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Research Article

Novel Identities for q-Genocchi Numbers and Polynomials

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The essential aim of this paper is to introduce novel identities for q-Genocchi numbers and polynomials by using the method by T. Kim et al. (article in press). We show that these polynomials are related to p-adic analogue of Bernstein polynomials. Also, we derive relations between q-Genocchi and q-Bernoulli numbers.

1. Preliminaries

Imagine that p be a fixed odd prime number. We now start with definition of the following notations. Let $\mathbb{Q}_p$ be the field p-adic rational numbers and let $\mathbb{C}_p$ be the completion of algebraic closure of $\mathbb{Q}_p$.

Thus,

$$\mathbb{Q}_p = \left\{ x = \sum_{n=-k}^{\infty} a_n p^n : 0 \leq a_n < p \right\}. \tag{1.1}$$

Then $\mathbb{Z}_p$ is integral domain, which is defined by

$$\mathbb{Z}_p = \left\{ x = \sum_{n=0}^{\infty} a_n p^n : 0 \leq a_n < p \right\}, \tag{1.2}$$

or

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}. \tag{1.3}$$

We assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ as an indeterminate. The p-adic absolute value $|\cdot|_p$, is normally defined by

$$|x|_p = p^{-r}, \tag{1.4}$$

where $x = p^r(s/t)$ with $(p,s) = (p,t) = (s,t) = 1$, and $r \in \mathbb{Q}$.

$[x]_q$ is a q-extension of x, which is defined by

$$[x]_q = \frac{1-q^x}{1-q}, \tag{1.5}$$

we note that $\lim_{q\to 1}[x]_q = x$ (see [1–17]).

Throughout this paper, we use notation of $\mathbb{N}^* := \mathbb{N}\cup\{0\}$, where $\mathbb{N}$ denotes set of Natural numbers.

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient,

$$F_f(x,y) = \frac{f(x)-f(y)}{x-y}, \tag{1.6}$$

has a limit $f'(a)$ as $(x,y) \to (a,a)$, and we denote this by $f \in UD(\mathbb{Z}_p)$. Then, for $f \in UD(\mathbb{Z}_p)$, we can start with the following expression:

$$\frac{1}{[p^N]_q}\sum_{0\le\xi<p^N} f(\xi)q^{\xi} = \sum_{0\le\xi<p^N} f(\xi)\mu_q\left(\xi + p^N\mathbb{Z}_p\right), \tag{1.7}$$

which represents a p-adic q-analogue of Riemann sums for f. The integral of f on $\mathbb{Z}_p$ will be defined as the limit $(N \to \infty)$ of these sums, when it exists. The p-adic q-integral of function $f \in UD(\mathbb{Z}_p)$ is defined by Kim in [7, 12] as

$$I_q(f) = \int_{\mathbb{Z}_p} f(\xi)d\mu_q(\xi) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{\xi=0}^{p^N-1} f(\xi)q^{\xi}. \tag{1.8}$$

The bosonic integral is considered as a bosonic limit $q \to 1$, $I_1(f) = \lim_{q\to 1} I_q(f)$. Similarly, the fermionic p-adic integral on $\mathbb{Z}_p$ is introduced by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(\xi)d\mu_{-q}(\xi) \tag{1.9}$$

(for more details, see [13–16]).

From (1.9), it is well-known equality that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \tag{1.10}$$

where $f_1(x) = f(x+1)$ (for details, see [2, 3, 8, 9, 12, 13, 15–17]).

The q-Genocchi polynomials with wegiht 0 are introduced as

$$\frac{\widetilde{G}_{n+1,q}(x)}{n+1} = \int_{\mathbb{Z}_p} (x+\xi)^n d\mu_{-q}(\xi). \tag{1.11}$$

From (1.11), we have

$$\widetilde{G}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^l \widetilde{G}_{n-l,q}, \tag{1.12}$$

where $\widetilde{G}_{n,q}(0) := \widetilde{G}_{n,q}$ are called q-Genocchi numbers with weight 0. Then, q-Genocchi numbers are defined as

$$\widetilde{G}_{0,q} = 0, \qquad q\left(\widetilde{G}_q + 1\right)^n + \widetilde{G}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1, \end{cases} \tag{1.13}$$

with the usual convention about replacing $(\widetilde{G}_q)^n$ by $\widetilde{G}_{n,q}$ is used (for details, see [3]).

Let $UD(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, p-adic analogue of Bernstein operator for f is defined by

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \tag{1.14}$$

where $n, k \in \mathbb{N}^*$. Here, $B_{k,n}(x)$ is called p-adic Bernstein polynomials, which are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1] \tag{1.15}$$

(for details, see [1, 4, 5, 7]).

The q-Bernoulli polynomials and numbers with weight 0 are defined by Kim et al., respectively,

$$\begin{gathered} c\widetilde{B}_{n,q}(x) = \lim_{n\to\infty} \frac{1}{[p^n]_q} \sum_{y=0}^{p^n-1} (x+y)^n q^y = \int_{\mathbb{Z}_p} (x+\xi)^n d\mu_q(\xi), \\ \widetilde{B}_{n,q} = \int_{\mathbb{Z}_p} \xi^n d\mu_q(\xi) \end{gathered} \tag{1.16}$$

(for more information, see [10]).

The author, by using derivative operator, will investigate some interesting identities on the q-Genocchi numbers and polynomials arising from their generating function. Also, the author derives some relations between q-Genocchi numbers and q-Bernoulli numbers by using Kim's q-Volkenborn integral and fermionic p-adic q-integral on $\mathbb{Z}_p$.

2. Novel Properties of q-Genocchi Numbers and Polynomials with Weight 0

Let $f(x) = e^{t(x+y)}$. Then, by using (1.10), we easily procure the following:

$$\int_{\mathbb{Z}_p} e^{t(x+\xi)} d\mu_{-q}(\xi) = \frac{[2]_q}{qe^t + 1} e^{xt}. \tag{2.1}$$

From the last equality, by (1.11), we get Araci, Acikgoz, and Qi's q-Genocchi polynomials with weight 0 in [3] as follows:

$$\frac{[2]_q t}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} \widetilde{G}_{n,q}(x) \frac{t^n}{n!}, \quad |\log q + t| < \pi. \tag{2.2}$$

Here, we assume that x is a fixed parameter. Let

$$\widetilde{F}_q(x,t) = \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} \widetilde{G}_{n,q}(x) \frac{t^{n-1}}{n!}. \tag{2.3}$$

Thus, by expression of (2.3), we can readily see the following:

$$qe^t \widetilde{F}_q(x,t) + \widetilde{F}_q(x,t) = [2]_q e^{xt}. \tag{2.4}$$

Last from equality, taking derivative operator D as $D = d/dt$ on the both sides of (2.4), then, we easily see that

$$qe^t (D + I)^k \widetilde{F}_q(x,t) + D^k \widetilde{F}_q(x,t) = [2]_q x^k e^{xt}, \tag{2.5}$$

where $k \in \mathbb{N}^*$ and I is identity operator. By multiplying e^{-t} on both sides of (2.5), we get

$$q(D + I)^k \widetilde{F}_q(x,t) + e^{-t} D^k \widetilde{F}_q(x,t) = [2]_q x^k e^{(x-1)t}. \tag{2.6}$$

Let us take derivative operator D^m $(m \in \mathbb{N})$ on the both sides of (2.6). Then, we get

$$qe^t D^m (D + I)^k \widetilde{F}_q(x,t) + D^k (D - I)^m \widetilde{F}_q(x,t) = [2]_q x^k (x-1)^m e^{xt}. \tag{2.7}$$

Let $G[0]$ (not $G(0)$) be the constant term in a Laurent series of $G(t)$ in (2.3). Then, we get

$$\sum_{j=0}^{k} \binom{k}{j} \left(qe^t D^{k+m-j} \widetilde{F}_q(x,t) \right)[0] + \sum_{j=0}^{m} \binom{m}{j} (-1)^j \left(D^{k+m-j} \widetilde{F}_q(x,t) \right)[0] = [2]_q x^k (x-1)^m. \tag{2.8}$$

By (2.3), we easily see

$$\left(D^N \widetilde{F}_q(x,t)\right)[0] = \frac{\widetilde{G}_{N+1,q}(x)}{N+1}, \qquad \left(e^t D^N \widetilde{F}_q(x,t)\right)[0] = \frac{\widetilde{G}_{N+1,q}(x)}{N+1}. \tag{2.9}$$

We see that the members of (2.11) are proportional to the Bernstein polynomials with the following theorem.

Theorem 2.1. *For $k, m \in \mathbb{N}$, one has*

$$[2]_q (-1)^m \frac{B_{k,m+k}(x)}{\binom{m+k}{k}} = \sum_{j=0}^{\max\{k,m\}} \frac{q\binom{k}{j} + (-1)^j \binom{m}{j}}{k+m-j+1} \widetilde{G}_{k+m-j+1,q}(x). \tag{2.10}$$

Proof. By expressions of (2.8) and (2.9), we see that

$$\sum_{j=0}^{\max\{k,m\}} \frac{q\binom{k}{j} + (-1)^j \binom{m}{j}}{k+m-j+1} \widetilde{G}_{k+m-j+1,q}(x) = [2]_q x^k (x-1)^m. \tag{2.11}$$

By applying basic operations to above equality, we can easily reach to the desired result. □

As a special case, we derive the following.

Corollary 2.2. *For $k \in \mathbb{N}$, one has*

$$[2]_q (-1)^k \frac{B_{k,2k}(x)}{\binom{2k}{k}} = [2]_q \sum_{j=0}^{[k/2]} \frac{\binom{k}{2j}}{2k-2j+1} \widetilde{G}_{2k-2j+1,q}(x) + \binom{2k}{k}(q-1) \sum_{j=0}^{[(k-1)/2]} \frac{\binom{k}{2j+1}}{2k-2j} \widetilde{G}_{2k-2j,q}(x). \tag{2.12}$$

Proof. When $k = m$ into (2.10), we derive the following identity:

$$\begin{aligned}
(-1)^k B_{k,2k}(x) &= \frac{\binom{2k}{k}}{1+q} \sum_{j=0}^{k} \left[q\binom{k}{j} + (-1)^j \binom{k}{j} \right] \frac{\widetilde{G}_{2k-j+1,q}(x)}{2k-j+1} \\
&= \binom{2k}{k} \sum_{j=0}^{[k/2]} \frac{\binom{k}{2j}}{2k-2j+1} \widetilde{G}_{2k-2j+1,q}(x) \\
&\quad + \binom{2k}{k} \frac{q-1}{q+1} \sum_{j=0}^{[(k-1)/2]} \frac{\binom{k}{2j+1}}{2k-2j} \widetilde{G}_{2k-2j,q}(x).
\end{aligned} \tag{2.13}$$

Here, $[x]$ is greatest integer $\leq x$. Then, we complete the proof of Theorem. □

From (2.2), we note that

$$\frac{d}{dx}\left(\widetilde{G}_{n,q}(x)\right) = n\sum_{l=0}^{n-1}\binom{n-1}{l}\widetilde{G}_{l,q}x^{n-1-l} = n\widetilde{G}_{n-1,q}(x). \tag{2.14}$$

By (2.14) and (1.11), we easily see that

$$\int_0^1 \widetilde{G}_{n,q}(x)dx = \frac{\widetilde{G}_{n+1,q}(1) - \widetilde{G}_{n+1,q}}{n+1} = -[2]_{q^{-1}}\frac{\widetilde{G}_{n+1,q}}{n+1} = -[2]_{q^{-1}}\int_{\mathbb{Z}_p}\xi^n d\mu_{-q}(\xi). \tag{2.15}$$

Now, let us consider definition of integral from 0 to 1 in (2.11), then we have

$$\begin{aligned} -[2]_{q^{-1}}\sum_{j=0}^{\max\{k,m\}}\frac{q\binom{k}{j}+(-1)^j\binom{m}{j}}{k+m-j+1}\frac{\widetilde{G}_{k+m-j+2,q}}{k+m-j+2} &= [2]_q(-1)^m B(k+1,m+1) \\ &= [2]_q(-1)^m\frac{\Gamma(k+1)\Gamma(m+1)}{\Gamma(k+m+2)}, \end{aligned} \tag{2.16}$$

where $B(k+1,m+1)$ is beta function which is defined by

$$B(k+1,m+1) = \int_0^1 x^k(1-x)^m dx = \frac{1}{(k+m+1)\binom{k+m}{m}}, \quad k>0,\ m>0. \tag{2.17}$$

As a result, we obtain the following theorem.

Theorem 2.3. *For $m,k\in\mathbb{N}$, one has*

$$\sum_{j=1}^{\max\{k,m\}}\frac{q\binom{k}{j}+(-1)^j\binom{m}{j}}{k+m-j+1}\frac{\widetilde{G}_{k+m-j+2,q}}{k+m-j+2} = q\frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} - \frac{[2]_q}{k+m+1}\frac{\widetilde{G}_{k+m+2,q}}{k+m+2}. \tag{2.18}$$

Proof. By taking integral from 0 to 1 in (2.11), we easily reach to desired result. □

Substituting $m=k+1$ into Theorem 2.1, we readily get

$$\sum_{j=1}^{k+1}\frac{q\binom{k}{j}+(-1)^j\binom{k+1}{j}}{2k-j+3}\widetilde{G}_{2k-j+3,q} = q\frac{(-1)^k}{(2k+2)\binom{2k+1}{k}} - \frac{[2]_q}{2k+2}\frac{\widetilde{G}_{2k+3,q}}{2k+3}. \tag{2.19}$$

By differentiating both sides of (2.11) with respect to t, we have the following:

$$\sum_{j=0}^{\max\{k,m\}}\left\{q\binom{k}{j}+(-1)^j\binom{m}{j}\right\}\widetilde{G}_{k+m-j,q}(x) = [2]_q x^{k-1}(x-1)^{m-1}((k+m)x-k). \tag{2.20}$$

We now give interesting theorem for q-Genocchi numbers with weight 0 as follows.

Theorem 2.4. *For $k \in \mathbb{N}$, one has*

$$[2]_q \sum_{j=0}^{[k/2]} \frac{\binom{k}{2j}}{2k-2j+1} \frac{\widetilde{G}_{2k-2j+2,q}}{2k-2j+2} + (q-1) \sum_{j=0}^{[(k-1)/2]} \frac{\binom{k}{2j+1}}{2k-2j} \frac{\widetilde{G}_{2k-2j+1,q}}{2k-2j+1} = \frac{q(-1)^{k+1}}{(2k+1)\binom{2k}{k}}. \tag{2.21}$$

Proof. It is proved by using definition of integral on the both sides in the following equality, that is,

$$\sum_{j=0}^{k} \frac{q\binom{k}{j} + (-1)^j \binom{k}{j}}{2k-j+1} \left\{ \int_0^1 \widetilde{G}_{2k-j+1,q}(x)dx \right\} = [2]_q \left\{ \int_0^1 x^k (x-1)^k dx \right\}. \tag{2.22}$$

Last from equality, we discover the following:

$$\begin{aligned} &[2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \frac{\left\{\int_0^1 \widetilde{G}_{2k-2j+1,q}(x)dx\right\}}{2k-2j+1} + (q-1) \sum_{j=0}^{[(k-1)/2]} \binom{k}{2j+1} \frac{\left\{\int_0^1 \widetilde{G}_{2k-2j,q}(x)dx\right\}}{2k-2j} \\ &\quad = [2]_q (-1)^k \left\{ \int_0^1 x^k (1-x)^k dx \right\}. \end{aligned} \tag{2.23}$$

Then, taking integral from 0 to 1 both sides of last equality, we get

$$\begin{aligned} &-[2]_{q^{-1}}[2]_q \sum_{j=0}^{[k/2]} \frac{\binom{k}{2j}}{2k-2j+1} \frac{\widetilde{G}_{2k-2j+2,q}}{2k-2j+2} + [2]_{q^{-1}}(1-q) \sum_{j=0}^{[(k-1)/2]} \frac{\binom{k}{2j+1}}{2k-2j} \frac{\widetilde{G}_{2k-2j+1,q}}{2k-2j+1} \\ &\quad = [2]_q (-1)^k B(k+1,k+1) = \frac{[2]_q (-1)^k}{(2k+1)\binom{2k}{k}}. \end{aligned} \tag{2.24}$$

Thus, we complete the proof of the theorem. □

Theorem 2.5. *For $k \in \mathbb{N}$, one has*

$$\begin{aligned} &[2]_q \sum_{j=0}^{[(k+1)/2]} \frac{\binom{k}{2j}}{2k-2j+1} \widetilde{G}_{2k-2j+1,q}(x) + \sum_{j=1}^{[k/2]} \frac{\binom{k}{2j-1}}{2k-2j+1} \widetilde{G}_{2k-2j+1,q}(x) - \sum_{j=0}^{[k/2]} \frac{\binom{k}{2j}}{2k-2j} \widetilde{G}_{2k-2j,q}(x) \\ &\quad + (q-1)[2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left\{ \frac{\widetilde{G}_{2k-2j,q}(x)}{[2]_q(2k-2j)} + \frac{\widetilde{G}_{2k-2j+1,q}(x)}{[2]_q^2(2k-2j+1)} \right\} = x^k (x-1)^k \left([2]_q x - q\right). \end{aligned} \tag{2.25}$$

Proof. In view of (2.2) and (2.23), we discover the following applications:

$$\begin{aligned}
\sum_{j=0}^{k+1}\left[q\binom{k}{j}+(-1)^j\binom{k+1}{j}\right]\frac{\widetilde{G}_{2k-j+1,q}(x)}{2k-j+1} &= [2]_q\frac{\widetilde{G}_{2k+1,q}(x)}{2k+1} \\
&+\sum_{j=1}^{[(k+1)/2]}\left[q\binom{k}{2j}+\binom{k}{2j}+\binom{k}{2j-1}\right]\frac{\widetilde{G}_{2k-2j+1,q}}{2k-2j+1}(x) \\
&+\sum_{j=0}^{[k/2]}\left[q\binom{k}{2j+1}-\binom{k}{2j+1}-\binom{k}{2j}\right]\frac{\widetilde{G}_{2k-2j,q}(x)}{2k-2j} \\
&=-\left\{\sum_{j=0}^{[(k+1)/2]}\binom{k}{2j}\frac{\widetilde{G}_{2k-2j,q}(x)}{2k-2j}+\frac{q-1}{1+q}\sum_{j=0}^{[k/2]}\binom{k}{2j+1}\frac{\widetilde{G}_{2k-2j+1,q}(x)}{2k-2j+1}\right\}+[2]_q\sum_{j=0}^{[(k+1)/2]}\binom{k}{2j} \\
&\times\frac{\widetilde{G}_{2k-2j+1,q}(x)}{2k-2j+1}+\sum_{j=1}^{[(k+1)/2]}\binom{k}{2j-1}\frac{\widetilde{G}_{2k-2j+1,q}(x)}{2k-2j+1}-\sum_{j=0}^{[k/2]}\binom{k}{2j}\frac{\widetilde{G}_{2k-2j,q}(x)}{2k-2j} \\
&+(q-1)\sum_{j=0}^{[k/2]}\binom{k}{2j+1}\frac{\widetilde{G}_{2k-2j,q}(x)}{2k-2j}+\frac{q-1}{1+q}\sum_{j=0}^{[k/2]}\binom{k}{2j+1}\frac{\widetilde{G}_{2k-2j+1,q}(x)}{2k-2j+1}.
\end{aligned} \tag{2.26}$$

Thus, we give evidence of the theorem. □

As $q \to 1$ into Theorem 2.5, it leads to the following interesting property.

Corollary 2.6. *For $k \in \mathbb{N}$, one has*

$$\begin{aligned}
&\sum_{j=0}^{[(k+1)/2]}\binom{k}{2j}\frac{G_{2k-2j+1}(x)}{2k+1-2j}+\sum_{j=1}^{[k/2]}\binom{k}{2j-1}\frac{G_{2k-2j+1}(x)}{4k-4j+2}-\sum_{j=0}^{[k/2]}\binom{k}{2j}\frac{G_{2k-2j,q}(x)}{4k-4j} \\
&\quad = x^k(x-1)^k\left(x-\frac{1}{2}\right),
\end{aligned} \tag{2.27}$$

where $G_n(x)$ is ordinary Genocchi polynomials, which is defined by the means of the following generating function [9]:

$$\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}=\frac{2t}{e^t+1}e^{xt},\quad |t|<\pi. \tag{2.28}$$

3. Some Identities q-Genocchi Numbers and q-Bernoulli Numbers by Using Kim's p-Adic q-Integrals on $\mathbb{Z}_p$

In this section, we consider q-Genocchi numbers and q-Bernoulli numbers by means of p-adic q-integral on $\mathbb{Z}_p$. Now, we start with the following theorem.

Theorem 3.1. *For $m,k \in \mathbb{N}$, one has*

$$\sum_{j=0}^{\max\{k,m\}} \frac{q\binom{k}{j} + (-1)^j \binom{m}{j}}{k+m-j+1} \sum_{l=0}^{k+m-j+1} \frac{\binom{k+m-j+1}{l}}{l+1} \widetilde{G}_{k+m-j-l+1,q} \widetilde{G}_{l+1,q} = [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \frac{\widetilde{G}_{l+k+1,q}}{l+k+1}. \tag{3.1}$$

Proof. For $m, k \in \mathbb{N}$, then by (2.11),

$$\begin{aligned} I_1 &= [2]_q \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu_{-q}(x) = [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{l+k} d\mu_{-q}(x) \\ &= [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \frac{\widetilde{G}_{l+k+1,q}}{l+k+1}. \end{aligned} \tag{3.2}$$

On the other hand, the right hand side of (2.11),

$$\begin{aligned} I_2 &= \sum_{j=0}^{\max\{k,m\}} \frac{q\binom{k}{j} + (-1)^j \binom{m}{j}}{k+m-j+1} \sum_{l=0}^{k+m-j+1} \binom{k+m-j+1}{l} \widetilde{G}_{k+m-j-l+1,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\ &= \sum_{j=0}^{\max\{k,m\}} \frac{q\binom{k}{j} + (-1)^j \binom{m}{j}}{k+m-j+1} \sum_{l=0}^{k+m-j+1} \frac{\binom{k+m-j+1}{l}}{l+1} \widetilde{G}_{k+m-j-l+1,q} \widetilde{G}_{l+1,q}. \end{aligned} \tag{3.3}$$

Combining I_1 and I_2, we arrive to the proof of the theorem. □

Theorem 3.2. *For $k \in \mathbb{N}$, one has*

$$\begin{aligned} &\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left\{ [2]_q \frac{\widetilde{G}_{k+l+2,q}}{k+l+2} - q \frac{\widetilde{G}_{k+l+1,q}}{k+l+1} \right\} \\ &\quad = [2]_q \sum_{j=0}^{[k/2]} \frac{\binom{k}{2j}}{2k-2j+1} \sum_{l=0}^{2k-2j+1} \frac{\binom{2k-2j+1}{l}}{l+1} \widetilde{G}_{2k+1-2j-l,q} \widetilde{G}_{l+1,q} \\ &\qquad + \sum_{j=1}^{[k/2]} \frac{\binom{k}{2j-1}}{2k-2j+1} \sum_{l=0}^{2k-2j+1} \frac{\binom{2k-2j+1}{l}}{l+1} \widetilde{G}_{2k+1-2j-l,q} \widetilde{G}_{l+1,q} + \frac{q-1}{1+q} \sum_{j=0}^{[k/2]} \binom{k}{2j+1} T_{k,j}^{q}, \end{aligned} \tag{3.4}$$

here $T_{k,j}^{q} = [2]_q \sum_{l=0}^{2k-2j} (\binom{2k-2j}{l}/(2k-2j))(\widetilde{G}_{l+1,q}\widetilde{G}_{2k-2j-l,q}/(l+1)) + \sum_{l=0}^{2k-2j+1} (\binom{2k-2j+1}{l}/(2k-2j+1))(\widetilde{G}_{l+1,q}\widetilde{G}_{2k-2j-l+1,q}/(l+1))$.

Proof. Let us take fermionic p-adic q-inetgral on $\mathbb{Z}_p$ left-hand side of Theorem 2.5, we get

$$\begin{aligned} I_3 &= \int_{\mathbb{Z}_p} x^k (x-1)^k \left([2]_q x - q\right) d\mu_{-q}(x) \\ &= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_{-q}(x) - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-q}(x) \\ &= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \frac{\widetilde{G}_{k+l+2,q}}{k+l+2} - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \frac{\widetilde{G}_{k+l+1,q}}{k+l+1}. \end{aligned} \tag{3.5}$$

In other word, we consider the right-hand side of Theorem 2.5 as follows:

$$\begin{aligned} I_4 &= [2]_q \sum_{j=0}^{[k/2]} \frac{\binom{k}{2j}}{2k-2j+1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{G}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\ &\quad + \sum_{j=1}^{[k/2]} \frac{\binom{k}{2j-1}}{2k-2j+1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{G}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\ &\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left\{ \begin{array}{l} (q-1) \displaystyle\sum_{j=0}^{2k-2j} \frac{\binom{2k-2j-l}{l}}{2k-2j} \widetilde{G}_{2k-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\ + \dfrac{q-1}{1+q} \displaystyle\sum_{l=0}^{2k-2j+1} \frac{\binom{2k-2j+1}{l}}{2k-2j+1} \widetilde{G}_{2k-2j-l+1} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \end{array} \right\} \\ &= [2]_q \sum_{j=0}^{[k/2]} \frac{\binom{k}{2j}}{2k-2j+1} \sum_{l=0}^{2k-2j+1} \frac{\binom{2k-2j-l+1}{l}}{l+1} \widetilde{G}_{2k+1-2j-l,q} \widetilde{G}_{l+1,q} \\ &\quad + \sum_{j=1}^{[k/2]} \frac{\binom{k}{2j-1}}{2k-2j+1} \sum_{l=0}^{2k-2j+1} \frac{\binom{2k-2j-l+1}{l}}{l+1} \widetilde{G}_{2k+1-2j-l,q} \widetilde{G}_{l+1,q} \\ &\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left\{ \begin{array}{l} (q-1) \displaystyle\sum_{j=0}^{2k-2j} \frac{\binom{2k-2j-l}{l}}{2k-2j} \widetilde{G}_{2k-2j-l,q} \frac{\widetilde{G}_{l+1,q}}{l+1} \\ + \dfrac{q-1}{1+q} \displaystyle\sum_{l=0}^{2k-2j+1} \frac{\binom{2k-2j-l+1}{l}}{2k-2j+1} \widetilde{G}_{2k-2j-l+1} \frac{\widetilde{G}_{l+1,q}}{l+1} \end{array} \right\}. \end{aligned} \tag{3.6}$$

Equating I_3 and I_4, we complete the proof of the theorem. □

As $q \to 1$ in the above theorem, we reach interesting property in Analytic Numbers Theory concerning ordinary Genocchi polynomials.

Corollary 3.3. *For $k \in \mathbb{N}$, one has*

$$\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left\{2\frac{G_{k+l+2}}{k+l+2}-\frac{G_{k+l+1}}{k+l+1}\right\}=2\sum_{j=0}^{[k/2]}\frac{\binom{k}{2j}}{2k-2j+1}\sum_{l=0}^{2k-2j+1}\frac{\binom{2k-2j+1}{l}}{l+1}G_{2k+1-2j-l}G_{l+1}$$
$$+\sum_{j=1}^{[k/2]}\frac{\binom{k}{2j-1}}{2k-2j+1}\sum_{l=0}^{2k-2j+1}\frac{\binom{2k-2j+1}{l}}{l+1}G_{2k+1-2j-l}G_{l+1}. \tag{3.7}$$

Theorem 3.4. *For $m, k \in \mathbb{N}$, one has*

$$[2]_q\sum_{l=0}^{m}\binom{m}{l}(-1)^{m-l}\widetilde{B}_{l+k,q}=\sum_{j=0}^{\max\{k,m\}}\frac{q\binom{k}{j}+(-1)^j\binom{m}{j}}{k+m-j+1}\sum_{l=0}^{k+m-j+1}\binom{k+m-j+1}{l}\widetilde{G}_{k+m+1-j-l,q}\widetilde{B}_{l,q}. \tag{3.8}$$

Proof. We consider (2.11) and (2.2) by means of q-Volkenborn integral. Then, by (2.11), we see

$$[2]_q\int_{\mathbb{Z}_p}x^k(x-1)^m d\mu_q(x)=[2]_q\sum_{l=0}^{m}\binom{m}{l}(-1)^{m-l}\int_{\mathbb{Z}_p}x^{l+k}d\mu_q(x)=[2]_q\sum_{l=0}^{m}\binom{m}{l}(-1)^{m-l}\widetilde{B}_{l+k,q}. \tag{3.9}$$

On the other hand,

$$\begin{aligned}&\sum_{j=0}^{\max\{k,m\}}\frac{q\binom{k}{j}+(-1)^j\binom{m}{j}}{k+m-j+1}\sum_{l=0}^{k+m-j+1}\binom{k+m-j+1}{l}\widetilde{G}_{k+m+1-j-l,q}\int_{\mathbb{Z}_p}x^l d\mu_q(x)\\&\quad=\sum_{j=0}^{\max\{k,m\}}\frac{q\binom{k}{j}+(-1)^j\binom{m}{j}}{k+m-j+1}\sum_{l=0}^{k+m-j+1}\binom{k+m-j+1}{l}\widetilde{G}_{k+m+1-j-l,q}\widetilde{B}_{l,q}.\end{aligned} \tag{3.10}$$

Therefore, we get the proof of theorem. □

Corollary 3.5. *For $k \in \mathbb{N}$, one gets*

$$\begin{aligned}&\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left\{[2]_q\widetilde{B}_{k+l+1,q}-q\widetilde{B}_{k+l,q}\right\}\\&\quad=[2]_q\sum_{j=0}^{[k/2]}\frac{\binom{k}{2j}}{2k-2j+1}\sum_{l=0}^{2k-2j+1}\binom{2k-2j+1}{l}\widetilde{G}_{2k+1-2j-l,q}\widetilde{B}_{l,q}+\sum_{j=1}^{[k/2]}\frac{\binom{k}{2j-1}}{2k-2j+1}\\&\qquad\times\sum_{l=0}^{2k-2j+1}\binom{2k-2j+1}{l}\widetilde{G}_{2k+1-2j-l,q}\widetilde{B}_{l,q}+\left(\frac{q-1}{q+1}\right)\sum_{j=0}^{[k/2]}\binom{k}{2j+1}S_{k,j}^{q},\end{aligned} \tag{3.11}$$

where $S_{k,j}^{q}=[2]_q\sum_{j=0}^{2k-2j}(1/(2k-2j))\binom{2k-2j}{l}\widetilde{G}_{2k-2j-l,q}\widetilde{B}_{l,q}+\sum_{l=0}^{2k-2j+1}(1/(2k-2j+1))\binom{2k-2j+1}{l}\widetilde{G}_{2k-2j-l+1}\widetilde{B}_{l,q}$.

Proof. By using p-adic q-integral on $\mathbb{Z}_p$ left-hand side of Theorem 2.5, we get

$$\begin{aligned} I_5 &= [2]_q \int_{\mathbb{Z}_p} x^k (x-1)^k ([2]x - q) d\mu_q(x) \\ &= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_q(x) - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x) \\ &= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \widetilde{B}_{k+l+1,q} - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \widetilde{B}_{k+l,q}. \end{aligned} \tag{3.12}$$

Also, we compute the right-hand side of Theorem 2.5 as follows:

$$\begin{aligned} I_6 &= [2]_q \sum_{j=0}^{[k/2]} \frac{1}{2k-2j+1} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{G}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\ &\quad + \sum_{j=1}^{[k/2]} \frac{1}{2k-2j+1} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{G}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\ &\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left\{ \begin{array}{c} (q-1) \sum_{j=0}^{2k-2j} \frac{1}{2k-2j} \binom{2k-2j}{l} \widetilde{G}_{2k-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\ + \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \frac{1}{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{G}_{2k-2j-l+1} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \end{array} \right\} \\ &= [2]_q \sum_{j=0}^{[k/2]} \frac{1}{2k-2j+1} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{G}_{2k+1-2j-l,q} \widetilde{B}_{l,q} \\ &\quad + \sum_{j=1}^{[k/2]} \frac{1}{2k-2j+1} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{G}_{2k+1-2j-l,q} \widetilde{B}_{l,q} \\ &\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left\{ \begin{array}{c} (q-1) \sum_{j=0}^{2k-2j} \frac{1}{2k-2j} \binom{2k-2j}{l} \widetilde{G}_{2k-2j-l,q} \widetilde{B}_{l,q} \\ + \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \frac{1}{2k-2j+1} \binom{2k-2j+1}{l} \widetilde{G}_{2k-2j-l+1} \widetilde{B}_{l,q} \end{array} \right\}. \end{aligned} \tag{3.13}$$

Equating I_5 and I_6, we get the proof of Corollary. □

As $q \to 1$ in the above theorem, we easily derive the following corollary.

Corollary 3.6. *For $k \in \mathbb{N}$, one has*

$$\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\{2B_{k+l+1}-B_{k+l}\} = 2\sum_{j=0}^{[k/2]}\frac{\binom{k}{2j}}{2k-2j+1}\sum_{l=0}^{2k-2j+1}\binom{2k-2j+1}{l}G_{2k+1-2j-l}B_l + \sum_{j=1}^{[k/2]}\frac{\binom{k}{2j-1}}{2k-2j+1}\sum_{l=0}^{2k-2j+1}\binom{2k-2j+1}{l}G_{2k+1-2j-l}B_l. \tag{3.14}$$

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