Research Article

On Univalence Criteria for a General Integral Operator

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We consider a new general integral operator, and we give sufficient conditions for the univalence of this integral operator in the open unit disk of the complex plane. Several consequences of the main results are also shown.

1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane and let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad (1.1)$$

that are analytic in U and satisfy the usual normalization conditions $f(0) = f'(0) - 1 = 0$. We denote by S the subclass of $\mathcal{A}$ consisting of all univalent functions f in U and consider the class P of functions h that are analytic in U and that satisfy $h(0) = 1$ and $\operatorname{Re} h(z) > 0$ for all $z \in U$.

In the present paper we obtain sufficient conditions for the following general integral operator to be in the class S (The univalent functions are of importance in geometric functions theory and may have some applications in fluid mechanics and physics):

$$H_{\alpha_1, \dots, \alpha_n, \beta, g_1, \dots, g_n, h_1, \dots, h_p}(z) = \left\{ \beta \int_0^z u^{\beta-1} \prod_{i=1}^n g_i'(u)^{\alpha_i} \prod_{j=1}^p h_j(u) du \right\}^{1/\beta}, \quad (1.2)$$

where α_i, β are complex numbers, $\beta \neq 0$, the functions $g_i(u) \in \mathcal{A}$ for all $i = 1, 2, \dots, n$ and $h_j(u) \in P$ for all $j = 1, 2, \dots, p$, where n, p are positive integers.

For proving our main results we need the following theorems.

Theorem 1.1 (see [1]). *Let α be a complex number, let $\operatorname{Re} \alpha > 0$; and let $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad (1.3)$$

for all $z \in U$, then for any complex number β with $\operatorname{Re} \beta \geq \operatorname{Re} \alpha$, the function

$$F_\beta(z) = \left\{ \beta \int_0^z u^{\beta-1} f'(u) du \right\}^{1/\beta} \quad (1.4)$$

is in the class S .

Theorem 1.2 (see [2]). *If the function $g(z)$ is regular in U and $|g(z)| < 1$ in U , then for all $\xi \in U$ and $z \in U$, the following inequalities hold:*

$$\left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)} \cdot g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \overline{z} \cdot \xi} \right|, \quad (1.5)$$

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}. \quad (1.6)$$

These are equalities if and only if $g(z) = \varepsilon(z + u)/(1 + \bar{u}z)$, where $|\varepsilon| = 1$ and $|u| < 1$.

Remark 1.3 (see [2]). For $z = 0$, from inequality (1.5) we have

$$\left| \frac{g(\xi) - g(0)}{1 - \overline{g(0)} \cdot g(\xi)} \right| \leq |\xi|, \quad (1.7)$$

and hence,

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)| \cdot |\xi|}. \quad (1.8)$$

Writing $g(0) = a$ and letting $\xi = z$, we get

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a| \cdot |z|}, \quad (1.9)$$

for all $z \in U$.

2. Main Results

Theorem 2.1. Let $\delta > 0$. For $i = 1, 2, \dots, n$, let α_i be a complex number, let $M_i > 0$, and let $g_i \in \mathcal{A}$. For $j = 1, 2, \dots, p$, let $N_j > 0$ and $h_j \in P$. If

$$\left| \frac{zg_i''(z)}{g_i'(z)} \right| \leq M_i, \quad \forall i = 1, 2, \dots, n \quad (z \in U), \quad (2.1)$$

$$\left| \frac{zh_j'(z)}{h_j(z)} \right| \leq N_j, \quad \forall j = 1, 2, \dots, p \quad (z \in U), \quad (2.2)$$

$$\sum_{i=1}^n |\alpha_i| \cdot M_i + \sum_{j=1}^p N_j \leq \delta, \quad (2.3)$$

then for every complex number β with $\operatorname{Re} \beta \geq \delta > 0$, the integral operator $H_{\alpha_1, \dots, \alpha_n, \beta, g_1, \dots, g_n, h_1, \dots, h_p}(z)$ given by (1.2) is in the class S .

Proof. Let us define the function:

$$f(z) = \int_0^z \prod_{i=1}^n g_i'(u)^{\alpha_i} \prod_{j=1}^p h_j(u) du, \quad (2.4)$$

with $g_i \in \mathcal{A}$, for all $i = 1, 2, \dots, n$ and $h_j \in P$, for all $j = 1, 2, \dots, p$, and, thus, we obtain

$$f'(z) = \prod_{i=1}^n g_i'(z)^{\alpha_i} \prod_{j=1}^p h_j(z). \quad (2.5)$$

The function f is regular in U and $f(0) = f'(0) - 1 = 0$. We have

$$\frac{zf''(z)}{f'(z)} = \sum_{i=1}^n \alpha_i \frac{zg_i''(z)}{g_i'(z)} + \sum_{j=1}^p \frac{zh_j'(z)}{h_j(z)} \quad (z \in U). \quad (2.6)$$

From (2.1), (2.2), and (2.6) we obtain

$$\frac{1 - |z|^{2\delta}}{\delta} \cdot \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1 - |z|^{2\delta}}{\delta} \left(\sum_{i=1}^n |\alpha_i| \cdot M_i + \sum_{j=1}^p N_j \right) \quad (z \in U), \quad (2.7)$$

and by (2.3), we have

$$\frac{1 - |z|^{2\delta}}{\delta} \cdot \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad \forall z \in U. \quad (2.8)$$

Using (2.8), by Theorem 1.1, it results that the integral operator $H_{\alpha_1, \dots, \alpha_n, \beta, g_1, \dots, g_n, h_1, \dots, h_p}(z)$ given by (1.2) is in the class S . $\square$

Letting $p = 1$ in Theorem 2.1, we have the following.

Corollary 2.2. *Let $\delta > 0$, let $N > 0$, and let $h \in P$. For $i = 1, 2, \dots, n$, let α_i be a complex number, let $M_i > 0$, and let $g_i \in \mathcal{A}$. If*

$$\begin{aligned} \left| \frac{zg_i''(z)}{g_i'(z)} \right| &\leq M_i, \quad \forall i = 1, 2, \dots, n \quad (z \in U), \\ \left| \frac{zh'(z)}{h(z)} \right| &\leq N \quad (z \in U) \\ \sum_{i=1}^n |\alpha_i| \cdot M_i + N &\leq \delta, \end{aligned} \quad (2.9)$$

then for every complex number β with $\operatorname{Re} \beta \geq \delta > 0$, the integral operator

$$F_{\alpha_1, \dots, \alpha_n, \beta, g_1, \dots, g_n, h}(z) = \left\{ \beta \int_0^z u^{\beta-1} \prod_{i=1}^n g_i'(u)^{\alpha_i} h(u) du \right\}^{1/\beta} \quad (2.10)$$

is in the class S .

Letting $n = 1$ in Theorem 2.1, we have the following.

Corollary 2.3. *Let $\delta > 0$, let $M > 0$, let $g \in \mathcal{A}$, and let α be a complex number. For $j = 1, 2, \dots, p$, let $N_j > 0$ and let $h_j \in P$. If*

$$\begin{aligned} \left| \frac{zg''(z)}{g'(z)} \right| &\leq M \quad (z \in U), \\ \left| \frac{zh_j'(z)}{h_j(z)} \right| &\leq N_j, \quad \forall j = 1, 2, \dots, p \quad (z \in U), \\ |\alpha| \cdot M + \sum_{j=1}^p N_j &\leq \delta, \end{aligned} \quad (2.11)$$

then for every complex number β with $\operatorname{Re} \beta \geq \delta > 0$, the integral operator

$$G_{\alpha, \beta, g, h_1, \dots, h_p}(z) = \left\{ \beta \int_0^z u^{\beta-1} g'(u)^\alpha \prod_{j=1}^p h_j(u) du \right\}^{1/\beta} \quad (2.12)$$

is in the class S .

For $M_1 = M_2 = \dots = M_n = M$ and $N_1 = N_2 = \dots = N_p = N$ in Theorem 2.1, we have the following.

Corollary 2.4. *Let $\delta > 0$, let $M > 0$, and let $N > 0$. For $i = 1, 2, \dots, n$, let α_i be a complex number and let $g_i \in \mathcal{A}$. For $j = 1, 2, \dots, p$, let $h_j \in P$. If*

$$\begin{aligned} \left| \frac{zg_i''(z)}{g_i'(z)} \right| &\leq M, \quad \forall i = 1, 2, \dots, n \ (z \in U), \\ \left| \frac{zh_j'(z)}{h_j(z)} \right| &\leq N, \quad \forall j = 1, 2, \dots, p \ (z \in U), \\ M \sum_{i=1}^n |\alpha_i| + pN &\leq \delta, \end{aligned} \quad (2.13)$$

then for every complex number β with $\operatorname{Re} \beta \geq \delta > 0$, the integral operator $H_{\alpha_1, \dots, \alpha_n, \beta, g_1, \dots, g_n, h_1, \dots, h_p}(z)$ given by (1.2) is in the class S .

Theorem 2.5. *Let $\delta > 0$. For $i = 1, 2, \dots, n$, let α_i be a complex number, let $M_i > 0$, and let $g_i \in \mathcal{A}$, $g_i(z) = z + a_2^i z^2 + a_3^i z^3 + \dots$. For $j = 1, 2, \dots, p$, let $N_j > 0$ and $h_j \in P$ with $h_j'(0) = 0$, and $c = \sum_{i=1}^n \alpha_i a_2^i / |\prod_{i=1}^n \alpha_i|$. If*

$$\left| \frac{g_i''(z)}{g_i'(z)} \right| < M_i, \quad \forall i = 1, 2, \dots, n \ (z \in U), \quad (2.14)$$

$$\left| \frac{h_j'(z)}{h_j(z)} \right| < N_j, \quad \forall j = 1, 2, \dots, p \ (z \in U), \quad (2.15)$$

$$\frac{\sum_{i=1}^n |\alpha_i| M_i + \sum_{j=1}^p N_j}{|\prod_{i=1}^n \alpha_i|} < 1, \quad (2.16)$$

$$\left| \prod_{i=1}^n \alpha_i \right| \leq \frac{1}{\max_{|z|<1} \left[\left((1 - |z|^{2\delta}) / \delta \right) \cdot |z| \cdot (|z| + 2|c|) / (1 + 2|c| \cdot |z|) \right] }, \quad (2.17)$$

then for every complex number β with $\operatorname{Re} \beta \geq \delta > 0$, the integral operator $H_{\alpha_1, \dots, \alpha_n, \beta, g_1, \dots, g_n, h_1, \dots, h_p}(z)$ given by (1.2) is in the class S .

Proof. We define the function:

$$f(z) = \int_0^z \prod_{i=1}^n g_i'(u)^{\alpha_i} \prod_{j=1}^p h_j(u) du \quad (z \in U), \quad (2.18)$$

with $g_i \in \mathcal{A}$, for all $i = 1, 2, \dots, n$ and $h_j \in P$, for all $j = 1, 2, \dots, p$.

We consider the function

$$K(z) = \frac{1}{|\prod_{i=1}^n \alpha_i|} \cdot \frac{f''(z)}{f'(z)}, \quad \forall z \in U. \quad (2.19)$$

We have

$$\frac{f''(z)}{f'(z)} = \sum_{i=1}^n \alpha_i \frac{g_i''(z)}{g_i'(z)} + \sum_{j=1}^p \frac{h_j'(z)}{h_j(z)} \quad (z \in U). \quad (2.20)$$

From (2.14), (2.15), and (2.20) we obtain

$$\frac{1}{|\prod_{i=1}^n \alpha_i|} \cdot \left| \frac{f''(z)}{f'(z)} \right| \leq \frac{\sum_{i=1}^n |\alpha_i| M_i + \sum_{j=1}^p N_j}{|\prod_{i=1}^n \alpha_i|} \quad (z \in U). \quad (2.21)$$

From (2.16), (2.19), and (2.21) we obtain $|K(z)| < 1$ for all $z \in U$.

We have $K(0) = 2 \sum_{i=1}^n \alpha_i a_2^i / |\prod_{i=1}^n \alpha_i| = 2c$, and, using Remark 1.3 we get

$$|K(z)| \leq \frac{|z| + 2|c|}{1 + 2|c| \cdot |z|} \quad (z \in U). \quad (2.22)$$

From (2.19) and (2.22), we obtain

$$\frac{1 - |z|^{2\delta}}{\delta} \cdot \left| \frac{zf''(z)}{f'(z)} \right| \leq \left| \prod_{i=1}^n \alpha_i \right| \cdot \max_{|z| < 1} \left[\frac{1 - |z|^{2\delta}}{\delta} \cdot |z| \cdot \frac{|z| + 2|c|}{1 + 2|c| \cdot |z|} \right], \quad (2.23)$$

for all $z \in U$.

From (2.17) and (2.23), we have

$$\frac{1 - |z|^{2\delta}}{\delta} \cdot \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (z \in U). \quad (2.24)$$

So, applying Theorem 1.1, we obtain that the integral operator $H_{\alpha_1, \dots, \alpha_n, \beta, g_1, \dots, g_n, h_1, \dots, h_p}(z)$ given by (1.2) is in the class S . $\square$

Letting $p = 1$ in Theorem 2.5, we have the following.

Corollary 2.6. Let $\delta > 0$, let $N > 0$, and let $h \in P$ with $h'(0) = 0$. For $i = 1, 2, \dots, n$, let α_i be a complex number, $M_i > 0$ and $g_i \in \mathcal{A}$, $g_i(z) = z + a_2^i z^2 + a_3^i z^3 + \dots$, and $c = \sum_{i=1}^n \alpha_i a_2^i / |\prod_{i=1}^n \alpha_i|$. If

$$\begin{aligned} \left| \frac{g_i''(z)}{g_i'(z)} \right| &< M_i, \quad \forall i = 1, 2, \dots, n \quad (z \in U), \\ \left| \frac{h'(z)}{h(z)} \right| &< N \quad (z \in U), \quad \frac{\sum_{i=1}^n |\alpha_i| M_i + N}{|\prod_{i=1}^n \alpha_i|} < 1, \\ \left| \prod_{i=1}^n \alpha_i \right| &\leq \frac{1}{\max_{|z|<1} \left[\left((1 - |z|^{2\delta}) / \delta \right) \cdot |z| \cdot ((|z| + 2|c|) / (1 + 2|c| \cdot |z|)) \right]}, \end{aligned} \quad (2.25)$$

then for every complex number β with $\operatorname{Re} \beta \geq \delta > 0$, the integral operator

$$F_{\alpha_1, \dots, \alpha_n, \beta, g_1, \dots, g_n, h}(z) = \left\{ \beta \int_0^z u^{\beta-1} \prod_{i=1}^n g_i'(u)^{\alpha_i} h(u) du \right\}^{1/\beta} \quad (2.26)$$

is in the class S .

Letting $n = 1$ in Theorem 2.5, we have the following.

Corollary 2.7. Let $\delta > 0$, $M > 0$, $g \in \mathcal{A}$, $g(z) = z + a_2 z^2 + a_3 z^3 + \dots$ and let α be a complex number. For $j = 1, 2, \dots, p$, let $N_j > 0$ and $h_j \in P$ with $h_j'(0) = 0$. If

$$\begin{aligned} \left| \frac{g''(z)}{g'(z)} \right| &< M \quad (z \in U), \\ \left| \frac{h_j'(z)}{h_j(z)} \right| &< N_j, \quad \forall j = 1, 2, \dots, p \quad (z \in U), \\ |\alpha| \cdot M + \sum_{j=1}^p N_j &< |\alpha| \leq \frac{1}{\max_{|z|<1} \left[\left((1 - |z|^{2\delta}) / \delta \right) \cdot |z| \cdot ((|z| + 2|a_2|) / (1 + 2|a_2| \cdot |z|)) \right]}, \end{aligned} \quad (2.27)$$

then for every complex number β with $\operatorname{Re} \beta \geq \delta > 0$, the integral operator

$$G_{\alpha, \beta, g, h_1, \dots, h_p}(z) = \left\{ \beta \int_0^z u^{\beta-1} g'(u)^\alpha \prod_{j=1}^p h_j(u) du \right\}^{1/\beta} \quad (2.28)$$

is in the class S .

For $M_1 = M_2 = \dots = M_n = M$ and $N_1 = N_2 = \dots = N_p = N$ in Theorem 2.5, we have the following.

Corollary 2.8. Let $\delta > 0$, $M > 0$, $N > 0$. For $i = 1, 2, \dots, n$, let α_i be a complex number and $g_i \in \mathcal{A}$, $g_i(z) = z + a_2^i z^2 + a_3^i z^3 + \dots$. For $j = 1, 2, \dots, p$, let $h_j \in P$ with $h_j'(0) = 0$, and $c = \sum_{i=1}^n \alpha_i a_2^i / |\prod_{i=1}^n \alpha_i|$.
If

$$\begin{aligned} \left| \frac{g_i''(z)}{g_i'(z)} \right| &< M \quad \forall i = 1, 2, \dots, n \quad (z \in U), \\ \left| \frac{h_j'(z)}{h_j(z)} \right| &< N \quad \forall j = 1, 2, \dots, p \quad (z \in U), \\ \frac{M \sum_{i=1}^n |\alpha_i| + pN}{|\prod_{i=1}^n \alpha_i|} &< 1, \end{aligned} \tag{2.29}$$

$$\left| \prod_{i=1}^n \alpha_i \right| \leq \frac{1}{\max_{|z|<1} \left[\left(\left(1 - |z|^{2\delta} \right) / \delta \right) \cdot |z| \cdot (|z| + 2|c|) / (1 + 2|c| \cdot |z|) \right]},$$

then for every complex number β with $\operatorname{Re} \beta \geq \delta > 0$, the integral operator $H_{\alpha_1, \dots, \alpha_n, \beta, g_1, \dots, g_n, h_1, \dots, h_p}(z)$ given by (1.2) is in the class S .

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