Research Article

On the Regularity of Solutions to an Adjoint Elliptic Equation with Partially VMO Coefficients

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We establish, in dimension two, a regularity result for nonnegative solutions to an adjoint elliptic equation, generalizing a previous result of Escauriaza (1994). We consider elliptic equations with coefficients $a_{ij}(x_1, x_2)$ which are measurable with respect to one variable and VMO with respect to the other.

1. Introduction

Let us consider a planar elliptic operator of nondivergence form:

$$
\mathcal{M} = \sum_{ij} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \quad (1.1)
$$

where $a_{ij} = a_{ji}$ for $i, j = 1, 2$ are measurable and the symmetric matrix

$$
A(x) = \begin{pmatrix}
a_{11}(x) & a_{12}(x) \\
a_{12}(x) & a_{22}(x)
\end{pmatrix} \quad (1.2)
$$

is uniformly elliptic, that is,

$$
\frac{|\xi|^2}{\sqrt{K}} \leq A(x)\xi, \xi \leq \sqrt{K}|\xi|^2, \quad (1.3)
$$
for all \( \xi \in \mathbb{R}^2 \) and a.e. \( x = (x_1, x_2) \in \Omega \), a bounded open subset of \( \mathbb{R}^2 \). Here the ratio \( \sqrt{K}/\sqrt{1} = K \) is the ellipticity constant.

The study of weak solutions \( v \) to the adjoint equation (adjoint solutions, for short)

\[
\mathcal{M}^*[v] = \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x)v(x)) = 0
\]  

(1.4)

often occurs in the literature (see Section 3, and for a very recent paper, see [1]).

We say that the function \( v \in L^1_{\text{loc}}(\Omega) \) is a weak solution to (1.4) if

\[
\int_{\Omega} v \mathcal{M} [\varphi] = 0 \quad \forall \varphi \in C^\infty_0(\Omega).
\]  

(1.5)

In this paper we make the assumption that the coefficients \( a_{ij}(x_1, x_2) \) are VMO with respect to one of the two variables (see Section 2). This kind of assumption has been recently considered mainly for divergence \( (L[u] = \text{div}(A(x)\nabla u) = 0) \) or nondivergence \( (M[w] = \text{Tr}(A(x)D^2w) = 0) \) elliptic equations.

On the other hand, in [2] Escauriaza gave a regularity result for nonnegative solutions to adjoint equation with VMO coefficients.

Here, in case \( n = 2 \), we give a generalized form of Theorem 1.2 in which he proves that, in particular,

\[
v \in \bigcap_{q > 1} G_q,
\]  

(1.6)

where \( G_q \) is the Gehring class, as defined in Section 2.

2. Definitions and Notations

In order to describe the results of the present paper, it is necessary to introduce some definitions. We start recalling basic definitions of the \( G_q \) classes, introduced by Gehring [3], in connection with local integrability properties of the gradient of quasiconformal mappings.

Let us assume that \( v \) is a weight, that is, a nonnegative locally integrable function on \( \mathbb{R}^2 \) and consider cubes \( Q \subset \mathbb{R}^2 \) with sides parallel to the coordinate axes. We will set

\[
v_Q = \int_Q v(x)dx = \frac{1}{|Q|} \int_Q v(x)dx
\]  

(2.1)

to denote the mean value of \( v \) over \( Q \), where \( |Q| \) denotes the 2-dimensional Lebesgue measure of a subset \( Q \) of \( \mathbb{R}^2 \).
Definition 2.1. A weight $v$ satisfies the $G_q$-condition if there exists a constant $G \geq 1$ such that, for all cubes $Q \subset \mathbb{R}^2$ as above, one has

$$\left(\frac{\int_Q v^q(x) \, dx}{\int_Q v(x) \, dx}\right)^{1/q} \leq G,$$

and one refers to (2.2) as a “reverse” Hölder inequality.

In the following, we will consider elliptic differential equations with coefficients $a_{ij}(x)$ of the matrix $A$ measurable with respect to one variable and vanishing mean oscillation (VMO) with respect to the other (we say partially-VMO, for short). We recall that the space VMO, introduced by Sarason [4], is a subspace of the functions in the John-Nirenberg space BMO. More precisely, VMO is defined as the closure in BMO of the subspace of uniformly continuous functions.

Definition 2.2. A locally integrable function $f : \mathbb{R}^2 \to \mathbb{R}$ is in VMO if

$$\lim_{r \to 0} \int_{B_r} |f(y) - f_{B_r}| \, dy = 0,$$

where $B_r = B(x, r)$ denotes a ball centered at $x \in \mathbb{R}^2$, with radius $r$. One will also assume that $f$ is defined at $\infty$ in the following average sense:

$$f(\infty) = \lim_{r \to \infty} \frac{1}{\pi r^2} \int_{B_r(0)} f(y) \, dy,$$

(see [5]).

3. Examples

In the present section we collect a certain number of examples where solutions $v$ to the adjoint equation

$$\mathcal{M}[v] = \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left( a_{ij}(x) v(x) \right) = 0$$

occur. The first example deals with adjoint solutions which are partial derivatives $h_{x_1}$ and $h_{x_2}$ of a very weak solution $h \in W^{1,2}_{\text{loc}}$ to a particular diagonal divergence type equation, and an interesting relation comes out between regularity results.
Example 3.1. For $1/\sqrt{K} \leq \beta \leq \sqrt{K}$, we consider the following elliptic operators in $\mathbb{R}^2$:

$$
\mathcal{M} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2},
\mathcal{L} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \left( \beta \frac{\partial}{\partial x_2} \right).
$$

(3.2)

Fortuitous relations occur between adjoint solutions to $\mathcal{M}$ and solutions $h$ to $\mathcal{L}[h] = 0$, as the following Lemma reveals (see [6]).

Lemma 3.2 (see [6]). Let $h \in W^{1,2}_{\text{loc}}(B_r)$, where $B_r$ denotes the open ball in $\mathbb{R}^2$ centered at 0 with radius $r$, such that

$$
\mathcal{L}[h] = 0.
$$

(3.3)

Set $w = \partial h/\partial x_1$, $v = \partial h/\partial x_2$. Then

$$
\mathcal{M}^*[w] = 0, \quad \mathcal{M}^*[v] = 0.
$$

(3.4)

Proof. We proceed similarly as in [6]. If $\phi \in C^\infty_0(B_r)$, we have

$$
\int_{B_r} w \cdot \mathcal{M}\phi \, dx = \int_{B_r} \frac{\partial h}{\partial x_1} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \beta \frac{\partial^2 \phi}{\partial x_2^2} \right) dx_1 dx_2
$$

$$
= \int_{B_r} \left( \frac{\partial h}{\partial x_1} \frac{\partial^2 \phi}{\partial x_1^2} + \beta \frac{\partial h}{\partial x_2} \frac{\partial^2 \phi}{\partial x_2^2} \right) dx_1 dx_2 = 0.
$$

(3.5)

Thus $\mathcal{M}^*[w] = 0$. In analogous way one checks that $\mathcal{M}^*[v] = 0$. \qed

Corollary 3.3. Let $h \in W^{1,2}_{\text{loc}}$ such that $\mathcal{L}[h] = 0$. If $\partial h/\partial x_1 \geq 0$ and $\partial h/\partial x_2 \geq 0$, then for any ball $B_r \subset B_{2r} \subset \mathbb{R}^2$, one has

$$
\left( \int_{B_r} |\nabla h|^p \, dx \right)^{1/p} \leq c(K,p) \int_{B_r} |\nabla h| \, dx,
$$

(3.6)

where $2 \leq p < 2K/(K-1)$.

Proof. See [7, Theorem 3.1]. \qed

Compare this with the following well-known result of Astala [8] (see also Leonetti-Nesi [9]).

Theorem 3.4 (see [8, 9]). Let $h \in W^{1,2}_{\text{loc}}(\Omega)$ be a local solution to the equation

$$
\text{div}(A(x) \nabla h(x)) = 0, \quad x \in \Omega,
$$

(3.7)
where $A$ is a real symmetric matrix satisfying the ellipticity bounds,

$$
\frac{|\xi|^2}{\sqrt{K}} \leq \langle A(x)\xi, \xi \rangle \leq \sqrt{K}|\xi|^2, \quad \forall \xi \in \mathbb{R}^2, \text{ for a.e. } x \in \Omega.
$$

Then, for any ball $B_{2r} \subset \Omega$ one has

$$
\left( \int_{B_r} |\nabla h|^s \, dx \right)^{1/s} \leq c(K,s) \int_{B_r} |\nabla h| \, dx,
$$

where $2 \leq s < 2\sqrt{K}/(\sqrt{K} - 1)$.

Notice that, while the exponent in the left-hand side of the reverse inequality (3.9) may be greater than the exponent in the reverse inequality (3.6), this one is stronger in another sense, because it involves the same support $B_r$ at both sides.

**Example 3.5.** In [10] (see also [11]) the Jacobian $v = \det DU$, where $U : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is a locally univalent $A$-harmonic mapping; that is, its components are $W^{1,2}_{loc}$ solution to (3.7), is shown to be solution to an adjoint equation for the elliptic operator

$$
\mathcal{M}[v] = \frac{\partial^2 v}{\partial x_1^2} + c \frac{\partial^2 v}{\partial x_2^2},
$$

where $1/K \leq c \leq K$.

**Example 3.6.** Very recently [12, 13], the reduced Beltrami differential equation

$$
\frac{\partial f}{\partial \bar{z}} = \lambda(z) \partial m \left( \frac{\partial f}{\partial z} \right), \quad |\lambda(z)| \leq k < 1, \quad k = \frac{K - 1}{K + 1}
$$

has been introduced and studied because it naturally arises in different contexts in the theory of quasiconformal mappings. It turns out that the partial derivatives of the components $u, v$ of $f(z) = u(z) + iv(z)$ solution to (3.11) satisfy the equation

$$
u_{x_2} = \frac{b(z) - 1}{b(z) + 1} v_{x_1}
$$

with $b(z) = \partial m \lambda(z)$. As a consequence of (3.12) in [13], it is proved that $u_{x_2} \neq 0$ a.e, and it is an adjoint solution for a suitable elliptic operator $\mathcal{M} = \sum_{ij} b_{ij}(z)(\partial^2 / \partial x_i \partial x_j)$.

Namely, it has been proved [13] that $u$ is a solution to an elliptic equation of divergence form

$$
\text{div} \ A(z) \nabla u = 0,
$$
where $A(z)$ is of the type

$$A(z) = \begin{pmatrix} 1 & a_{12}(z) \\ 0 & a_{22}(z) \end{pmatrix}.$$  \hfill (3.14)

As a consequence, the function $v = u_{x_2}$ is a solution to the adjoint equation

$$\mathcal{M}^*[v] = 0,$$  \hfill (3.15)

where

$$\mathcal{M} = \frac{\partial^2}{\partial x_1^2} + a_{12} \frac{\partial^2}{\partial x_1 \partial x_2} + a_{22} \frac{\partial^2}{\partial x_2^2}.$$  \hfill (3.16)

Note that the matrix $A(z)$ is not symmetric; however, the operator $\mathcal{M}$ can also be represented by the symmetric and uniformly elliptic matrix

$$B(x) = \begin{pmatrix} 1 & \frac{a_{12}(z)}{2} \\ \frac{a_{12}(z)}{2} & a_{22}(z) \end{pmatrix}.$$  \hfill (3.17)

Notice also that $v = u_{x_2} > 0$ a.e. (see [13]), and moreover, by general properties of nonnegative adjoint solutions, $v$ satisfies a reverse Hölder inequality [7, 14, 15]

$$\left( \frac{1}{B_r} \int_{B_r} v(z)^2 \, dz \right)^{1/2} \leq c(K) \int_{B_r} v(z) \, dz,$$  \hfill (3.18)

in every ball $B_r \subset \Omega$ such that $B_{2r} \subset \Omega$. Hence $v$ is identically zero or $v > 0$ a.e. [13].

Example 3.7. The properties of the adjoint solutions are also very useful for studying the $G$-convergence of non divergence operators, as shown, for example, in a paper of D’Onofrio and Greco [16]. In that paper the authors consider elliptic operators $\mathcal{M}$ of non divergence type, defined by

$$\mathcal{M}[u] = \text{Tr} \left( AD^2 u \right), \quad \text{for } u \in W^{2,2}(\Omega), \ \Omega \subset \mathbb{R}^2,$$  \hfill (3.19)

where $A = (a_{ij}) \in \mathbb{M}$, the set of all symmetric $2 \times 2$ real matrices and satisfy the ellipticity condition (1.3).

The adjoint to the operator $\mathcal{M}$ is given by $\mathcal{M}^*[v] = (a_{11}v)_{x_1} + 2(a_{12}v)_{x_1x_2} + (a_{22}v)_{x_2x_2}$ and reveals useful behaviour with respect to $G$-convergence of sequence of operators of the form (3.19).
Proposition 3.8 (see [16]). Let $\mathcal{M}_k$, $k = 1, 2, \ldots, \mathcal{M}$ be operators whose coefficient matrices $A_k$, $A \in \mathbb{M}$ and satisfying (1.3). Assume that $v_k \in L^2(\Omega)$ are solutions to the adjoint equations $\mathcal{M}_k^* [v_k] = 0$ and verify that

$$
\begin{align*}
  v_k &\rightharpoonup v \quad \text{in} \quad L^2(\Omega), \\
  v_k A_k &\rightharpoonup v A \quad \text{in} \quad L^2(\Omega; \mathbb{M}),
\end{align*}
$$

(3.20)

where $v(x) > 0$ a.e. in $\Omega$. Then, one has $\mathcal{M}_k \overset{C}{\rightarrow} \mathcal{M}$.

In order to prove Proposition 3.8, the following lemma is crucial.

Lemma 3.9 (see [16]). Let $\mathcal{M}_k$, $k = 1, 2, \ldots, \mathcal{M}$ be operators with coefficient matrices $A_k$, $A \in \mathbb{M}$ and satisfy (1.3), $v_k \in L^2(\Omega)$ satisfying $\mathcal{M}_k^* [v_k] = 0$, and let $v_k \in W^{2,2}_{\text{loc}}(\Omega)$ be given. If

$$
\begin{align*}
  u_k &\rightharpoonup u \quad \text{in} \quad W^{2,2}_{\text{loc}}(\Omega), \\
  v_k A_k &\rightharpoonup v A \quad \text{in} \quad L^2_{\text{loc}}(\Omega; \mathbb{M}),
\end{align*}
$$

(3.21)

then

$$
\text{Tr} \left( v_k A_k D^2 u_k \right) \longrightarrow \text{Tr} \left( v A D^2 u \right)
$$

(3.22)

in the sense of distributions.

Moreover, if we consider the Hessian matrix of any $w \in W^{2,2}(\Omega)$,

$$
D^2 \omega = \begin{pmatrix}
  w_{x_1 x_1} & w_{x_1 x_2} \\
  w_{x_1 x_2} & w_{x_2 x_2}
\end{pmatrix}.
$$

(3.23)

In [17] it is proved that $w$ is a solution to

$$
\mathcal{M} [w] = \text{Tr} \left( BD^2 \omega \right) = 0,
$$

(3.24)

where $B$ is a suitable coefficient matrix, if and only if

$$
\left( w_{x_1 x_2}^2 - w_{x_1 x_1} w_{x_2 x_2} \right) \left( K + \frac{1}{K} \right) \geq w_{x_1 x_1}^2 + 2 w_{x_1 x_2}^2 + w_{x_2 x_2}^2,
$$

(3.25)

where $\sqrt{K}, K \leq 1$, is the elliptic constant. In the case where the Hessian matrix is diagonal, that is, $w_{x_1 x_2} = 0$, it is easy to see that a solution of $\mathcal{M}^* [v] = 0$ is the positive function $v = \sqrt{- \det D^2 \omega} = \sqrt{|w_{x_1 x_1} w_{x_2 x_2}|}$. 


4. The Coefficients Measurable with Respect to One Variable and VMO with Respect to the Other

It is well known that, for linear elliptic operators in nondivergence form with continuous coefficients, the $W^{2,p}$ estimates hold for all $p > 1$. It was shown that these estimates still hold in the same range when the coefficients are in VMO [18] or partially in VMO [19]. Our aim here is to generalize a regularity result of Escauriaza (Theorem 1.2, [2]) for the nonnegative adjoint solutions $v$ to

$$\mathcal{M}^* [v] = 0 \quad \text{in } \Omega \subset \mathbb{R}^2,$$

as defined in (1.4), with

$$A = (a_{ij}) = {}^t A, \quad \frac{|\xi|^2}{\sqrt{K}} \leq \langle A(x) \xi, \xi \rangle \leq \sqrt{K} |\xi|^2,$$

for a.e. $x = (x_1, x_2) \in \Omega$ and for $\xi \in \mathbb{R}^2$.

**Theorem 4.1.** If $v \in L^1(\Omega)$ is a nonnegative solution to (4.1) and the coefficient matrix $A(x)$ satisfies (4.2), and moreover

$$A(x_1, \cdot) \in \text{VMO},$$

then

$$v \in \bigcap_{q>1} G_q.$$  \hspace{1cm} (4.4)

Let us begin with the following $L^p$-global regularity (for all $p \geq 2$) result for the complex Beltrami equation

$$F_{\bar{z}} + \mu(z) F_z + \overline{\mu(z)} F_{\bar{z}} = H(z) \quad z \in \mathbb{R}^2,$$  \hspace{1cm} (4.5)

under a partially-VMO assumption on the Beltrami coefficients $\mu$, as defined in Section 2.

**Proposition 4.2.** Let $B_r = B(0, r) \subset \mathbb{R}^2$, and let $\mu(z) = \mu(x_1, x_2)$ be measurable, such that $|\mu(z)| + |\overline{\mu(z)}| = 2 |\mu(z)| \leq k < 1$ with $k = (K - 1)/(K + 1)$ and $\mu(z) = 0$ for $|z| \geq r > 0$. Moreover, assume that $\mu(x_1, \cdot) \in \text{VMO}(\mathbb{R}, \mathbb{R})$ for a.e. $x_1 \in \mathbb{R}$. Then for any $p \geq 2$ and for $H \in L^p(\mathbb{R}^2)$ ($H(z) = 0$ for $|z| > r$), there exists a unique solution $F$ to the Beltrami equation (4.5) such that $F_z \in L^p$ and

$$\|F_{\bar{z}}\|_{L^p(\mathbb{R}^2)} \leq c(p, k) \|H\|_{L^p(\mathbb{R}^2)}.$$  \hspace{1cm} (4.6)
Remark 4.3. We note that in general, elliptic Beltrami operator

\[ I - \mu T - \nu \overline{T}, \tag{4.7} \]

where \( T \) is the Beurling transform defined via the relation

\[ T(F_z) = F_z \tag{4.8} \]

under the assumption

\[ \mu, \nu \in \text{VMO}(\mathbb{C}), \tag{4.9} \]

is invertible in all \( L^p(\mathbb{C}) \) spaces, \( p > 1 \). The proof is much the same [5, 20], considering the complex Beltrami equation

\[ F_z - \mu(z)F_z - \nu(z)\overline{F_z} = h, \quad h \in L^p(\mathbb{C}). \tag{4.10} \]

The meaning of the condition (4.9) is that \( \mu \) and \( \nu \) have vanishing mean oscillation in the usual sense, that is, belong to the closure of \( C_0^\infty(\mathbb{C}) \) in \( \text{BMO}(\mathbb{C}) \) and that \( \mu \) and \( \nu \) are defined at infinity in the following average sense:

\[ \mu(\infty) = \lim_{r \to \infty} \int_{B_r(0)} \mu(z)|dz|^2, \]
\[ \nu(\infty) = \lim_{r \to \infty} \int_{B_r(0)} \nu(z)|dz|^2. \tag{4.11} \]

The following example, due to T. Iwaniec, shows that without such condition the result fails.

**Example 4.4.** There exists a function \( f \in \text{VMO}(\mathbb{R}^n) \), \( 0 \leq f(x) \leq 1 \) everywhere, such that

\[ 0 = \lim \inf_{|B| \to \infty} \int_B f(x)dx < \lim \sup_{|B| \to \infty} \int_B f(x)dx = 1, \tag{4.12} \]

where \( B \) stands for a ball centered at the origin.

**Preliminaries**

Let \( L : \mathbb{R} \to [0, 1] \) be a Lipschitz function given by

\[ L(t) = \frac{1}{2}(1 + |t| - |t - 1|). \tag{4.13} \]
The Lipschitz constant of $L$ equals 1, and, therefore, for each $\varphi \in \text{BMO}(\mathbb{R}^n)$, we have

$$
\| L \circ \varphi \|_{\text{BMO}} \leq 2\| \varphi \|_{\text{BMO}}. 
$$

(4.14)

Next, denote by $C(n)$ the BMO-norm of the function $x \rightarrow \log |x|$. We will truncate this function to make building blocks to our construction.

The Building Blocks

For a nonnegative integer $k$, we set

$$
\varphi_k(x) = L \left( 7 - 2^{-k} \log |x| \right), \\
\psi_k(x) = L \left( 5 - 2^{-k} \log |x| \right).
$$

(4.15)

We define the building block as $f_k = \varphi_k - \psi_k$. Note that each $f_k$ is continuous and supported in the ball $|x| \leq e^{7 \cdot 2^k}$, whereas $f_{k+1}$ vanishes on this ball. The BMO-norm of $f_k$ can be estimated as

$$
\| f_k \|_{\text{BMO}} \leq \| \varphi_k \|_{\text{BMO}} + \| \psi_k \|_{\text{BMO}} \leq 2 \cdot 2^{-k} C(n) + 2 \cdot 2^{-k} C(n) = 2^{-k+2} C(n). 
$$

(4.16)

Thus the infinite series

$$
f = \sum_{k=0}^{\infty} f_k
$$

(4.17)

represents a VMO function.

Computation of $L^1$-Averages

Given any positive integer $N$, we consider concentric balls $B_r \subset B_R$ centered at the origin and with radii $r = e^{k \cdot 2^N} < e^{b \cdot 2^N} = R$. Elementary geometric observation reveals that

$$
\int_{B_r} f \geq \frac{1}{|B_R|} \int_{B_R} f_N \geq \frac{1}{|B_R|} \int_{e^{3 \cdot 2^N} \leq |x| \leq e^{5 \cdot 2^N}} f_N
$$

$$
= \frac{e^{6n2^N} - e^{5n2^N}}{e^{6n2^N}} = 1 - e^{-n2^N} \rightarrow 1, \quad \text{as } N \rightarrow \infty.
$$

(4.18)
On the other hand

\[ \int_{B_R} f = \frac{1}{|B_R|} \sum_{k=1}^{N-1} \int_{|x| \leq e^{2N}} f_k(x) \, dx \leq \frac{1}{|B_R|} \sum_{k=1}^{N-1} \int_{|x| \leq e^{2N}} \, dx \]

\[ \leq \sum_{k=1}^{N-1} \frac{e^{7n^2}}{e^{kn^2}} \leq (N-1) \frac{e^{7n^2}}{e^{kn^2}} = (N-1)e^{-n^2} \rightarrow 0, \quad \text{as } N \rightarrow \infty, \]

as desired.

**Proof of Proposition 4.2.** If we set \( K = (1+k)/(1-k) \), then there exists a symmetric matrix

\[ A(z) = a_{ij}(z) \]  \hspace{1cm} (4.20)

such that

\[ \frac{I}{\sqrt{K}} \leq A(z) \leq \sqrt{K} I, \]

\[ \mu(z) = \frac{1}{2} \left[ \frac{a_{11}(z) - a_{22}(z) + 2a_{12}(z)i}{a_{11}(z) + a_{22}(z)} \right]. \]  \hspace{1cm} (4.21)

Moreover \( a_{ij}(z) = 0 \) for \( |z| \geq r \).

We may assume the following familiar normalization:

\[ \text{Tr} \, A(z) = a_{11}(z) + a_{22}(z) = 1. \]  \hspace{1cm} (4.22)

With the previous prescriptions we easily check that for \( \omega \in W^{2,1}_{\text{loc}}(B_r) \) if we define the complex gradient of \( \omega \) as

\[ F(z) = \omega_z = \frac{1}{2}(\omega x_1 - i\omega x_2), \]  \hspace{1cm} (4.23)

we have

\[ \text{Tr} \left( A(z)D^2 \omega \right) = 2 \left( F_z + \mu F_z + \bar{\mu} \bar{F}_z \right) \]  \hspace{1cm} (4.24)

(see [5]). Hence (4.5) is equivalent to

\[ \text{Tr} \left( A(z)D^2 \omega \right) = H \quad \text{in } \mathbb{R}^2 \]  \hspace{1cm} (4.25)

with coefficient matrix \( A(z) = A(x_1, x_2) \) allowed to be only measurable with respect to \( x_1 \) and VMO with respect to \( x_2 \in \mathbb{R} \), (thanks to (4.22)).
Under these assumptions, in [19, Theorem 2.4], the existence of a unique solution \( w \in W_{2,p} \) to (4.25) for \( H \in L^p \) has been established \((p \geq 2)\), together with the estimate

\[
\|D^2w\|_{L^p(B_r)} \leq c(p, k)\|H\|_{L^p(B_r)}.
\] (4.26)

Hence (4.6) follows.

Let us now give the following sharp version of the Alexandrov-Bakelman-Pucci maximum principle for non divergence elliptic operators

\[
\mathcal{M}[w] = \sum a_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j}
\] (4.27)

with partially VMO coefficients.

**Lemma 4.5.** Under the assumptions (4.2), (4.3), (4.22) on \( A \), suppose \( B_r = B(0, r) \subset \mathbb{R}^2 \) and that \( w \in W^{2,1}_{\text{loc}}(B_r) \cap C^0(B_r) \) satisfies, for \( h \in L^p(B_r), p > 1 \),

\[
\mathcal{M}[w] = h \quad \text{in } B_r,
\]

\[
w = 0 \quad \text{on } \partial B_r.
\] (4.28)

Then one has

\[
\|w\|_{L^\infty(B_r)} \leq c(K, p) r^{2-2/p}\|h\|_{L^p(B_r)}.
\] (4.29)

**Proof.** In view of [19, Theorem 2.4], we know that the Dirichlet problem (4.28) always has a unique solution \( w \in W^{2,p}(B_r) \cap W_0^{1,2}(B_r) \) for every \( h \in L^p(B_r), p > 1 \).

Define \( h(z) = 0 \) for \( z = (x_1, x_2) \in \mathbb{R}^2 \setminus B_r \) and

\[
\mu(z) = \frac{1}{2}[a_{11}(z) - a_{22}(z) + 2a_{12}(z)i],
\] (4.30)

and set \( \mu(z) = 0 \) for \( z \in \mathbb{R}^2 \setminus B_r \). According to Proposition 4.2, the equation

\[
F_\bar{z} + \mu F_z + \overline{\mu} F_{\bar{z}} = \frac{h}{2}
\] (4.31)

has unique solution \( F \in W^{1,p}_{\text{loc}}(B_r) \) such that

\[
\|F_z\|_{L^p(\mathbb{R}^2)} \leq c(p, K)\|h\|_{L^p(\mathbb{R}^2)}.
\] (4.32)

Now, let us see that

\[
F|_{B_r} \in W^{1,2}_{\text{loc}}(B_r).
\] (4.33)
Define \( f = w_z = (1/2)(w_{x_1} - i w_{x_2}) \). Then
\[
(f - F)_z + \mu(f - F)_z + \overline{\mu} (f - F)_z = 0 \quad \text{in } B_r.
\] (4.34)

This means that the mapping
\[
g = f - F
\] (4.35)
is weakly \( K \)-quasiregular, and since \( F \in W^{1,p}_{\text{loc}} \) and \( f \in W^{1,2}_{\text{loc}} \), we deduce \( g \in W^{1,p}_{\text{loc}} \) and actually
\[
g \in W^{1,2}_{\text{loc}}(B_r).
\] (4.36)

Then (4.33) follows.

Now, let us introduce the solution \( U \) to the problem
\[
\begin{align*}
\Delta U &= 4F_z, \\
U(0) &= 0.
\end{align*}
\] (4.37)

We have \( \Delta U \in L^p(\mathbb{R}^2) \cap L^2_{\text{loc}}(B_r) \) and \( \mathcal{M}[U] = h \) a.e. \( z \in \mathbb{R}^2 \). Moreover, classically
\[
\|\Delta U\|_{L^p(\mathbb{R}^2)} \leq c(K,p)\|h\|_{L^p}.
\] (4.38)

(Notice that \( c(K,p) \) is optimal in Talenti [21].)

Finally, let us introduce
\[
u = U - w.
\] (4.39)

Then \( u \) is continuous in \( \overline{B_r} \), by the Sobolev imbedding, and it is the solution to the Dirichlet problem
\[
\begin{align*}
\mathcal{M}[u] &= 0 \quad \text{a.e. } z \in B_r, \\
u_{/\partial B_r} &= U_r
\end{align*}
\] (4.40)

and \( u \in W^{2,2}_{\text{loc}}(B_r) \cap C^0(\overline{B_r}) \). By the classical maximum principle [22],
\[
\|U - w\|_{L^\infty(B_r)} = \|U\|_{L^\infty(\partial B_r)}.
\] (4.41)

Hence, we use Sobolev and the condition \( U(0) = 0 \) to conclude
\[
\|w\|_{L^\infty(B_r)} \leq 2\|U\|_{L^\infty(B_r)} \leq cr^{2-2/p}\|\Delta U\|_{L^p(B_r)} \leq cr^{2-2/p}\|h\|_{L^p(B_r)}.
\] (4.42)
Proof of Theorem 4.1. Let us fix \( q > 1 \), set \( p = q/(q-1) \), and fix a ball \( B_r \) such that \( B_{2r} \subset \Omega \). As in [7, 15], we make use of the dual formulation of the \( L^q \)-norm

\[
\left( \int_{B_r} \psi^q \right)^{1/q} = \sup \left\{ \int_{B_r} \psi h : h \geq 0, \ h \in C_0^1(B_r), \|h\|_{L^q(B^2)} \leq 1 \right\}.
\]

(4.43)

Fix \( h \in C_0^1(B_r) \), \( \|h\|_{L^q(B^2)} \leq 1 \), \( h \geq 0 \), and solve the Dirichlet problem

\[
\mathcal{M}[w] = \sum a_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} = h \quad \text{in} \ B_{2r},
\]

\[ w = 0 \quad \text{on} \ \partial B_{2r}. \quad (4.44) \]

Since \( A(x_1, \cdot) \in \text{VMO} \), the problem has a unique solution \( w \in W^{2,p}(B_{2r}) \) vanishing on \( \partial B_{2r} \), satisfying the estimate

\[
\left\| D^2 w \right\|_{L^q(B_{2r})} \leq c \|h\|_{L^p(B_{2r})},
\]

(4.45)

\[ c = c(K, p, \|A\|_{\text{VMO}}). \]

Fix a nonnegative function \( \varphi_r \in C_0^1(B_{3r/2}) \) such that \( \varphi_r = 1 \) on \( B_r \) and \( |\partial^\alpha \varphi_r/\partial x^\alpha| \leq C(\alpha)/r^{3|\alpha|} \).

Then, we have

\[
\int_{B_r} \psi h \leq \int_{B_{2r}} \psi \mathcal{M}[w] \varphi_r = - \int_{B_r} \psi \mathcal{M}[\varphi_r] - 2 \int_{B_{2r}} \psi \langle A \nabla w, \nabla \varphi_r \rangle
\]

\[ \leq \frac{c}{r^2} \|w\|_{L^q(B_{2r})} \int_{B_{3r/2}} \psi + \frac{c\sqrt{K}}{r} \int_{B_{3r/2}} |\nabla \psi|. \quad (4.46) \]

By (4.29) \( \|w\|_{L^q(B_r)} \leq c(K,p)r^{2-2/q}\|h\|_{L^q(B_{2r})} \leq c r^{2/q} \); hence, (4.46) implies

\[
\int_{B_r} \psi h \leq \frac{c}{r^2} r^{2/q} \int_{B_{3r/2}} \psi + \frac{c}{r} \left( \int_{B_{3r/2}} \psi \right)^{1/2} \left( \int_{B_{2r}} |\nabla \psi|^2 \right)^{1/2}. \quad (4.47)
\]

Now, we estimate the last integral in the right-hand side. By (1.3), one has

\[
\int_{B_{2r}} |\nabla \psi|^2 \leq \sqrt{K} \int_{B_{2r}} \psi \langle A \nabla \psi, \nabla \psi \rangle = \sqrt{K} \int_{B_{2r}} \psi \left[ \frac{M[w^2]}{2} - wh \right]. \quad (4.48)
\]

Since \( w^2 = 0 \) and \( \nabla (w^2) = 0 \) on \( \partial B_{2r} \), then we deduce

\[
\int_{B_r} \psi \mathcal{M}[w^2] = 0 \quad \text{as} \ \mathcal{M}^*[\psi] = 0. \quad (4.49)
\]
Using again (4.29) yields
\[
\int_{B_{2r}} v|\nabla w|^2 \leq 2\sqrt{K} \int_{B_{2r}} v|h| \leq 2\sqrt{K} \|w\|_{L^\infty(B_{2r})} \int_{B_r} vh
\]
\[
\leq 2\sqrt{K} r^{2/q} \int_{B_r} vh.
\]

By (4.47) and (4.50), it follows that
\[
\int_{B_r} vh \leq \frac{c}{r^{2(1-1/q)}} \int_{B_{3r/2}} v + \frac{c}{r^{2(1-1/q)}} \left( \int_{B_{3r/2}} v \right)^{1/2} \left( \int_{B_r} vh \right)^{1/2}.
\]

By elementary inequality \(\sqrt{a \sqrt{b}} \leq a/2 + b/2\), we obtain
\[
\int_{B_r} vh \leq \frac{c}{r^{2(1-1/q)}} \int_{B_{3r/2}} v + \frac{c}{r^{2(1-1/q)}} \int_{B_{3r/2}} v + \frac{1}{2} \int_{B_r} vh.
\]

Rearranging yields
\[
\int_{B_r} vh \leq \frac{c}{r^{2(1-1/q)}} \int_{B_{3r/2}} v.
\]

Since \(h\) is arbitrary, by (4.43) and (4.53), we obtain
\[
\left( \int_{B_r} v^p \right)^{1/p} \leq c \int_{B_{3r/2}} v,
\]
with \(c = c(K,p)\). An application of [14, Lemma 2.0] concludes the proof. \(\square\)

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**References**


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