1,k})|^p}{((k-1)!)^p |t_{j,2k} - t_{j,1}|^{p-1}} \\ &= \sum_{j=1}^m \left| \int_{\xi_{j-1,k}}^{\xi_{j,2k}} u^{(k)}(\sigma) d\sigma \right|^p \frac{1}{((k-1)!)^p |t_{j,2k} - t_{j,1}|^{p-1}}, \end{aligned} \quad (3.16)$$

where $\xi_{j,2k} \in \text{conv}\{t_{j,k+1}, \dots, t_{j,2k}\}$ and $\xi_{j-1,k} \in \text{conv}\{t_{j,1}, \dots, t_{j,k}\}$. By Hölder's inequality, we get then the estimates

$$\begin{aligned} &\sum_{j=1}^m \left| \int_{\xi_{j-1,k}}^{\xi_{j,2k}} u^{(k)}(t) dt \right|^p \frac{1}{((k-1)!)^p |t_{j,2k} - t_{j,1}|^{p-1}} \\ &\leq \sum_{j=1}^m \int_{\xi_{j-1,k}}^{\xi_{j,2k}} |u^{(k)}(t)|^p dt \frac{|\xi_{j,2k} - \xi_{j-1,k}|^{p-1}}{((k-1)!)^p |t_{j,2k} - t_{j,1}|^{p-1}} \\ &\leq \sum_{j=1}^m \int_{\xi_{j-1,k}}^{\xi_{j,2k}} \frac{|u^{(k)}(t)|^p}{((k-1)!)^p} dt \leq \sum_{j=1}^m \int_{t_{j,1}}^{t_{j,2k}} \frac{|u^{(k)}(t)|^p}{((k-1)!)^p} dt = \int_a^b \frac{|u^{(k)}(t)|^p}{((k-1)!)^p} dt. \end{aligned} \quad (3.17)$$

This implies that $u \in RV_{(p,k)}[a, b]$ and

$$\int_a^b \frac{|u^{(k)}(t)|^p}{((k-1)!)^p} dt \geq V_{(p,k)}(u; [a, b]). \quad (3.18)$$

Combining estimates (3.15) and (3.18), we conclude that (1.2) is true, and so the proof is complete. $\square$

The characterization of functions in $RV_{(p,k)}[a, b]$ included in Theorem 3.4 can be considered as a natural generalization of that given by Riesz [5] for the class $A_p[a, b]$ ($1 < p < \infty$), and by Merentes [4] for the class $A_{(p,2)}[a, b]$.

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