Research Article

Novel Identities for $q$-Genocchi Numbers and Polynomials

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The essential aim of this paper is to introduce novel identities for $q$-Genocchi numbers and polynomials by using the method by T. Kim et al. (article in press). We show that these polynomials are related to $p$-adic analogue of Bernstein polynomials. Also, we derive relations between $q$-Genocchi and $q$-Bernoulli numbers.

1. Preliminaries

Imagine that $p$ be a fixed odd prime number. We now start with definition of the following notations. Let $\mathbb{Q}_p$ be the field $p$-adic rational numbers and let $\mathbb{C}_p$ be the completion of algebraic closure of $\mathbb{Q}_p$.

Thus,

$$\mathbb{Q}_p = \left\{ x = \sum_{n=-k}^{\infty} a_n p^n : 0 \leq a_n < p \right\}.$$  \hspace{1cm} (1.1)

Then $\mathbb{Z}_p$ is integral domain, which is defined by

$$\mathbb{Z}_p = \left\{ x = \sum_{n=0}^{\infty} a_n p^n : 0 \leq a_n < p \right\},$$  \hspace{1cm} (1.2)

or

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}.$$  \hspace{1cm} (1.3)
We assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ as an indeterminate. The $p$-adic absolute value $|\cdot|_p$, is normally defined by
\begin{equation}
|x|_p = p^{-r},
\end{equation}
where $x = p^r (s/t)$ with $(p, s) = (p, t) = (s, t) = 1$, and $r \in \mathbb{Q}$.

$[x]_q$ is a $q$-extension of $x$, which is defined by
\begin{equation}
[x]_q = \frac{1 - q^x}{1 - q},
\end{equation}
we note that $\lim_{q \rightarrow 1}[x]_q = x$ (see [1–17]).

Throughout this paper, we use notation of $\mathbb{N}^* := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes set of Natural numbers.

We say that $f$ is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient,
\begin{equation}
F_f(x, y) = \frac{f(x) - f(y)}{x - y},
\end{equation}
has a limit $f'(a)$ as $(x, y) \rightarrow (a, a)$, and we denote this by $f \in UD(\mathbb{Z}_p)$. Then, for $f \in UD(\mathbb{Z}_p)$, we can start with the following expression:
\begin{equation}
\frac{1}{[p]^N} \sum_{0 \leq \xi < p^N} f(\xi)q^\xi = \sum_{0 \leq \xi < p^N} f(\xi)\mu_q(\xi + p^N\mathbb{Z}_p),
\end{equation}
which represents a $p$-adic $q$-analogue of Riemann sums for $f$. The integral of $f$ on $\mathbb{Z}_p$ will be defined as the limit $(N \rightarrow \infty)$ of these sums, when it exists. The $p$-adic $q$-integral of function $f \in UD(\mathbb{Z}_p)$ is defined by Kim in [7, 12] as
\begin{equation}
I_q(f) = \int_{\mathbb{Z}_p} f(\xi)d\mu_q(\xi) = \lim_{N \rightarrow \infty} \frac{1}{[p]^N} \sum_{\xi=0}^{p^N-1} f(\xi)q^\xi.
\end{equation}

The bosonic integral is considered as a bosonic limit $q \rightarrow 1$, $I_1(f) = \lim_{q \rightarrow 1}I_q(f)$. Similarly, the fermionic $p$-adic integral on $\mathbb{Z}_p$ is introduced by Kim as follows:
\begin{equation}
I_{-q}(f) = \int_{\mathbb{Z}_p} f(\xi)d\mu_{-q}(\xi)
\end{equation}
(for more details, see [13–16]).

From (1.9), it is well-known equality that
\begin{equation}
qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0),
\end{equation}
where $f_1(x) = f(x + 1)$ (for details, see [2, 3, 8, 9, 12, 13, 15–17]).
The $q$-Genocchi polynomials with weight 0 are introduced as
\[
\tilde{G}_{n+1,q}(x) = \frac{1}{n+1} \int_{\mathbb{Z}_p} (x + \xi)^n d\mu_q(\xi).
\] (1.11)

From (1.11), we have
\[
\tilde{G}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^l \tilde{G}_{n-l,q},
\] (1.12)

where $\tilde{G}_{n,q}(0) := \tilde{G}_{n,q}$ are called $q$-Genocchi numbers with weight 0. Then, $q$-Genocchi numbers are defined as
\[
\tilde{G}_{0,q} = 0, \quad q(\tilde{G}_{q} + 1)^n + \tilde{G}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1, \end{cases}
\] (1.13)

with the usual convention about replacing $(\tilde{G}_q)^n$ by $\tilde{G}_{n,q}$ is used (for details, see [3]).

Let $UD(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, $p$-adic analogue of Bernstein operator for $f$ is defined by
\[
B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k},
\] (1.14)

where $n, k \in \mathbb{N}^*$. Here, $B_{k,n}(x)$ is called $p$-adic Bernstein polynomials, which are defined by
\[
B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1]
\] (1.15)

(for details, see [1, 4, 5, 7]).

The $q$-Bernoulli polynomials and numbers with weight 0 are defined by Kim et al., respectively,
\[
c \tilde{B}_{n,q}(x) = \lim_{n \to \infty} \frac{1}{[p^n]_q} \sum_{y=0}^{p^n-1} (x + y)^n q^y = \int_{\mathbb{Z}_p} (x + \xi)^n d\mu_q(\xi),
\] (1.16)

\[
\tilde{B}_{n,q} = \int_{\mathbb{Z}_p} \xi^n d\mu_q(\xi)
\]

(for more information, see [10]).

The author, by using derivative operator, will investigate some interesting identities on the $q$-Genocchi numbers and polynomials arising from their generating function. Also, the author derives some relations between $q$-Genocchi numbers and $q$-Bernoulli numbers by using Kim’s $q$-Volkenborn integral and fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$. 
2. Novel Properties of \( q \)-Genocchi Numbers and Polynomials with Weight 0

Let \( f(x) = e^{t(x+t)} \). Then, by using (1.10), we easily procure the following:

\[
\int_{\mathbb{R}} e^{t(x+t)} d\mu_{-q}(\xi) = \frac{[2]_q e^{xt}}{qe^t + 1}. \tag{2.1}
\]

From the last equality, by (1.11), we get Araci, Acikgoz, and Qi’s \( q \)-Genocchi polynomials with weight 0 in [3] as follows:

\[
\frac{[2]_q e^{xt}}{qe^t + 1} = \sum_{n=0}^{\infty} \bar{c}_{n,q}(x) \frac{t^n}{n!} \quad |\log q + t| < \pi. \tag{2.2}
\]

Here, we assume that \( x \) is a fixed parameter. Let

\[
\tilde{F}_q(x,t) = \frac{[2]_q e^{xt}}{qe^t + 1} = \sum_{n=0}^{\infty} \bar{c}_{n,q}(x) \frac{t^n}{n!}. \tag{2.3}
\]

Thus, by expression of (2.3), we can readily see the following:

\[
qe^t \tilde{F}_q(x,t) + \tilde{F}_q(x,t) = [2]_q e^{xt}. \tag{2.4}
\]

Last from equality, taking derivative operator \( D \) as \( D = d/dt \) on the both sides of (2.4), then, we easily see that

\[
qe^{t(D + I)^k \tilde{F}_q(x,t)} + D^k \tilde{F}_q(x,t) = [2]_q e^{x(t)} x^k \tag{2.5}
\]

where \( k \in \mathbb{N}^* \) and \( I \) is identity operator. By multiplying \( e^{-t} \) on both sides of (2.5), we get

\[
q(D + I)^k \tilde{F}_q(x,t) + e^{-t} D^k \tilde{F}_q(x,t) = [2]_q e^{x(t-1)}. \tag{2.6}
\]

Let us take \( D^m \) (\( m \in \mathbb{N} \)) on the both sides of (2.6). Then, we get

\[
qe^{D^m(D + I)^k \tilde{F}_q(x,t)} + D^k (D - I)^m \tilde{F}_q(x,t) = [2]_q e^{x(t-1)} x^m. \tag{2.7}
\]

Let \( G[0] \) (not \( G(0) \)) be the constant term in a Laurent series of \( G(t) \) in (2.3). Then, we get

\[
\sum_{j=0}^{k} \binom{k}{j} \left(qe^t D^{k+m-j} \tilde{F}_q(x,t)[0] + \sum_{j=0}^{m} \binom{m}{j} (-1)^j (D^{k+m-j} \tilde{F}_q(x,t))[0] = [2]_q x^k (x-1)^m. \tag{2.8}
\]


By (2.3), we easily see

\[ \left( D^N \tilde{F}_q(x,t) \right)[0] = \frac{\tilde{G}_{N+1,q}(x)}{N+1}, \quad \left( e^t D^N \tilde{F}_q(x,t) \right)[0] = \frac{\tilde{G}_{N+1,q}(x)}{N+1}. \]  

(2.9)

We see that the members of (2.11) are proportional to the Bernstein polynomials with the following theorem.

**Theorem 2.1.** For \( k, m \in \mathbb{N} \), one has

\[ [2]_q(-1)^m B_{k,m+k}(x) = \sum_{j=0}^{\max\{k,m\}} \frac{q^j}{k^j} (-1)^j \binom{m}{j} \tilde{G}_{k+m-j+1,q}(x). \]  

(2.10)

**Proof.** By expressions of (2.8) and (2.9), we see that

\[ \sum_{j=0}^{\max\{k,m\}} \frac{q^j}{k^j} (-1)^j \binom{m}{j} \tilde{G}_{k+m-j+1,q}(x) = [2]_q x^k (x-1)^m. \]  

(2.11)

By applying basic operations to above equality, we can easily reach to the desired result.

As a special case, we derive the following.

**Corollary 2.2.** For \( k \in \mathbb{N} \), one has

\[ [2]_q(-1)^k B_{2k}(x) = [2]_q \sum_{j=0}^{[k/2]} \frac{\binom{k}{2j}}{2k-2j+1} \tilde{G}_{2k-2j+1,q}(x) + \binom{2k}{k} (q-1)^{[k/2]} \sum_{j=0}^{[k/2]} \frac{\binom{k}{2j+1}}{2k-2j} \tilde{G}_{2k-2j,q}(x). \]  

(2.12)

**Proof.** When \( k = m \) into (2.10), we derive the following identity:

\[ (-1)^k B_{k,2k}(x) = \frac{2k^2}{1+q} \sum_{j=0}^{k} \frac{q^j}{2k-2j+1} \frac{\tilde{G}_{2k-2j+1,q}(x)}{2k-2j+1} \]

\[ = \frac{2k}{k} \sum_{j=0}^{[k/2]} \frac{\binom{k}{2j}}{2k-2j+1} \tilde{G}_{2k-2j+1,q}(x) \]

\[ + \frac{2k}{k} \frac{q-1}{q+1} \sum_{j=0}^{[k/2]} \frac{\binom{k}{2j+1}}{2k-2j} \tilde{G}_{2k-2j,q}(x). \]  

(2.13)

Here, \([x]\) is greatest integer \( \leq x \). Then, we complete the proof of Theorem.
From (2.2), we note that
\[
\frac{d}{dx}(\tilde{G}_{n,q}(x)) = n \sum_{i=0}^{n-1} \binom{n-1}{i} \tilde{G}_{i,q} x^{n-1-i} = n \tilde{G}_{n-1,q}(x). \quad (2.14)
\]

By (2.14) and (1.11), we easily see that
\[
\int_0^1 \tilde{G}_{n,q}(x) dx = \frac{\tilde{G}_{n+1,q}(1) - \tilde{G}_{n+1,q}}{n+1} = -[2]_q^{-1} \tilde{G}_{n+1,q} = -[2]_q^{-1} \int_{\mathbb{Z}_q} \xi^n d\mu_q(\xi). \quad (2.15)
\]

Now, let us consider definition of integral from 0 to 1 in (2.11), then we have
\[
-[2]_q^{-1} \sum_{j=0}^{\max[k,m]} q^j \binom{m}{j} \tilde{G}_{k+m-j+2,q} = [2]_q (-1)^m B(k+1,m+1) = [2]_q (-1)^m \frac{\Gamma(k+1) \Gamma(m+1)}{\Gamma(k+m+2)},
\]

where \(B(k+1,m+1)\) is beta function which is defined by
\[
B(k+1,m+1) = \int_0^1 x^k (1-x)^m dx = \frac{1}{(k+m+1) \binom{k+m}{m}}, \quad k > 0, \ m > 0. \quad (2.17)
\]

As a result, we obtain the following theorem.

**Theorem 2.3.** For \(m, k \in \mathbb{N}\), one has
\[
\sum_{j=1}^{\max[k,m]} q^j \binom{m}{j} \frac{\tilde{G}_{k+m-j+2,q}}{k+m-j+1} = q^k \frac{(-1)^{m+1}}{(k+m+1) \binom{k+m}{k}} - [2]_q \tilde{G}_{k+m+2,q}. \quad (2.18)
\]

**Proof.** By taking integral from 0 to 1 in (2.11), we easily reach to desired result. \(\square\)

Substituting \(m = k + 1\) into Theorem 2.1, we readily get
\[
\sum_{j=1}^{k+1} q^j \binom{k+1}{j} \frac{\tilde{G}_{2k-j+3,q}}{2k-j+3} = q^k \frac{(-1)^k}{(2k+2) \binom{2k+1}{k}} - [2]_q \tilde{G}_{2k+3,q}. \quad (2.19)
\]

By differentiating both sides of (2.11) with respect to \(t\), we have the following:
\[
\sum_{j=0}^{\max[k,m]} \{ q^j \binom{m}{j} \} \tilde{G}_{k+m-j,q}(x) = [2]_q x^{k-1}(x-1)^{m-1}((k+m)x - k). \quad (2.20)
\]

We now give interesting theorem for \(q\)-Genocchi numbers with weight 0 as follows.
For Theorem 2.5. For Theorem 2.4. Journal of Function Spaces and Applications 7

\[
[k/2]_q \sum_{j=0}^{[k/2]} \left( \frac{k}{2j} \right) \tilde{G}_{2k-2j+2,q} + (q-1) \sum_{j=1}^{[(k-1)/2]} \left( \frac{k}{2j+1} \right) \tilde{G}_{2k-2j+1,q} = \frac{q(-1)^{k+1}}{(2k+1)\left( \frac{2}{k} \right)}. \tag{2.21}
\]

**Proof.** It is proved by using definition of integral on the both sides in the following equality, that is,

\[
\sum_{j=0}^{k} \frac{q\left( \frac{1}{j} \right) + (-1)^j\left( \frac{1}{j} \right)}{2k-j+1} \left\{ \int_0^1 \tilde{G}_{2k-j+1,q}(x) dx \right\} = [2]_q \left\{ \int_0^1 x^k(x-1)^k dx \right\}. \tag{2.22}
\]

Last from equality, we discover the following:

\[
[2]_q \sum_{j=0}^{[k/2]} \left( \frac{k}{2j} \right) \tilde{G}_{2k-2j+2,q} + (q-1) \sum_{j=1}^{[(k-1)/2]} \left( \frac{k}{2j+1} \right) \tilde{G}_{2k-2j+1,q} = \frac{q(-1)^k}{(2k+1)\left( \frac{2}{k} \right)}. \tag{2.23}
\]

Then, taking integral from 0 to 1 both sides of last equality, we get

\[
- [2]_q \sum_{j=0}^{[k/2]} \left( \frac{k}{2j} \right) \tilde{G}_{2k-2j+2,q} + [2]_q \left( 1-q \right) \sum_{j=1}^{[(k-1)/2]} \left( \frac{k}{2j+1} \right) \tilde{G}_{2k-2j+1,q}
\]

\[
= [2]_q (-1)^k B(k+1, k+1) = \frac{[2]_q (-1)^k}{(2k+1)\left( \frac{2}{k} \right)}. \tag{2.24}
\]

Thus, we complete the proof of the theorem. \qed

**Theorem 2.5.** For \( k \in \mathbb{N} \), one has

\[
[2]_q \sum_{j=0}^{[(k+1)/2]} \left( \frac{k}{2j} \right) \tilde{G}_{2k-2j+1,q}(x) + [2]_q \sum_{j=1}^{[k/2]} \left( \frac{k}{2j+1} \right) \tilde{G}_{2k-2j+1,q}(x) - \sum_{j=0}^{[k/2]} \left( \frac{k}{2j+1} \right) \tilde{G}_{2k-2j,q}(x)
\]

\[
+ (q-1) \sum_{j=0}^{[k/2]} \left( \frac{k}{2j+1} \right) \left\{ \tilde{G}_{2k-2j,q}(x) \left[ \frac{1}{[2]_q(2k-2j)} + \frac{\tilde{G}_{2k-2j+1,q}(x)}{[2]_q(2k-2j+1)} \right] \right\} = x^k(x-1)^k \left[ 2]_q x - q \right). \tag{2.25}
\]
Proof. In view of (2.2) and (2.23), we discover the following applications:

\[
\sum_{j=0}^{[k/2]} \left\{ \sum_{j=0}^{[(k+1)/2]} q \binom{k}{2j} \tilde{G}_{2k-2j+1,q}(x) - 1 \right\} + q \sum_{j=0}^{[k/2]} \frac{\tilde{G}_{2k-2j+1,q}(x)}{2k-2j+1}
\]

\[
+ \sum_{j=0}^{[k/2]} \left\{ \sum_{j=0}^{[(k+1)/2]} q \binom{k}{2j+1} \tilde{G}_{2k-2j+1,q}(x) - 1 \right\} + q \sum_{j=0}^{[k/2]} \frac{\tilde{G}_{2k-2j+1,q}(x)}{2k-2j+1}
\]

\[
= - \frac{\tilde{G}_{2k-2j+1,q}(x)}{2k-2j+1} + \sum_{j=0}^{[(k+1)/2]} q \binom{k}{2j} \tilde{G}_{2k-2j+1,q}(x) + q \sum_{j=0}^{[k/2]} \frac{\tilde{G}_{2k-2j+1,q}(x)}{2k-2j+1}
\]

(2.26)

Thus, we give evidence of the theorem.

As \( q \to 1 \) into Theorem 2.5, it leads to the following interesting property.

Corollary 2.6. For \( k \in \mathbb{N} \), one has

\[
\sum_{j=0}^{[(k+1)/2]} \frac{\binom{k}{2j}}{2k+1-2j} \tilde{G}_{2k-2j+1,q}(x) + \sum_{j=1}^{[k/2]} \frac{\binom{k}{2j-1}}{4k-4j+2} \tilde{G}_{2k-2j,q}(x) - \sum_{j=0}^{[k/2]} \frac{k \binom{k}{2j}}{4k-4j} \tilde{G}_{2k-2j+1,q}(x)
\]

\[
= x^k (x-1)^k \left( x - \frac{1}{2} \right),
\]

(2.27)

where \( G_n(x) \) is ordinary Genocchi polynomials, which is defined by the means of the following generating function [9]:

\[
\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t + 1} e^{xt}, \quad |t| < \pi.
\]

(2.28)

3. Some Identities \( q \)-Genocchi Numbers and \( q \)-Bernoulli Numbers by Using Kim’s \( p \)-Adic \( q \)-Integrals on \( \mathbb{Z}_p \)

In this section, we consider \( q \)-Genocchi numbers and \( q \)-Bernoulli numbers by means of \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \). Now, we start with the following theorem.
Theorem 3.1. For $m,k \in \mathbb{N}$, one has

$$\max_{j=0}^{\max\{k,m\}} q\binom{k}{j} \frac{(-1)^j}{k+m-j+1} \sum_{l=0}^{k+m-j+1} \binom{k+m-j+1}{l} \tilde{G}_{k+m-j-l+1,q} \tilde{G}_{l+1,q} = [2]^m \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \tilde{G}_{l+1,q} \frac{l+k+1}{l+1}.$$  

(3.1)

Proof. For $m,k \in \mathbb{N}$, then by (2.11),

$$I_1 = [2^q \int_{\mathbb{R}^d} x^k (x-1)^m d\mu_q(x) = [2^q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \int_{\mathbb{R}^d} x^l d\mu_q(x)$$

$$= [2^q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \tilde{G}_{l+1,q} \frac{l+k+1}{l+1}.$$  

(3.2)

On the other hand, the right hand side of (2.11),

$$I_2 = \max_{j=0}^{\max\{k,m\}} q\binom{k}{j} \frac{(-1)^j}{k+m-j+1} \sum_{l=0}^{k+m-j+1} \binom{k+m-j+1}{l} \tilde{G}_{k+m-j-l+1,q} \int_{\mathbb{R}^d} x^l d\mu_q(x)$$

$$= \max_{j=0}^{\max\{k,m\}} q\binom{k}{j} \frac{(-1)^j}{k+m-j+1} \sum_{l=0}^{k+m-j+1} \binom{k+m-j+1}{l} \tilde{G}_{k+m-j-l+1,q} \tilde{G}_{l+1,q}.$$  

(3.3)

Combining $I_1$ and $I_2$, we arrive to the proof of the theorem. \[\square\]

Theorem 3.2. For $k \in \mathbb{N}$, one has

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left\{ [2]^q \frac{\tilde{G}_{k+1+2,q}}{k+1+2} - q \frac{\tilde{G}_{k+1,q}}{k+1+1} \right\}$$

$$= [2]^q \sum_{j=1}^{\lfloor k/2 \rfloor} \binom{k}{2j} \frac{2k-2j+1}{2k-2j+1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{G}_{2k+1-2j-l,q} \tilde{G}_{l+1,q}$$

$$+ \sum_{j=1}^{\lfloor k/2 \rfloor} \binom{k}{2j-1} \frac{2k-2j+1}{2k-2j+1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{G}_{2k+1-2j-l,q} \tilde{G}_{l+1,q} + \frac{q-1}{1+q} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j+1} T_{k,j}.$$  

(3.4)

Here $T_{k,j} = [2]^q \sum_{l=0}^{2k-2j} \binom{2k-2j}{l} (2k-2j-l) \frac{\tilde{G}_{l+1,q} \tilde{G}_{2k-2j-l,q}}{(l+1)} + \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{G}_{l+1,q} \tilde{G}_{2k-2j-l,q} \frac{2k-2j+1}{(l+1)}$. 


Proof. Let us take fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ left-hand side of Theorem 2.5, we get

$$I_3 = \int_{\mathbb{Z}_p} x^k(x - 1)^k \left[ [2]_q x - q \right] d\mu_q(x)$$

$$= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{j} (-1)^{k-j-1} \int_{\mathbb{Z}_p} x^{k+j+1} d\mu_q(x) - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x)$$

$$= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{j} (-1)^{k-j} \tilde{G}_{k+j+2,q} \frac{k!}{k+j+2} - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \tilde{G}_{k+l+1,q} \frac{k!}{k+l+1}$$

In other word, we consider the right-hand side of Theorem 2.5 as follows:

$$I_4 = [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \left( \frac{1}{2j+1} \right) 2^{k-2j+1} \sum_{l=0}^{k-2j+1} \binom{2k-2j+1}{l} \tilde{G}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x)$$

$$+ \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \left( \frac{1}{2j} \right) 2^{k-2j+1} \sum_{l=0}^{k-2j+1} \binom{2k-2j+1}{l} \tilde{G}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x)$$

$$+ \sum_{j=0}^{[k/2]} \left( \frac{k}{2j+1} \right) \left\{ (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j-l}{l} \tilde{G}_{2k-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \right\}$$

$$+ \sum_{j=1}^{[k/2]} \left( \frac{k}{2j} \right) \left\{ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j-1} \binom{2k-2j-1-l}{l} \tilde{G}_{2k-2j-1-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \right\}$$

$$= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \left( \frac{1}{2j+1} \right) 2^{k-2j+1} \sum_{l=0}^{k-2j+1} \binom{2k-2j+1}{l} \tilde{G}_{2k+1-2j-l,q} \tilde{G}_{l+1,q}$$

$$+ \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \left( \frac{1}{2j} \right) 2^{k-2j+1} \sum_{l=0}^{k-2j+1} \binom{2k-2j+1}{l} \tilde{G}_{2k+1-2j-l,q} \tilde{G}_{l+1,q}$$

$$+ \sum_{j=0}^{[k/2]} \left( \frac{k}{2j+1} \right) \left\{ (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j-l}{l} \tilde{G}_{2k-2j-l,q} \frac{l}{l+1} \right\}$$

$$+ \sum_{j=1}^{[k/2]} \left( \frac{k}{2j} \right) \left\{ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j-1} \binom{2k-2j-1-l}{l} \tilde{G}_{2k-2j-1-l,q} \frac{l}{l+1} \right\}$$

Equating $I_3$ and $I_4$, we complete the proof of the theorem. \qed

As $q \to 1$ in the above theorem, we reach interesting property in Analytic Numbers Theory concerning ordinary Genocchi polynomials.
Corollary 3.3. For $k \in \mathbb{N}$, one has
\[
\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left\{ 2 \frac{G_{k+l+2}}{k+l+2} - G_{k+l+1} \right\} = 2 \sum_{j=0}^{[k/2]} \frac{\binom{k/2}{j}}{2k - 2j + 1} \sum_{l=0}^{2k-2j+1} \frac{\binom{2k-2j+1}{l}}{l+1} G_{2k+1-2j-l} G_{l+1}
\]
\[+ \sum_{j=1}^{[k/2]} \frac{\binom{k}{2j-1}}{2k - 2j + 1} \sum_{l=0}^{2k-2j+1} \frac{\binom{2k-2j+1}{l}}{l+1} G_{2k+1-2j-l} G_{l+1}.
\]
(3.7)

Theorem 3.4. For $m, k \in \mathbb{N}$, one has
\[
[2]_{q} \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} B_{l+k,q} = \sum_{l=0}^{\max(k,m)} \frac{q^{(k)}}{k+m-j+1} \sum_{j=0}^{l} \frac{\binom{k+m-j+1}{l}}{l} G_{k+m+1-j-l} B_{l,q}.
\]
(3.8)

Proof. We consider (2.11) and (2.2) by means of $q$-Volkenborn integral. Then, by (2.11), we see
\[
[2]_{q} \int_{\mathbb{Z}} x^{k} (x-1)^{m} d\mu_{q}(x) = [2]_{q} \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}} x^{l+k} d\mu_{q}(x) = [2]_{q} \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} B_{l+k,q}.
\]
(3.9)

On the other hand,
\[
\sum_{j=0}^{\max(k,m)} \frac{q^{(k)}}{k+m-j+1} \sum_{l=0}^{j} \frac{\binom{k+m-j+1}{l}}{l} G_{k+m+1-j-l} B_{l,q} = \sum_{j=0}^{\max(k,m)} \frac{q^{(k)}}{k+m-j+1} \sum_{l=0}^{j} \frac{\binom{k+m-j+1}{l}}{l} G_{k+m+1-j-l} B_{l,q}.
\]
(3.10)

Therefore, we get the proof of theorem. \hfill \Box

Corollary 3.5. For $k \in \mathbb{N}$, one gets
\[
\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left\{ [2]_{q} B_{k+l+1,q} - q B_{k+l,q} \right\}
\]
\[= [2]_{q} \sum_{j=0}^{[k/2]} \frac{\binom{k}{j}}{2k - 2j + 1} \sum_{l=0}^{2k-2j+1} \frac{\binom{2k-2j+1}{l}}{l} G_{2k+1-2j-l} B_{l,q} + \sum_{j=1}^{[k/2]} \frac{\binom{2j-1}{j}}{2k - 2j + 1}
\]
\[\times \sum_{l=0}^{2k-2j+1} \frac{\binom{2k-2j+1}{l}}{l} G_{2k+1-2j-l} B_{l,q} + \left( \frac{q-1}{q+1} \right) \sum_{j=0}^{[k/2]} \frac{k}{2j+1} S_{k,j}^{q}
\]
(3.11)

where
\[S_{k,j}^{q} = [2]_{q} \sum_{j=0}^{2k-2j} \frac{1}{l(2k-2j)} \left( \frac{2k-2j}{l} \right) G_{2k-2j-l} B_{l,q} + \sum_{l=0}^{2k-2j} \frac{1}{l(2k-2j+1)} \left( \frac{2k-2j+1}{l} \right) G_{2k-2j-l} B_{l,q}.
\]
Proof. By using \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) left-hand side of Theorem 2.5, we get

\[
I_5 = [2]_q \int_{\mathbb{Z}_p} x^k(x - 1)^k ([2]x - q) \, d\mu_q(x) \\
= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} \, d\mu_q(x) - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} \, d\mu_q(x) \\
= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \tilde{B}_{k+l,1,q} - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \tilde{B}_{k+l,1,q}. \\
\tag{3.12}
\]

Also, we compute the right-hand side of Theorem 2.5 as follows:

\[
I_6 = [2]_q \sum_{j=0}^{[k/2]} \frac{1}{2k - 2j + 1} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{G}_{2k-2j-1,q} \int_{\mathbb{Z}_p} x^l \, d\mu_q(x) \\
+ \sum_{j=1}^{[k/2]} \frac{1}{2k - 2j + 1} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{G}_{2k-2j-1,q} \int_{\mathbb{Z}_p} x^l \, d\mu_q(x) \\
+ \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left\{ \begin{array}{l}
\frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \frac{1}{2k - 2j + 1} \binom{2k-2j+1}{l} \tilde{G}_{2k-2j-1,q} \int_{\mathbb{Z}_p} x^l \, d\mu_q(x) \\
\end{array} \right\} \\
= [2]_q \sum_{j=0}^{[k/2]} \frac{1}{2k - 2j + 1} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{G}_{2k+1-2j-1,q} \tilde{B}_{l,q} \\
+ \sum_{j=1}^{[k/2]} \frac{1}{2k - 2j + 1} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{G}_{2k+1-2j-1,q} \tilde{B}_{l,q} \\
+ \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left\{ \begin{array}{l}
\frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \frac{1}{2k - 2j + 1} \binom{2k-2j+1}{l} \tilde{G}_{2k-2j-1,q} \tilde{B}_{l,q} \\
\end{array} \right\}. \\
\tag{3.13}
\]

Equating \( I_5 \) and \( I_6 \), we get the proof of Corollary. \qed

As \( q \to 1 \) in the above theorem, we easily derive the following corollary.
Corollary 3.6. For \( k \in \mathbb{N} \), one has
\[
\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} [2B_{k+l+1} - B_{k+l}] = 2 \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} G_{2k+1-2j-l} B_l^{(l)} + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} G_{2k+1-2j-l} B_l^{(l)}
\]

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References


