Research Article

Weighted $\bar{\partial}$ -Integral Representations of C^1 -Functions in C^n

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For C^1 -functions f , given in the complex space C^n , integral representations of the form $f = P(f) - T(\bar{\partial}f)$ are obtained. Here, P is the orthogonal projector of the space $L^2\{C^n; e^{-\sigma|z|^\rho}|z|^\gamma dm(z)\}$ onto its subspace of entire functions and the integral operator T appears by means of explicitly constructed kernel Φ which is investigated in detail.

1. Introduction

Let $n \geq 1$, $1 < p < +\infty$, and $0 < \rho, \sigma < \infty$, $\gamma > -2n$. Denote by $L_{\rho, \sigma, \gamma}^p(C^n)$ the space of all measurable complex-valued functions $f(z)$, $z \in C^n$, satisfying the condition

$$\int_{C^n} |f(z)|^p e^{-\sigma|z|^\rho} |z|^\gamma dm(z) < +\infty. \quad (1.1)$$

Let $H_{\rho, \sigma, \gamma}^p(C^n)$ be the corresponding subspace of entire functions. It was established in [1, 2] (when $n = 1$) and [3] (when $n > 1$) that arbitrary function $f \in H_{\rho, \sigma, \gamma}^p(C^n)$ has an integral representation of the form

$$f(z) = \frac{\rho \sigma^\mu}{2\pi^n} \int_{C^n} f(w) e^{-\sigma|w|^\rho} |w|^\gamma \cdot E_{\rho/2}^{(n)}(\sigma^{2/\rho} \langle z, w \rangle; \mu) dm(w), \quad z \in C^n, \quad (1.2)$$

where $\mu = (\gamma + 2n)/\rho$ and

$$E_{\rho/2}^{(n)}(\eta; \mu) = \sum_{k=0}^{\infty} \frac{\Gamma(k+n)}{\Gamma(k+1)} \cdot \frac{\eta^k}{\Gamma(\mu + 2k/\rho)}, \quad \eta \in C, \quad (1.3)$$

is the Mittag-Leffler type function. Moreover, the integral operator generated by the right-hand side of the formula (1.2) is an orthogonal projection of the space $L^2_{\rho,\sigma,\gamma}(C^n)$ onto its subspace $H^2_{\rho,\sigma,\gamma}(C^n)$. Certainly, the condition (1.1) and corresponding properties of $E^{(n)}_{\rho/2}(\eta; \mu)$ ensure an absolute convergence of the integral in (1.2).

Note that for $p = 2$, $\rho = 2$, $\gamma = 0$, $H^p_{\rho,\sigma,\gamma}(C^n)$ coincides with the well-known Fock space of entire functions and (1.2) takes the form

$$f(z) = \frac{\sigma^n}{\pi^n} \int_{C^n} f(w) \cdot e^{-\sigma|w|^2} \cdot e^{\sigma\langle z,w \rangle} dm(w), \quad z \in C^n. \quad (1.4)$$

In [4] general weighted integral representations were obtained for differential forms. In particular, for functions $f \in C^1(C^n)$ (satisfying certain growth conditions), the following generalization of the formula (1.4) was established:

$$\begin{aligned} f(z) = & \frac{\sigma^n}{\pi^n} \int_{C^n} f(w) \cdot e^{\sigma\langle z,w \rangle} \cdot e^{-\sigma|w|^2} dm(w) \\ & - \frac{\Gamma(n)}{\pi^n} \int_{C^n} \frac{\langle (\partial f / \partial \bar{w})(w), w - z \rangle}{|w - z|^{2n}} \cdot e^{\sigma\langle z,w \rangle} \cdot \sum_{v=0}^{n-1} \frac{\sigma^v}{v!} |w - z|^{2v} \cdot e^{-\sigma|w|^2} dm(w), \quad z \in C^n, \end{aligned} \quad (1.5)$$

where

$$\frac{\partial f}{\partial \bar{w}}(w) = \left(\frac{\partial f}{\partial \bar{w}_1}(w), \frac{\partial f}{\partial \bar{w}_2}(w), \dots, \frac{\partial f}{\partial \bar{w}_n}(w) \right) \quad (1.6)$$

and, consequently,

$$\left\langle \frac{\partial f}{\partial \bar{w}}(w), w - z \right\rangle = \sum_{k=1}^n \frac{\partial f}{\partial \bar{w}_k}(w) \cdot (\bar{w}_k - \bar{z}_k). \quad (1.7)$$

In [5] a canonical operator is constructed for $\bar{\partial}$ -solution in a space of differential forms square integrable with the weight $e^{-|w|^2}$.

The following natural question arises: as good as (1.4) is generalized for the case of smooth (not necessarily holomorphic) functions by the representation (1.5), is it possible to generalize the representation (1.2) in the similar way? Of course, such generalization should contain (1.5) as a particular case. Let us note (before we discuss this question) that in the case of bounded domains $\bar{\partial}$ (and $\partial\bar{\partial}$) integral representations are well investigated: for unit ball of C^n see [6–8]; for general strictly pseudoconvex domains see [9–13]; for Cartan matrix domain (“matrix disc”) see [14]. The whole space C^n essentially differs from strictly pseudoconvex and bounded symmetric domains (the last ones have rich group of automorphisms!), so the above-mentioned generalization requires other methods. For $n = 1$ it was done in [15]. For the case $n > 1$, two essentially different generalizations are possible. In [16] “polycylindric” weight function of the type $\prod_{k=1}^n e^{-\sigma_k |w_k|^{\rho_k}} |w_k|^{\gamma_k}$, $w = (w_1, w_2, \dots, w_n) \in C^n$, was considered. In the present paper, the corresponding weighted $\bar{\partial}$ -integral representations are obtained for the case of radial weight function of the type $e^{-\sigma|w|^\rho} |w|^\gamma$, $w = (w_1, w_2, \dots, w_n) \in C^n$,

$|w| = \sqrt{\sum_{k=1}^n |w_k|^2}$ (see the condition (1.1)). More precisely, for functions $f \in C^1(C^n)$ (satisfying certain growth conditions), the integral representation of the form

$$\begin{aligned} f(z) = & \frac{\rho\sigma^\mu}{2\pi^n} \int_{C^n} f(w) e^{-\sigma|w|^\rho} |w|^\gamma \cdot E_{\rho/2}^{(n)}(\sigma^{2/\rho} \langle z, w \rangle; \mu) dm(w) \\ & - \frac{\Gamma(n)}{\pi^n} \int_{C^n} \frac{\langle (\partial f / \partial \bar{w})(w), w - z \rangle}{|w - z|^{2n}} \cdot \Phi(z; w) dm(w), \quad z \in C^n, \end{aligned} \quad (1.8)$$

is established. Moreover, the kernel Φ is written in an explicit form. Also, we prove certain important differential and integral properties of this kernel. As it will be seen below, in the case of radial weight function, a new approach is requested and significant analytical difficulties are arisen.

Remark 1.1. In [4], instead of $|w|^\gamma \exp\{-\sigma|w|^\rho\}$, the case of weight function of the type $\exp\{-\varphi(w)\}$ was considered in the assumption that φ is a convex function of class C^2 . In this case, a formula of type (1.8) was obtained. But in that formula the operator of orthogonal projection of the space $L^2(C^n; \exp\{-\varphi(w)\} dm(w))$ onto its subspace of entire functions does not appear, except of the special case $\varphi(w) = \sigma|w|^2$ when we again obtain (1.5).

2. Heuristic Argument: Revealing of the Kernel Φ

In what follows, it is supposed that $n \geq 1$ and $0 < \rho, \sigma < \infty$, $\gamma > -2n$, $\mu = (\gamma + 2n)/\rho$.

We intend to reveal a formula of type (1.2) but this time for C^1 -functions. This means that the formula we search needs to have a second summand containing $\bar{\partial}$ -“part” of functions. In other words, for functions $f \in C^1(C^n)$ satisfying certain (indefinite yet) growth conditions at infinity, we search a formula of the type (1.8). Besides, it will be desirable for the kernel $\Phi(z; w)$, $z, w \in C^n$, to have the following (or similar) properties:

$$\Phi(z; z) = 1, \quad \forall z \in C^n, \quad (2.1)$$

$$\Phi(z; \infty) = 0, \quad \forall z \in C^n. \quad (2.2)$$

Denote by $K_{MB}(z; w)$ (for all $z \in C^n$, $w \in C^n \setminus \{z\}$) the well-known Martinelli-Bochner kernel, which has the following useful properties (see, for instance, [17, Chapter 16]):

$$\bar{\partial}_w K_{MB}(z; w) \equiv 0, \quad w \in C^n \setminus \{z\}; \quad (2.3)$$

$$\bar{\partial}_w \{f(w) \cdot K_{MB}(z; w)\} = \bar{\partial}f(w) \wedge K_{MB}(z; w) \equiv \frac{\Gamma(n)}{\pi^n} \cdot \frac{\langle (\partial f / \partial \bar{w})(w), w - z \rangle}{|w - z|^{2n}} \times dm(w) \quad (2.4)$$

for arbitrary function $f \in C^1(\Omega)$, where $\Omega \subset C^n \setminus \{z\}$ is an open set;

$$\lim_{\varepsilon \downarrow 0} \int_{|w-z|=\varepsilon} u(w) \cdot K_{MB}(z; w) = u(z) \quad (2.5)$$

for arbitrary function u continuous in a neighborhood of z ;

$$\int_{S_n} f(\zeta) \cdot K_{\text{MB}}(0; \zeta) = \frac{\Gamma(n)}{2\pi^n} \cdot \int_{S_n} f(\zeta) d\sigma(\zeta), \quad (2.6)$$

where $S_n = \{\zeta \in C^n : |\zeta| = 1\}$ is the unit sphere in C^n , $f \in C(S_n)$, and σ is the surface measure on S_n .

Let us fix an arbitrary $z \in C^n$ and consider the following differential form:

$$\varphi(w) = f(w) \cdot \Phi(z; w) \cdot K_{\text{MB}}(z; w), \quad w \in C^n \setminus \{z\}. \quad (2.7)$$

Then apply the Stokes formula to this form and to the domain $\{w \in C^n : 0 < \varepsilon < |w - z| < R < +\infty\}$:

$$\int_{|w-z|=R} \varphi - \int_{|w-z|=\varepsilon} \varphi = \int_{\varepsilon < |w-z| < R} d\varphi. \quad (2.8)$$

The integral $\int_{|w-z|=R} \varphi \rightarrow 0$ (as $R \rightarrow +\infty$) due to the property (2.2) (our argument is heuristic !!!). When $\varepsilon \rightarrow 0$, then $\int_{|w-z|=\varepsilon} \varphi \rightarrow f(z)$ due to the properties (2.5) and (2.1). Thus after $R \rightarrow +\infty$ and $\varepsilon \rightarrow 0$, we have

$$-f(z) = \int_{C^n} d\varphi, \quad (2.9)$$

or, in view of (2.3)-(2.4),

$$\begin{aligned} f(z) = & -\frac{\Gamma(n)}{\pi^n} \int_{C^n} \frac{f(w) \cdot \langle (\partial\Phi/\partial\bar{w})(z; w), w - z \rangle}{|w - z|^{2n}} dm(w) \\ & - \frac{\Gamma(n)}{\pi^n} \int_{C^n} \frac{\langle (\partial f/\partial\bar{w})(w), w - z \rangle \cdot \Phi(z; w)}{|w - z|^{2n}} dm(w). \end{aligned} \quad (2.10)$$

Comparing (1.8) and (2.10), we arrive at the equality

$$\begin{aligned} \left\langle \frac{\partial\Phi}{\partial\bar{w}}(z; w), w - z \right\rangle = & -\frac{\rho\sigma^\mu}{2\Gamma(n)} E_{\rho/2}^{(n)} \left(\sigma^{2/\rho} \langle z, w \rangle; \mu \right) e^{-\sigma|w|^\rho} |w|^\gamma |w - z|^{2n}, \\ & \forall w \in C^n \setminus \{z\}. \end{aligned} \quad (2.11)$$

Let us fix arbitrary $z, w \in C^n$, $z \neq w$ and consider the function

$$\varphi(\lambda) = \Phi(z; z + \lambda(w - z)), \quad \lambda \in C. \quad (2.12)$$

Then we have

$$\begin{aligned}
 \frac{\partial \varphi}{\partial \bar{\lambda}} &= \sum_{k=1}^n \frac{\partial \Phi}{\partial \bar{w}_k}(z; z + \lambda(w - z)) \cdot (\bar{w}_k - \bar{z}_k) \\
 &= \left\langle \frac{\partial \Phi}{\partial \bar{w}}(z; z + \lambda(w - z)), w - z \right\rangle \\
 &= -\frac{\rho \sigma^\mu}{\lambda \cdot 2\Gamma(n)} \times E_{\rho/2}^{(n)}\left(\sigma^{2/\rho} \langle z, z + \lambda(w - z) \rangle; \mu\right) e^{-\sigma|z + \lambda(w - z)|^\rho} \\
 &\quad \times |z + \lambda(w - z)|^\gamma |\lambda|^{2n} |w - z|^{2n}.
 \end{aligned} \tag{2.13}$$

The condition (2.2) implies $\varphi(\infty) = 0$; hence due to Cauchy-Green-Pompeiu formula,

$$\varphi(\lambda_0) = -\frac{1}{\pi} \int_C \frac{\partial \varphi / \partial \bar{\lambda}}{\lambda - \lambda_0} dm(\lambda), \quad \forall \lambda_0 \in C. \tag{2.14}$$

Since $\varphi(1) = \Phi(z; w)$, we finally have

$$\begin{aligned}
 \Phi(z; w) &= \frac{\rho \sigma^\mu}{2\pi\Gamma(n)} |w - z|^{2n} \\
 &\quad \times \int_C \frac{|\lambda|^{2n}}{\lambda(\lambda - 1)} E_{\rho/2}^{(n)}\left(\sigma^{2/\rho} \langle z, z + \lambda(w - z) \rangle; \mu\right) e^{-\sigma|z + \lambda(w - z)|^\rho} \\
 &\quad \times |z + \lambda(w - z)|^\gamma dm(\lambda), \quad \forall z \in C^n, \forall w \in C^n \setminus \{z\}.
 \end{aligned} \tag{2.15}$$

This formula can be also written in the following (may be, more convenient) form:

$$\begin{aligned}
 \Phi(z; z + w) &= \frac{\rho \sigma^\mu}{2\pi\Gamma(n)} |w|^{2n} \\
 &\quad \times \int_C \frac{|\lambda|^{2n}}{\lambda(\lambda - 1)} E_{\rho/2}^{(n)}\left(\sigma^{2/\rho} \langle z, z + \lambda w \rangle; \mu\right) e^{-\sigma|z + \lambda w|^\rho} |z + \lambda w|^\gamma dm(\lambda), \\
 &\quad \forall z \in C^n, \forall w \in C^n \setminus \{0\}.
 \end{aligned} \tag{2.16}$$

Now it is natural to investigate the properties of the kernel introduced. But first we need some auxiliary results.

3. Auxiliary Results

First of all let us put for brevity (see (1.3))

$$E(\eta) \equiv E_{\rho/2}^{(n)}(\eta; \mu), \quad \eta \in C. \tag{3.1}$$

It is an entire function of order $\rho/2$ and of type 1. The same is true for its derivative $E'(\eta)$, $\eta \in \mathbb{C}$. Consequently, we have

$$|E(\eta)| + |E'(\eta)| \leq \text{const}(\rho; n; \mu) \cdot e^{2|\eta|^{\rho/2}}, \quad \eta \in \mathbb{C}. \quad (3.2)$$

Let us introduce a convenient notation

$$\varphi(x) \equiv e^{-\sigma x^{\rho/2}} \cdot x^{\gamma/2}, \quad x \in (0; +\infty). \quad (3.3)$$

If $\gamma \geq 0$, we can suppose that $x \in [0; +\infty)$ in (3.3). Then obviously the function $\varphi \in C[0; +\infty)$ and

$$\varphi(x) \leq \text{const}(\rho; \sigma; \gamma) \cdot e^{(-\sigma/2)x^{\rho/2}}, \quad x \in [0; +\infty). \quad (3.4)$$

Note that under additional assumptions $\gamma \geq 2, \rho > 0$ or $\gamma = 0, \rho \geq 2$, the function $\varphi \in C^1[0; +\infty)$ and, moreover,

$$\begin{aligned} \varphi'(x) &= e^{-\sigma x^{\rho/2}} \cdot \left(\frac{\gamma}{2} x^{\gamma/2-1} - \sigma \frac{\rho}{2} x^{\rho/2-1+\gamma/2} \right), \quad x \in [0; +\infty), \\ |\varphi'(x)| &\leq \text{const}(\rho; \sigma; \gamma) \cdot e^{(-\sigma/2)x^{\rho/2}}, \quad x \in [0; +\infty). \end{aligned} \quad (3.5)$$

In view of (3.1) and (3.3), the formula (2.16) can be written as follows:

$$\begin{aligned} \Phi(z; z + w) &= \frac{\rho \sigma^\mu}{2\pi \Gamma(n)} |w|^{2n} \cdot \int_{\mathbb{C}} \frac{|\lambda|^{2n}}{\lambda(\lambda - 1)} E\left(\sigma^{2/\rho} \langle z, z + \lambda w \rangle\right) \varphi(|z + \lambda w|^2) dm(\lambda), \\ &\quad \forall z \in \mathbb{C}^n, \forall w \in \mathbb{C}^n \setminus \{0\}. \end{aligned} \quad (3.6)$$

Further, assume that $z \in \mathbb{C}^n, w \in \mathbb{C}^n \setminus \{0\}, \lambda \in \mathbb{C}$, then evidently

$$|z + \lambda w|^2 = |w|^2 \cdot |\lambda + \tilde{a}|^2 + \tilde{\delta}, \quad (3.7)$$

where

$$\tilde{a} = \frac{\langle z, w \rangle}{|w|^2}, \quad \tilde{\delta} = \frac{|z|^2 |w|^2 - |\langle z, w \rangle|^2}{|w|^2} \geq 0. \quad (3.8)$$

Note that $\tilde{\delta} = 0 \Leftrightarrow z$ and w lies on the same complex “straight line” (i.e., complex plane) of $\mathbb{C}^n$ passing through the origin.

Lemma 3.1. Assume that $0 < R < +\infty$ and $0 < m < M < +\infty$, then

$$\begin{aligned}
& \left| E\left(\sigma^{2/\rho}\langle z, z + \lambda w \rangle\right) \right| \cdot e^{-\sigma|z + \lambda w|^\rho} |z + \lambda w|^\gamma \\
& \equiv \left| E\left(\sigma^{2/\rho}\langle z, z + \lambda w \rangle\right) \right| \cdot \varphi(|z + \lambda w|^2) \\
& \leq \begin{cases} c \cdot e^{-b|\lambda|^\rho}, & \lambda \in C, \\ c_1 \cdot e^{-b_1|\lambda|^\rho} \cdot \left(|\lambda + \tilde{a}|^2 + \frac{\tilde{\delta}}{|w|^2} \right)^{\gamma/2} \leq c_1 \cdot e^{-b_1|\lambda|^\rho} \cdot |\lambda + \tilde{a}|^\gamma, & \lambda \in C \setminus \{-\tilde{a}\}, \quad (-2n < \gamma < 0), \end{cases} \quad (\gamma \geq 0), \\
& \hspace{25em} (3.9)
\end{aligned}$$

uniformly in z and w with $|z| \leq R$, $m \leq |w| \leq M$, where c, c_1, b, b_1 are positive constants and depend, in general, on $n, \rho, \sigma, \gamma, R, m, M$.

Proof. According to (3.2),

$$\begin{aligned}
\left| E\left(\sigma^{2/\rho}\langle z, z + \lambda w \rangle\right) \right| & \leq \text{const} \cdot e^{2\sigma|z|^{\rho/2}(|z| + |\lambda||w|)^{\rho/2}} \leq \text{const} \cdot e^{2\sigma|z|^{\rho/2}2^{\rho/2}(|z|^{\rho/2} + |\lambda|^{\rho/2}|w|^{\rho/2})} \\
& = \text{const} \cdot e^{2^{(2+\rho)/2}\sigma|z|^\rho} \cdot e^{2^{(2+\rho)/2}\sigma|z|^{\rho/2}|w|^{\rho/2}|\lambda|^{\rho/2}} \leq \text{const} \cdot e^{2^{(2+\rho)/2}\sigma R^\rho} \\
& \quad \cdot e^{2^{(2+\rho)/2}\sigma R^{\rho/2}M^{\rho/2}|\lambda|^{\rho/2}} \\
& \equiv \text{const} \cdot e^{k|\lambda|^{\rho/2}}, \quad \lambda \in C,
\end{aligned} \tag{3.10}$$

where k is a positive number.

Further, if $\gamma \geq 0$, then in view of (3.4) and (3.7)

$$\begin{aligned}
\varphi(|z + \lambda w|^2) & \leq \text{const}(\rho, \sigma, \gamma) \cdot e^{(-\sigma/2)|z + \lambda w|^\rho} \\
& = \text{const} \cdot e^{(-\sigma/2)|w|^\rho(|\lambda + \tilde{a}|^2 + \tilde{\delta}/|w|^2)^{\rho/2}} \\
& \leq \text{const} \cdot e^{(-\sigma/2)m^\rho|\lambda + \tilde{a}|^\rho}.
\end{aligned} \tag{3.11}$$

Due to the conditions on z and w , we have $|\tilde{a}| \leq R/m$. Let us choose $T > 0$ such that $R/mT \leq 1/2$, then for $|\lambda| \geq T$

$$|\lambda + \tilde{a}| \geq |\lambda| - |\tilde{a}| = |\lambda| \left(1 - \frac{|\tilde{a}|}{|\lambda|} \right) \geq |\lambda| \left(1 - \frac{R}{mT} \right) \geq \frac{|\lambda|}{2}. \tag{3.12}$$

Hence

$$e^{(-\sigma/2)m^\rho|\lambda + \tilde{a}|^\rho} \leq \text{const} \cdot e^{(-\sigma/2)m^\rho(|\lambda|^\rho/2^\rho)} \quad (\lambda \in C). \tag{3.13}$$

Combining (3.11) and (3.13), we obtain

$$\varphi(|z + \lambda w|^2) \leq \text{const} \cdot e^{-d|\lambda|^\rho}, \quad \lambda \in \mathbb{C}, \quad (3.14)$$

where d is a positive number.

Combination of (3.10) and (3.14) easily implies (3.9) for the case $\gamma \geq 0$. If $-2n < \gamma < 0$, then similarly to (3.11)–(3.14) we have

$$e^{-\sigma|z + \lambda w|^\rho} \leq \text{const} \cdot e^{-d_1|\lambda|^\rho}, \quad \lambda \in \mathbb{C}, \quad (3.15)$$

where d_1 is a positive number. Also, in view of (3.7),

$$\begin{aligned} |z + \lambda w|^\gamma &= \left(|w|^2 \cdot |\lambda + \tilde{a}|^2 + \tilde{\delta}\right)^{\gamma/2} \\ &= |w|^\gamma \left(|\lambda + \tilde{a}|^2 + \frac{\tilde{\delta}}{|w|^2}\right)^{\gamma/2} \leq m^\gamma \left(|\lambda + \tilde{a}|^2 + \frac{\tilde{\delta}}{|w|^2}\right)^{\gamma/2}. \end{aligned} \quad (3.16)$$

Combination of (3.10), (3.15), and (3.16) establishes (3.9) for the case $-2n < \gamma < 0$. The proof is complete. $\square$

Taking into account (3.2), (3.4), (3.5) and repeating “word by word” the argument of Lemma 3.1, we obtain the following lemma.

Lemma 3.2. *Assume that $0 < R < +\infty$ and $0 < m < M < +\infty$. Then there exist positive constants c, b (depending, in general, on $n, \rho, \sigma, \gamma, R, m, M$) such that uniformly in z and w with $|z| \leq R, m \leq |w| \leq M$.*

(a) *If $\gamma \geq 0$, then*

$$\frac{|E(\sigma^{2/\rho}\langle z, z + \lambda w \rangle)|}{|E'(\sigma^{2/\rho}\langle z, z + \lambda w \rangle)|} \times |\varphi(|z + \lambda w|^2)| \leq c \cdot e^{-b|\lambda|^\rho}, \quad \lambda \in \mathbb{C}. \quad (3.17)$$

(b) *If $\gamma \geq 2, \rho > 0$ or $\gamma = 0, \rho \geq 2$, then*

$$\frac{|E(\sigma^{2/\rho}\langle z, z + \lambda w \rangle)|}{|E'(\sigma^{2/\rho}\langle z, z + \lambda w \rangle)|} \times |\varphi'(|z + \lambda w|^2)| \leq c \cdot e^{-b|\lambda|^\rho}, \quad \lambda \in \mathbb{C}. \quad (3.18)$$

Lemma 3.3. *Let $\beta(\eta), \eta \in \mathbb{C}$, be an arbitrary function of class $C^1(\mathbb{C})$ such that it together with its first-order partial derivatives decreases (modulo) at infinity. For instance, the decreasing of type $O(1/|\eta|^{1+\varepsilon}), |\eta| \rightarrow +\infty$, (for arbitrary small $\varepsilon > 0$) is quite sufficient for us. Then the function*

$$\alpha(\eta) \equiv -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\beta(\xi)}{\xi - \eta} dm(\xi), \quad \eta \in \mathbb{C}, \quad (3.19)$$

is of class $C^1(\mathbb{C})$ and $(\partial\alpha/\partial\bar{\eta})(\eta) \equiv \beta(\eta), \eta \in \mathbb{C}$.

This assertion is of standard type so we omit the proof. Similar results one can find in [18, page 10, Theorem 1.1.3] for *bounded* open sets or in [19, page 300, Lemma] for *arbitrary simply connected* domains, but for C^∞ -functions.

Lemma 3.4. *Assume that $\psi(w)$, $w \in C^n \setminus \{0\}$, is continuously differentiable (i.e., of class C^1) and $g(w)$, $w \in C^n \setminus \{0\}$, is continuous. Then the following two relations are equivalent:*

$$\left\langle \frac{\partial \psi}{\partial \bar{w}}(w), w \right\rangle \equiv g(w), \quad w \in C^n \setminus \{0\}, \quad (3.20)$$

$$\frac{\partial}{\partial \bar{\eta}} \psi(\eta \cdot w) \equiv \frac{g(\eta \cdot w)}{\bar{\eta}}, \quad \eta \in C \setminus \{0\} \quad (\forall w \in C^n \setminus \{0\}). \quad (3.21)$$

Proof. Let us fix an arbitrary $w \in C^n \setminus \{0\}$; then

$$\frac{\partial}{\partial \bar{\eta}} \psi(\eta \cdot w) = \sum_{k=1}^n \frac{\partial \psi}{\partial \bar{w}_k}(\eta \cdot w) \cdot \bar{w}_k = \left\langle \frac{\partial \psi}{\partial \bar{w}}(\eta \cdot w), w \right\rangle = \frac{\langle (\partial \psi / \partial \bar{w})(\eta \cdot w), \eta \cdot w \rangle}{\bar{\eta}}, \quad (3.22)$$

$$\eta \in C \setminus \{0\}.$$

This immediately gives the implication (3.20) $\Rightarrow$ (3.21). On the contrary, if (3.21) is valid, then

$$\frac{\langle (\partial \psi / \partial \bar{w})(\eta \cdot w), \eta \cdot w \rangle}{\bar{\eta}} = \frac{g(\eta \cdot w)}{\bar{\eta}}, \quad \eta \in C \setminus \{0\}. \quad (3.23)$$

Substitution of $\eta = 1$ into the last relation gives (3.20). Thus, the assertion is proved. $\square$

4. The Main Properties of the Kernel Φ

Proposition 4.1. *If $\gamma > -1$, then for fixed $z \in C^n$, $w \in C^n \setminus \{0\}$ and for arbitrary $\eta \in C \setminus \{0\}$*

$$\Phi(z; z + \eta \cdot w) = -\frac{1}{\pi} \cdot \int_C \frac{\beta(\xi)}{\xi - \eta} dm(\xi), \quad (4.1)$$

where

$$\begin{aligned} \beta(\xi) &= -\frac{\rho \sigma^\mu}{2\Gamma(n)} |w|^{2n} \cdot \frac{|\xi|^{2n}}{\bar{\xi}} E\left(\sigma^{2/\rho} \langle z, z + \xi w \rangle\right) e^{-\sigma|z+\xi w|^\rho} |z + \xi w|^\gamma \\ &\equiv -\frac{\rho \sigma^\mu}{2\Gamma(n)} |w|^{2n} \cdot \frac{|\xi|^{2n}}{\bar{\xi}} E\left(\sigma^{2/\rho} \langle z, z + \xi w \rangle\right) \varphi\left(|z + \xi w|^2\right), \quad \xi \in C \setminus \{-\tilde{a}\}. \end{aligned} \quad (4.2)$$

Moreover, assume that $0 < R < +\infty$ and $0 < m < M < +\infty$. Then there exist positive constants c, q (depending, in general, on $n, \rho, \sigma, \gamma, R, m, M$) such that uniformly in z and w with $|z| \leq R$, $m \leq |w| \leq M$.

(a) If $\gamma \geq 0$, then β is a continuous function in C and

$$|\beta(\xi)| \leq c \cdot e^{-q|\xi|^p}, \quad \xi \in C. \quad (4.3)$$

(b) If $\gamma \geq 2$, $\rho > 0$ or $\gamma = 0$, $\rho \geq 2$, then β is a C^1 -function in C and, in addition to (4.3),

$$\left| \frac{\partial \beta(\xi)}{\partial \xi} \right| \leq c \cdot e^{-q|\xi|^p}, \quad \xi \in C. \quad (4.3')$$

Proof. Indeed, according to (2.16)

$$\begin{aligned} \Phi(z; z + \eta \cdot w) &= \frac{\rho \sigma^\mu}{2\pi \Gamma(n)} |w|^{2n} |\eta|^{2n} \\ &\times \int_C \frac{|\lambda|^{2n}}{\bar{\lambda}(\lambda - 1)} E\left(\sigma^{2/\rho} \langle z, z + \lambda \eta w \rangle\right) e^{-\sigma|z + \lambda \eta w|^\rho} |z + \lambda \eta w|^\gamma dm(\lambda) \stackrel{\lambda \eta \rightarrow \xi}{=} \\ &= \frac{\rho \sigma^\mu}{2\pi \Gamma(n)} |w|^{2n} \cdot \int_C \frac{|\xi|^{2n}}{\bar{\xi}(\xi - \eta)} E\left(\sigma^{2/\rho} \langle z, z + \xi w \rangle\right) e^{-\sigma|z + \xi w|^\rho} |z + \xi w|^\gamma dm(\xi) \\ &\equiv -\frac{1}{\pi} \cdot \int_C \frac{\beta(\xi)}{\xi - \eta} dm(\xi). \end{aligned} \quad (4.4)$$

As to (4.3)-(4.3'), these inequalities immediately follow from Lemma 3.2 and the following relations:

$$\begin{aligned} \frac{\partial}{\partial \xi} E\left(\sigma^{2/\rho} \langle z, z + \xi w \rangle\right) &= E'\left(\sigma^{2/\rho} \langle z, z + \xi w \rangle\right) \cdot 0 \equiv 0, \\ \frac{\partial}{\partial \bar{\xi}} E\left(\sigma^{2/\rho} \langle z, z + \xi w \rangle\right) &= E'\left(\sigma^{2/\rho} \langle z, z + \xi w \rangle\right) \cdot \sigma^{2/\rho} \langle z, w \rangle, \\ \frac{\partial}{\partial \xi} \varphi(|z + \xi w|^2) &= \varphi'(|z + \xi w|^2) \cdot (\langle w, z \rangle + \bar{\xi} |w|^2), \\ \frac{\partial}{\partial \bar{\xi}} \varphi(|z + \xi w|^2) &= \varphi'(|z + \xi w|^2) \cdot (\langle z, w \rangle + \xi |w|^2). \end{aligned} \quad (4.5)$$

□

Proposition 4.2. *If $\gamma \geq 0$, then the kernel $\Phi(z; z + w)$ is continuous in $z \in C^n$, $w \in C^n \setminus \{0\}$.*

Proof. Let us write $\Phi(z; z + w)$ as follows:

$$\Phi(z; z + w) = \frac{\rho \sigma^\mu}{2\pi \Gamma(n)} \cdot |w|^{2n} \cdot \int_C H(z; w; \lambda) dm(\lambda), \quad (4.6)$$

where

$$\begin{aligned} H(z; w; \lambda) &= \frac{|\lambda|^{2n}}{\lambda(\lambda - 1)} E\left(\sigma^{2/\rho} \langle z, z + \lambda w \rangle\right) e^{-\sigma|z + \lambda w|^\rho} |z + \lambda w|^\gamma \\ &\equiv \frac{|\lambda|^{2n}}{\lambda(\lambda - 1)} E\left(\sigma^{2/\rho} \langle z, z + \lambda w \rangle\right) \\ &\quad \cdot \varphi\left(|z + \lambda w|^2\right), \quad z \in C^n, \quad w \in C^n \setminus \{0\}, \quad \lambda \in C \setminus \{1\}. \end{aligned} \quad (4.7)$$

For arbitrary fixed positive numbers R, m, M ($m < M$), it suffices to construct a function $h(\lambda) \in L^1(C \setminus \{1\})$, such that

$$|H(z; w; \lambda)| \leq h(\lambda), \quad \lambda \in C \setminus \{1\}, \quad (4.8)$$

uniformly in z and w with $|z| \leq R$, $m \leq |w| \leq M$. According to Lemma 3.1 (the case $\gamma \geq 0$), the function

$$c \cdot \frac{|\lambda|^{2n-1}}{|\lambda - 1|} e^{-b|\lambda|^\rho} \quad (4.9)$$

is suitable for a function h we seek. □

Proposition 4.3. *If $\gamma \geq 0$ and $z \in C^n$ is arbitrary, then*

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{|w-z|=\varepsilon} f(w) \cdot \Phi(z; w) \cdot K_{\text{MB}}(z; w) \\ \equiv \lim_{\varepsilon \downarrow 0} \int_{|w|=\varepsilon} f(z + w) \cdot \Phi(z; z + w) \cdot K_{\text{MB}}(z; z + w) = f(z) \end{aligned} \quad (4.10)$$

for arbitrary function f continuous in a neighborhood of z .

Proof. For sufficiently small $\varepsilon > 0$, put

$$I_\varepsilon(z) = \int_{|w|=\varepsilon} f(z + w) \cdot \Phi(z; z + w) \cdot K_{\text{MB}}(z; z + w). \quad (4.11)$$

Taking into account the explicit formula (2.16) for $\Phi(z; z + w)$ and the well-known explicit form of the Martinelli-Bochner kernel $K_{\text{MB}}(z; z + w)$, we obtain

$$\begin{aligned}
I_\varepsilon(z) &= \frac{\rho\sigma^\mu}{2\pi\Gamma(n)} \cdot \frac{(-1)^{n(n-1)/2}\Gamma(n)}{(2\pi i)^n} \cdot \int_{|w|=\varepsilon} f(z+w) \\
&\quad \times |w|^{2n} \cdot \left\{ \int_C \frac{|\lambda|^{2n}}{\bar{\lambda}(\lambda-1)} E\left(\sigma^{2/\rho}\langle z, z + \lambda w \rangle\right) \cdot \varphi(|z + \lambda w|^2) dm(\lambda) \right\} \\
&\quad \times \frac{1}{|w|^{2n}} \cdot \sum_{j=1}^n (-1)^{j-1} \bar{w}_j d\bar{w}_{[j]} \wedge dw \stackrel{w=\varepsilon\cdot\zeta(\zeta\in S_n)}{=} \\
&= \frac{\rho\sigma^\mu}{2\pi\Gamma(n)} \cdot \frac{(-1)^{n(n-1)/2}\Gamma(n)}{(2\pi i)^n} \cdot \int_{S_n} f(z + \varepsilon \cdot \zeta) \\
&\quad \times \left\{ \int_C \frac{|\lambda|^{2n}}{\bar{\lambda}(\lambda-1)} E\left(\sigma^{2/\rho}\langle z, z + \lambda\varepsilon \cdot \zeta \rangle\right) \cdot \varphi(|z + \lambda\varepsilon \cdot \zeta|^2) dm(\lambda) \right\} \\
&\quad \times \varepsilon^{2n} \cdot \sum_{j=1}^n (-1)^{j-1} \bar{\zeta}_j d\bar{\zeta}_{[j]} \wedge d\zeta \\
&\equiv \frac{\rho\sigma^\mu}{2\pi\Gamma(n)} \cdot \varepsilon^{2n} \cdot \int_{S_n} f(z + \varepsilon \cdot \zeta) \\
&\quad \times \left\{ \int_C \frac{|\lambda|^{2n}}{\bar{\lambda}(\lambda-1)} E\left(\sigma^{2/\rho}\langle z, z + \lambda\varepsilon \cdot \zeta \rangle\right) \cdot \varphi(|z + \lambda\varepsilon \cdot \zeta|^2) dm(\lambda) \right\} \cdot K_{\text{MB}}(0; \zeta).
\end{aligned} \tag{4.12}$$

In view of (2.6), we obtain

$$\begin{aligned}
I_\varepsilon(z) &= \frac{\rho\sigma^\mu \varepsilon^{2n}}{4\pi^{n+1}} \cdot \int_{S_n} f(z + \varepsilon \cdot \zeta) \\
&\quad \times \left\{ \int_C \frac{|\lambda|^{2n}}{\bar{\lambda}(\lambda-1)} E\left(\sigma^{2/\rho}\langle z, z + \lambda\varepsilon \cdot \zeta \rangle\right) \cdot \varphi(|z + \lambda\varepsilon \cdot \zeta|^2) dm(\lambda) \right\} d\sigma(\zeta).
\end{aligned} \tag{4.13}$$

After the change of variable $\lambda \rightarrow \lambda/\varepsilon$ in the inner integral in (4.13), we have

$$\begin{aligned}
I_\varepsilon(z) &= \frac{\rho\sigma^\mu}{4\pi^{n+1}} \cdot \int_{S_n} f(z + \varepsilon \cdot \zeta) \\
&\quad \times \left\{ \int_C \frac{|\lambda|^{2n}}{\bar{\lambda}(\lambda-\varepsilon)} E\left(\sigma^{2/\rho}\langle z, z + \lambda \cdot \zeta \rangle\right) \cdot \varphi(|z + \lambda \cdot \zeta|^2) dm(\lambda) \right\} d\sigma(\zeta) \stackrel{\lambda \rightarrow \lambda+\varepsilon}{=}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\rho\sigma^\mu}{4\pi^{n+1}} \cdot \int_{S_n} f(z + \varepsilon \cdot \zeta) \\
&\quad \times \left\{ \int_C \frac{(\lambda + \varepsilon)^n (\bar{\lambda} + \varepsilon)^{n-1}}{\lambda} E\left(\sigma^{2/\rho} \langle z, z + (\lambda + \varepsilon)\zeta \rangle\right) \cdot \varphi\left(|z + (\lambda + \varepsilon)\zeta|^2\right) dm(\lambda) \right\} d\sigma(\zeta) \\
&\equiv \frac{\rho\sigma^\mu}{4\pi^{n+1}} \cdot \int_{S_n} f(z + \varepsilon \cdot \zeta) \cdot I_\varepsilon(z; \zeta) d\sigma(\zeta),
\end{aligned} \tag{4.14}$$

where

$$I_\varepsilon(z; \zeta) = \int_C \rho_\varepsilon(z; \zeta; \lambda) dm(\lambda), \quad \zeta \in S_n, \tag{4.15}$$

where

$$\begin{aligned}
\rho_\varepsilon(z; \zeta; \lambda) &= \frac{(\lambda + \varepsilon)^n (\bar{\lambda} + \varepsilon)^{n-1}}{\lambda} E\left(\sigma^{2/\rho} \langle z, z + (\lambda + \varepsilon)\zeta \rangle\right) \cdot \varphi\left(|z + (\lambda + \varepsilon)\zeta|^2\right), \\
&\quad \zeta \in S_n, \lambda \in C \setminus \{0\}.
\end{aligned} \tag{4.16}$$

Obviously, without loss of generality, we can suppose that $0 < \varepsilon \leq 1$.

According to Lemma 3.1 (the case $\gamma \geq 0$) or, equivalently, to Lemma 3.2(a), there exist positive constants c, b (depending on n, ρ, σ, γ) such that

$$\begin{aligned}
|\rho_\varepsilon(z; \zeta; \lambda)| &\leq \frac{(|\lambda| + 1)^{2n-1}}{|\lambda|} \cdot c \cdot e^{-b|\lambda + \varepsilon|^\rho} \\
&\leq h(\lambda) \equiv \begin{cases} c \cdot \frac{(|\lambda| + 1)^{2n-1}}{|\lambda|}, & 0 < |\lambda| \leq 1 \\ c \cdot \frac{(|\lambda| + 1)^{2n-1}}{|\lambda|} \cdot e^{-b(|\lambda| - 1)^\rho}, & |\lambda| > 1 \end{cases} \in L^1(C \setminus \{0\})
\end{aligned} \tag{4.17}$$

uniformly in $\zeta \in S_n, 0 < \varepsilon \leq 1$. Hence, due to the Lebesgue dominated convergence theorem, we can conclude that

- (i) the functions $I_\varepsilon(z; \zeta)$ are continuous in $\zeta \in S_n$;
- (ii) $|I_\varepsilon(z; \zeta)| \leq M < +\infty, \zeta \in S_n$ (uniformly in ε);
- (iii) For all $\zeta \in S_n : \lim_{\varepsilon \downarrow 0} I_\varepsilon(z; \zeta) = I(z; \zeta)$, where

$$I(z; \zeta) = \int_C |\lambda|^{2n-2} \cdot E\left(\sigma^{2/\rho} \langle z, z + \lambda\zeta \rangle\right) \cdot \varphi\left(|z + \lambda\zeta|^2\right) dm(\lambda). \tag{4.18}$$

Note that the function $I(z; \zeta)$ is also continuous in $\zeta \in S_n$ and $|I(z; \zeta)| \leq M < +\infty, \zeta \in S_n$.

Now remember (see (4.14)) that

$$I_\varepsilon(z) = \frac{\rho\sigma^\mu}{4\pi^{n+1}} \cdot \int_{S_n} f(z + \varepsilon \cdot \zeta) \cdot I_\varepsilon(z; \zeta) d\sigma(\zeta). \quad (4.19)$$

Therefore, the application of the Lebesgue dominated convergence theorem once again gives

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} I_\varepsilon(z) &= f(z) \cdot \frac{\rho\sigma^\mu}{4\pi^{n+1}} \int_{S_n} I(z; \zeta) d\sigma(\zeta) \\ &= f(z) \cdot \frac{\rho\sigma^\mu}{4\pi^{n+1}} \int_{S_n} d\sigma(\zeta) \int_C |\lambda|^{2n-2} \cdot E\left(\sigma^{2/\rho} \langle z, z + \lambda\zeta \rangle\right) \cdot \varphi\left(|z + \lambda\zeta|^2\right) dm(\lambda) \\ &= f(z) \cdot \frac{\rho\sigma^\mu}{4\pi^{n+1}} \int_{S_n} d\sigma(\zeta) \int_0^{+\infty} \int_0^{2\pi} r^{2n-1} \cdot E\left(\sigma^{2/\rho} \langle z, z + re^{i\vartheta} \cdot \zeta \rangle\right) \\ &\quad \cdot \varphi\left(|z + re^{i\vartheta} \cdot \zeta|^2\right) dr d\vartheta \stackrel{r\cdot\zeta=w}{=} \\ &= f(z) \cdot \frac{\rho\sigma^\mu}{4\pi^{n+1}} \int_0^{2\pi} d\vartheta \int_{C^n} E\left(\sigma^{2/\rho} \langle z, z + e^{i\vartheta} \cdot w \rangle\right) \\ &\quad \cdot \varphi\left(|z + e^{i\vartheta} \cdot w|^2\right) dm(w) \stackrel{z+e^{i\vartheta}\cdot w \rightarrow w}{=} \\ &= f(z) \cdot \frac{\rho\sigma^\mu}{4\pi^{n+1}} \int_0^{2\pi} d\vartheta \int_{C^n} E\left(\sigma^{2/\rho} \langle z, w \rangle\right) \cdot \varphi(|w|^2) dm(w) \\ &= f(z) \cdot \frac{\rho\sigma^\mu}{2\pi^n} \int_{C^n} E\left(\sigma^{2/\rho} \langle z, w \rangle\right) \cdot e^{-\sigma|w|^\rho} |w|^\gamma dm(w) = f(z) \cdot 1 = f(z) \end{aligned} \quad (4.20)$$

in view of (1.2). Thus (4.10) is established. $\square$

Remark 4.4. Assume for a moment that $\Phi(z; z+w)$ can be defined at $w = 0$ such that $\Phi(z; z) = 1$ and, moreover, that after this $\Phi(z; z+w)$ becomes continuous at $w = 0$ (unfortunately, this is not, in general, true, as it will be mentioned below). Then the relation (4.10) is a simple consequence of (2.5). Hence (4.10) can be considered as a substitute for the natural (and “very desired”) property $\Phi(z; z) = 1$.

Proposition 4.5. *If $\gamma \geq 2$, $\rho > 0$ or $\gamma = 0$, $\rho \geq 2$, the kernel $\Phi(z; z+w)$ is a function of class C^1 in $z \in C^n$, $w \in C^n \setminus \{0\}$.*

Proof. In view of (4.6)-(4.7), we have to show that $\int_C H(z; w; \lambda) dm(\lambda)$ is of class C^1 in $z \in C^n$, $w \in C^n \setminus \{0\}$. In other words, for arbitrary fixed positive numbers R, m, M ($m < M$), it suffices to construct a function $h(\lambda) \in L^1(C \setminus \{1\})$, such that

$$\begin{aligned} \left| \frac{\partial}{\partial w_k} H(z; w; \lambda) \right|, \left| \frac{\partial}{\partial \bar{w}_k} H(z; w; \lambda) \right| &\leq h(\lambda), \quad \lambda \in C \setminus \{1\}, \\ \left| \frac{\partial}{\partial z_k} H(z; w; \lambda) \right|, \left| \frac{\partial}{\partial \bar{z}_k} H(z; w; \lambda) \right| &\leq h(\lambda), \quad \lambda \in C \setminus \{1\}, \end{aligned} \quad (4.21)$$

uniformly in z and w with $|z| \leq R$, $m \leq |w| \leq M$, and $k = 1, \dots, n$.

Explicitly computing the corresponding partial derivatives and taking note of Lemma 3.2, we find that we are reduced to the question of the finiteness of integrals of the type

$$\int_C \frac{|\lambda|^{2n+\tau}}{|\lambda-1|} e^{-b|\lambda|^\rho} dm(\lambda) \quad (b > 0, \tau = -1; 0; 1), \quad (4.22)$$

which is evident. $\square$

Proposition 4.6. *If $\gamma \geq 2$, $\rho > 0$ or $\gamma = 0$, $\rho \geq 2$, then for arbitrary fixed $z \in C^n$ we have*

$$\left\langle \frac{\partial \Phi}{\partial \bar{w}}(z; z+w), w \right\rangle = -\frac{\rho \sigma^\mu}{2\Gamma(n)} |w|^{2n} E\left(\sigma^{2/\rho} \langle z, z+w \rangle\right) e^{-\sigma|z+w|^\rho} |z+w|^\gamma, \quad (4.23)$$

$$\forall w \in C^n \setminus \{0\},$$

or, equivalently,

$$\left\langle \frac{\partial \Phi}{\partial \bar{w}}(z; w), w-z \right\rangle = -\frac{\rho \sigma^\mu}{2\Gamma(n)} |w-z|^{2n} E\left(\sigma^{2/\rho} \langle z, w \rangle\right) e^{-\sigma|w|^\rho} |w|^\gamma, \quad (4.23')$$

$$\forall w \in C^n \setminus \{z\}.$$

Proof. We intend to use Lemma 3.4. To this end, let us put for $w \in C^n \setminus \{0\}$

$$\begin{aligned} \psi(w) &= \Phi(z; z+w), \\ g(w) &= -\frac{\rho \sigma^\mu}{2\Gamma(n)} |w|^{2n} E\left(\sigma^{2/\rho} \langle z, z+w \rangle\right) e^{-\sigma|z+w|^\rho} |z+w|^\gamma. \end{aligned} \quad (4.24)$$

It suffices to establish (3.21) for the introduced functions ψ and g . Fix $w \in C^n \setminus \{0\}$, then in view of Proposition 4.1 we have

$$\psi(\eta \cdot w) = -\frac{1}{\pi} \cdot \int_C \frac{\beta(\xi)}{\xi - \eta} dm(\xi), \quad \forall \eta \in C \setminus \{0\}, \quad (4.25)$$

where $\beta(\xi) \in C^1(C)$ is defined by (4.2) and satisfies (4.3)-(4.3'). Consequently, β satisfies all the conditions of Lemma 3.3. Hence

$$\begin{aligned} \frac{\partial}{\partial \bar{\eta}} \psi(\eta \cdot w) &\equiv \beta(\eta) = -\frac{\rho \sigma^\mu}{2\Gamma(n)} |w|^{2n} \cdot \frac{|\eta|^{2n}}{\eta} E\left(\sigma^{2/\rho} \langle z, z+\eta w \rangle\right) e^{-\sigma|z+\eta w|^\rho} |z+\eta w|^\gamma \\ &\equiv \frac{g(\eta \cdot w)}{\bar{\eta}}, \quad \forall \eta \in C \setminus \{0\}. \end{aligned} \quad (4.26)$$

By this (3.21) has been established for the pair of introduced functions ψ and g . The proof is complete. $\square$

The next assertion describes the behaviour of the kernel $\Phi(z; z + w)$ when $w \rightarrow 0$ or $w \rightarrow \infty$. Since these properties will not be used in what follows, we omit the proof, which, by the way, is not easy.

Proposition 4.7. *If $\gamma \geq 0$, then*

(a) *for arbitrary $z \in C^n$ and for arbitrary $w \in C^n \setminus \{0\}$, we have*

$$\lim_{\eta \rightarrow 0} \Phi(z; z + \eta \cdot w) = \frac{\rho \sigma^\mu \cdot |w|^{2n}}{2\pi \cdot \Gamma(n)} \times \int_C |\xi|^{2n-2} \cdot E\left(\sigma^{2/\rho} \langle z, z + \xi w \rangle\right) \cdot e^{-\sigma|z+\xi w|^\rho} |z + \xi w|^\gamma dm(\xi); \quad (4.27)$$

(b) *if $z \neq 0$ and $[z]$ is the complex plane generated by the vector z , then*

$$\lim_{w \rightarrow 0, w \perp [z]} \Phi(z; z + w) = \frac{\rho \sigma^\mu}{2\Gamma(n)} \cdot E\left(\sigma^{2/\rho} |z|^2\right) \cdot \int_0^{+\infty} x^{n-1} \cdot e^{-\sigma(|z|^2+x)^{\rho/2}} \left\{|z|^2 + x\right\}^{\gamma/2} dx, \quad (4.28)$$

and for arbitrary $R > 0$ and for (arbitrary small) $\varepsilon > 0$, there exist a positive constant $c = c(n, \rho, \sigma, \gamma, \varepsilon, R)$ such that

$$|\Phi(z; z + w)| \leq c \cdot e^{-(\sigma-\varepsilon)|w|^\rho}, \quad \forall w \perp [z] (w \neq 0), \quad (4.29)$$

uniformly in z with $|z| < R$;

(c) *if $z \neq 0$, then*

$$\lim_{w \rightarrow 0, w \in [z]} \Phi(z; z + w) = \sum_{m=0}^{n-1} (-1)^{n-1-m} \cdot C_{n-1}^m \cdot \left(|z|^2 \sigma^{2/\rho}\right)^{n-1-m} \cdot \mathcal{L}_{n,m}^{\gamma,\rho}, \quad (4.30)$$

where the coefficients $\mathcal{L}_{n,m}^{\gamma,\rho} (m = 1, 2, \dots, n-1)$ can be written in an explicit form;

(d) *for arbitrary $R > 0$, there exist positive constants c, δ_1 (depending, in general, on $n, \rho, \sigma, \gamma, R$) such that*

$$|\Phi(z; z + w)| \leq c \cdot \left(|w| + e^{-\delta_1|w|^\rho}\right) \quad (4.31)$$

uniformly in $w \in C^n \setminus \{0\}$ and z with $|z| < R$. In particular, the kernel $\Phi(z; z + w)$ remains bounded (uniformly in z with $|z| < R$) as $w \rightarrow 0$.

Remark 4.8. As it follows from (4.28) and (4.30), in general, $\lim_{w \rightarrow 0} \Phi(z; z + w)$ (when w tends to zero arbitrarily) cannot be properly defined (i.e., this limit does not exist). In view of this fact, it seems surprising the existence of the limit in (4.27). In fact, it only means that, nevertheless, the restrictions of the kernel $\Phi(z; z + w)$ on complex planes $[w]$ (generated by arbitrary $w \in C^n \setminus \{0\}$) have limit values at the origin.

5. The Main Integral Representation

Now we are ready to formulate and prove the main result: an integral representation of the type (1.8). To this end, we have to repeat the heuristic argument of Section 2, but this time it should be well reasoned.

In what follows, we need a function $\chi(t)$, $t \in (-\infty; +\infty)$, satisfying the following conditions:

- (i) $\chi \in C^1(R)$;
- (ii) $0 \leq \chi(t) \leq 1$, $t \in (-\infty; +\infty)$;
- (iii) $\chi(t) \equiv 1$, $t \in (-\infty; 0]$;
- (iv) $\chi(t) \equiv 0$, $t \in [1; +\infty)$;
- (v) $\chi \downarrow [0; 1]$;
- (vi) $|\chi'(t)| \leq M < +\infty$, $t \in [0; 1]$ and (obviously) $\chi'(t) \equiv 0$ otherwise.

The existence of such functions is evident. Then put

$$\chi_R(w) \equiv \chi(|w|^2 - R^2) \begin{cases} = 1, & 0 \leq |w| \leq R, \\ \in [0; 1], & R \leq |w| \leq \sqrt{R^2 + 1}, \\ = 0, & |w| \geq \sqrt{R^2 + 1}. \end{cases} \quad (5.1)$$

Note that $\chi_R \in C^1(C^n)$ and

$$\begin{aligned} \frac{\partial \chi_R}{\partial \bar{w}}(w) &\equiv \left(\frac{\partial \chi_R}{\partial \bar{w}_1}(w), \frac{\partial \chi_R}{\partial \bar{w}_2}(w), \dots, \frac{\partial \chi_R}{\partial \bar{w}_n}(w) \right) \\ &= \begin{cases} (0, 0, \dots, 0), & 0 \leq |w| \leq R, \\ \chi'(|w|^2 - R^2) \cdot (w_1, w_2, \dots, w_n), & R \leq |w| \leq \sqrt{R^2 + 1}, \\ (0, 0, \dots, 0), & |w| \geq \sqrt{R^2 + 1}. \end{cases} \end{aligned} \quad (5.2)$$

Moreover,

$$\left| \frac{\partial \chi_R}{\partial \bar{w}}(w) \right| \equiv \sqrt{\sum_{k=1}^n \left| \frac{\partial \chi_R}{\partial \bar{w}_k}(w) \right|^2} \begin{cases} = 0, & 0 \leq |w| \leq R, \\ \leq M \cdot |w|, & R \leq |w| \leq \sqrt{R^2 + 1}, \\ = 0, & |w| \geq \sqrt{R^2 + 1}. \end{cases} \quad (5.3)$$

Theorem 5.1. Assume that $n \geq 1$, $1 < p < +\infty$, $\sigma > 0$ and $\mu = (\gamma + 2n)/\rho$, where either $\gamma \geq 2$, $\rho > 0$ or $\gamma = 0$, $\rho \geq 2$. If the kernel $\Phi(z; w)$, for all $z \in C^n$, for all $w \in C^n \setminus \{z\}$, is defined by the formula (2.15), then the integral representation of the form

$$\begin{aligned} f(z) &= \frac{\rho\sigma^\mu}{2\pi^n} \int_{C^n} f(w) \cdot E_{\rho/2}^{(n)}(\sigma^{2/\rho} \langle z, w \rangle; \mu) \cdot e^{-\sigma|w|^\rho} |w|^\gamma dm(w) \\ &\quad - \frac{\Gamma(n)}{\pi^n} \int_{C^n} \frac{\langle (\partial f / \partial \bar{w})(w), w - z \rangle}{|w - z|^{2n}} \Phi(z; w) dm(w), \quad z \in C^n, \end{aligned} \quad (5.4)$$

is valid for each function $f \in C^1(C^n)$ satisfying the following conditions:

- (a) $f \in L_{\rho, \sigma, \gamma}^p(C^n)$;
- (b) for any fixed $z \in C^n$,

$$\frac{|f(w)| \cdot \Phi(z; w)}{|w - z|^{2n-2}} \in L^1(C^n; dm(w)) \iff \frac{|f(w)| \cdot \Phi(z; w)}{|w|^{2n-2}} \in L^1(C^n; dm(w)); \quad (5.5)$$

- (c) for any fixed $z \in C^n$

$$\frac{|(\partial f / \partial \bar{w})(w)| \cdot \Phi(z; w)}{|w - z|^{2n-1}} \in L^1(C^n; dm(w)) \iff \frac{|(\partial f / \partial \bar{w})(w)| \cdot \Phi(z; w)}{|w|^{2n-1}} \in L^1(C^n; dm(w)). \quad (5.6)$$

Proof. Let us fix an arbitrary $z \in C^n$, and for $\forall R > 0$ consider the following differential form:

$$\varphi(z; w) = f(w) \cdot \chi_R(w - z) \cdot \Phi(z; w) \cdot K_{MB}(z; w), \quad w \in C^n \setminus \{z\}. \quad (5.7)$$

Then choose an $\varepsilon \in (0; R)$ and apply the Stokes formula to this form and to the domain $\{w \in C^n : 0 < \varepsilon < |w - z| < \sqrt{R^2 + 1}\}$:

$$\int_{|w-z|=\sqrt{R^2+1}} \varphi - \int_{|w-z|=\varepsilon} \varphi = \int_{\varepsilon < |w-z| < \sqrt{R^2+1}} d\varphi. \quad (5.8)$$

In view of (5.1) and (2.4), the last relation can be written as follows:

$$\begin{aligned} & - \int_{|w-z|=\varepsilon} f(w) \cdot \Phi(z; w) \cdot K_{MB}(z; w) \\ &= \frac{\Gamma(n)}{\pi^n} \int_{\varepsilon < |w-z| < \sqrt{R^2+1}} \frac{\langle (\partial f / \partial \bar{w})(w), w - z \rangle}{|w - z|^{2n}} \cdot \chi_R(w - z) \cdot \Phi(z; w) dm(w) \end{aligned}$$

$$\begin{aligned}
& + \frac{\Gamma(n)}{\pi^n} \int_{\varepsilon < |w-z| < \sqrt{R^2+1}} \frac{\langle (\partial \chi_R / \partial \bar{w})(w-z), w-z \rangle}{|w-z|^{2n}} \cdot f(w) \cdot \Phi(z; w) dm(w) \\
& + \frac{\Gamma(n)}{\pi^n} \int_{\varepsilon < |w-z| < \sqrt{R^2+1}} \frac{\langle (\partial \Phi / \partial \bar{w})(z; w), w-z \rangle}{|w-z|^{2n}} \cdot f(w) \cdot \chi_R(w-z) dm(w).
\end{aligned} \tag{5.9}$$

Moreover, (5.2) and (4.23') imply

$$\begin{aligned}
& - \int_{|w-z|=\varepsilon} f(w) \cdot \Phi(z; w) \cdot K_{MB}(z; w) \\
& = \frac{\Gamma(n)}{\pi^n} \int_{\varepsilon < |w-z| < \sqrt{R^2+1}} \frac{\langle (\partial f / \partial \bar{w})(w), w-z \rangle}{|w-z|^{2n}} \cdot \chi_R(w-z) \cdot \Phi(z; w) dm(w) \\
& + \frac{\Gamma(n)}{\pi^n} \int_{R < |w-z| < \sqrt{R^2+1}} \frac{\chi'(|w-z|^2 - R^2)}{|w-z|^{2n-2}} \cdot f(w) \cdot \Phi(z; w) dm(w) \\
& - \frac{\rho \sigma^\mu}{2\pi^n} \int_{\varepsilon < |w-z| < \sqrt{R^2+1}} E_{\rho/2}^{(n)} \left(\sigma^{2/\rho} \langle z, w \rangle; \mu \right) e^{-\sigma|w|^\rho} |w|^\gamma \cdot f(w) \cdot \chi_R(w-z) dm(w).
\end{aligned} \tag{5.10}$$

When $\varepsilon \rightarrow 0$, then due to (4.10) we obtain

$$\begin{aligned}
f(z) & = \frac{\rho \sigma^\mu}{2\pi^n} \int_{|w-z| < \sqrt{R^2+1}} E_{\rho/2}^{(n)} \left(\sigma^{2/\rho} \langle z, w \rangle; \mu \right) e^{-\sigma|w|^\rho} |w|^\gamma \cdot f(w) \cdot \chi_R(w-z) dm(w) \\
& - \frac{\Gamma(n)}{\pi^n} \int_{|w-z| < \sqrt{R^2+1}} \frac{\langle (\partial f / \partial \bar{w})(w), w-z \rangle}{|w-z|^{2n}} \cdot \chi_R(w-z) \cdot \Phi(z; w) dm(w) \\
& - \frac{\Gamma(n)}{\pi^n} \int_{R < |w-z| < \sqrt{R^2+1}} \frac{\chi'(|w-z|^2 - R^2)}{|w-z|^{2n-2}} \cdot f(w) \cdot \Phi(z; w) dm(w) \\
& \equiv A_1(R) - A_2(R) - A_3(R).
\end{aligned} \tag{5.11}$$

Further,

$$\begin{aligned}
A_1(R) & = \frac{\rho \sigma^\mu}{2\pi^n} \int_{|w-z| < R} E_{\rho/2}^{(n)} \left(\sigma^{2/\rho} \langle z, w \rangle; \mu \right) e^{-\sigma|w|^\rho} |w|^\gamma \cdot f(w) dm(w) \\
& + \frac{\rho \sigma^\mu}{2\pi^n} \int_{R < |w-z| < \sqrt{R^2+1}} E_{\rho/2}^{(n)} \left(\sigma^{2/\rho} \langle z, w \rangle; \mu \right) e^{-\sigma|w|^\rho} |w|^\gamma \cdot f(w) \cdot \chi_R(w-z) dm(w) \\
& \equiv A_1^{(1)}(R) + A_1^{(2)}(R).
\end{aligned} \tag{5.12}$$

Since (via the condition (a))

$$f(w) \cdot E_{\rho/2}^{(n)}\left(\sigma^{2/\rho}\langle z, w \rangle; \mu\right) e^{-\sigma|w|^\rho} |w|^\gamma \in L^1(C^n; dm(w)), \quad (5.13)$$

the summand $A_1^{(1)}(R)$ tends to the same integral but taken now over the whole space C^n and $A_1^{(2)}(R)$ tends to zero (as $R \rightarrow +\infty$).

In other words,

$$\lim_{R \rightarrow +\infty} A_1(R) = \frac{\rho\sigma^\mu}{2\pi^n} \int_{C^n} f(w) \cdot E_{\rho/2}^{(n)}\left(\sigma^{2/\rho}\langle z, w \rangle; \mu\right) e^{-\sigma|w|^\rho} |w|^\gamma dm(w). \quad (5.14)$$

Similarly,

$$\begin{aligned} A_2(R) &= \frac{\Gamma(n)}{\pi^n} \int_{|w-z|<R} \frac{\langle (\partial f / \partial \bar{w})(w), w-z \rangle}{|w-z|^{2n}} \cdot \Phi(z; w) dm(w) \\ &\quad + \frac{\Gamma(n)}{\pi^n} \int_{R<|w-z|<\sqrt{R^2+1}} \frac{\langle (\partial f / \partial \bar{w})(w), w-z \rangle}{|w-z|^{2n}} \cdot \chi_R(w-z) \cdot \Phi(z; w) dm(w) \\ &\equiv A_2^{(1)}(R) + A_2^{(2)}(R). \end{aligned} \quad (5.15)$$

In view of the condition (c) of the theorem

$$\frac{\langle (\partial f / \partial \bar{w})(w), w-z \rangle}{|w-z|^{2n}} \cdot \Phi(z; w) \in L^1(C^n; dm(w)), \quad (5.16)$$

so $A_2^{(1)}(R)$ tends to the same integral but taken now over the whole space C^n and $A_2^{(2)}(R)$ tends to zero (as $R \rightarrow +\infty$). In other words,

$$\lim_{R \rightarrow +\infty} A_2(R) = \frac{\Gamma(n)}{\pi^n} \int_{C^n} \frac{\langle (\partial f / \partial \bar{w})(w), w-z \rangle}{|w-z|^{2n}} \cdot \Phi(z; w) dm(w). \quad (5.17)$$

Finally,

$$\begin{aligned} |A_3(R)| &\leq \frac{\Gamma(n)}{\pi^n} \cdot M \cdot \int_{R<|w-z|<\sqrt{R^2+1}} \frac{|f(w)| \cdot \Phi(z; w)}{|w-z|^{2n-2}} dm(w) \\ &\leq \frac{\Gamma(n)}{\pi^n} \cdot M \cdot \int_{R<|w-z|<+\infty} \frac{|f(w)| \cdot \Phi(z; w)}{|w-z|^{2n-2}} dm(w). \end{aligned} \quad (5.18)$$

Hence, due to the condition (b) of the theorem,

$$\lim_{R \rightarrow +\infty} A_3(R) = 0. \quad (5.19)$$

Combining (5.11)–(5.19), we ultimately obtain

$$\begin{aligned} f(z) &= \frac{\rho\sigma^\mu}{2\pi^n} \int_{C^n} f(w) \cdot E_{\rho/2}^{(n)}\left(\sigma^{2/\rho}\langle z, w \rangle; \mu\right) \cdot e^{-\sigma|w|^\rho} |w|^\gamma dm(w) \\ &\quad - \frac{\Gamma(n)}{\pi^n} \int_{C^n} \frac{\langle (\partial f / \partial \bar{w})(w), w - z \rangle}{|w - z|^{2n}} \cdot \Phi(z; w) dm(w), \end{aligned} \quad (5.20)$$

which coincides with (5.4). Thus the theorem is proved. $\square$

Remark 5.2. The idea of introducing an auxiliary function χ is borrowed from [16].

6. The Computable Form of the Kernel Φ

Up to now, we base on the formulae (2.15)–(2.16) defining the kernel Φ , and this makes it possible to investigate the properties of the kernel (see Section 4). Now we intend to simplify (2.15) or, more precisely, to bring the formula to a more *algorithmical* form (in the sense of explicit computability). To this end, let us start with several notations.

For arbitrary $z, w \in C^n (z \neq w)$, put (compare with (3.8))

$$\begin{aligned} a &= \frac{\langle z, w - z \rangle}{|w - z|}, \quad c = \frac{\langle w, w - z \rangle}{|w - z|}, \\ \delta &= \frac{|z|^2 |w - z|^2 - |\langle z, w - z \rangle|^2}{|w - z|^2} \equiv \frac{|z|^2 |w|^2 - |\langle z, w \rangle|^2}{|w - z|^2} \geq 0. \end{aligned} \quad (6.1)$$

Note that a slight change of (3.7) yields

$$|z + \lambda(w - z)|^2 \equiv |w - z|^2 \cdot \left| \lambda + \frac{a}{|w - z|} \right|^2 + \delta. \quad (6.2)$$

Let $\rho, \sigma > 0$, $\gamma \geq 0$, and $\mu = (\gamma + 2n)/\rho$. Consider the following functions:

$$\varphi(s) = (s - \bar{a})^{n-1} E_{\rho/2}^{(n)}\left(\sigma^{2/\rho}(a \cdot s + \delta); \mu\right), \quad s \in C, \quad (6.3)$$

$$\varphi^{(\nu)}(s) \equiv \frac{d^\nu \varphi(s)}{ds^\nu}, \quad \nu = 0, 1, 2, 3, \dots \quad (6.4)$$

Besides, put ($x \geq 0$):

$$\phi(x) = e^{-\sigma(x+\delta)^{\rho/2}} (x + \delta)^{\gamma/2}, \quad (6.5)$$

$$\Phi_k(x) = \frac{1}{\Gamma(k)} \int_x^{+\infty} \phi(t) (t - x)^{k-1} dt, \quad k = 1, 2, 3, \dots \quad (6.6)$$

Note that Φ_k is the k -th primitive of the function ϕ , that is,

$$\Phi_k(x) = \underbrace{\int_x^{+\infty} dx \int_x^{+\infty} dx \cdots \int_x^{+\infty}}_{k \text{ times}} \phi(x) dx. \quad (6.7)$$

In what follows we also put $\Phi_0(x) \equiv \phi(x)$, $x \geq 0$. Obviously, the following simple relations are valid:

$$\frac{d}{dx} [\Phi_{k+1}(x)] = -\Phi_k(x) \implies \frac{d^k \Phi_k(x)}{dx^k} = (-1)^k \phi(x), \quad x \geq 0, \quad k = 0, 1, 2, 3, \dots, \quad (6.8)$$

$$\Phi_k(0) = \frac{1}{\Gamma(k)} \int_0^{+\infty} \phi(t) t^{k-1} dt, \quad k = 1, 2, 3, \dots, \quad (6.9)$$

$$\int_{r < |\lambda| < R} \lambda^l \bar{\lambda}^m \phi(|\lambda|^2) dm(\lambda) = 0, \quad (6.10)$$

where $0 \leq r < R \leq \infty$ and $l, m = 0, 1, 2, 3, \dots$, $l \neq m$. Certainly, here ϕ can be replaced by any other radial (i.e., depending only on $|\lambda|$) function if only the corresponding integrals exist.

Theorem 6.1. *The kernel Φ can be computed by the following formula:*

$$\begin{aligned} \Phi(z; w) = & \frac{\rho \sigma^\mu}{2\Gamma(n)} \cdot \left\{ \sum_{\nu=0}^{n-1} \psi^{(\nu)}(0) \cdot \Phi_{\nu+1}(0) \cdot \sum_{k=0}^{n-1-\nu} C_n^{k+1+\nu} (-a)^{n-1-\nu-k} c^k \right\} \\ & - \frac{\rho \sigma^\mu}{2\Gamma(n)} \cdot \bar{c}(c-a)^n \cdot \int_0^1 \psi(\bar{c} \cdot t) \phi(|c|^2 \cdot t) dt. \end{aligned} \quad (6.11)$$

Proof. In view of (2.15) and (6.2), we have

$$\Phi(z; w) = \frac{\rho \sigma^\mu}{2\Gamma(n)} \cdot I, \quad (6.12)$$

where

$$\begin{aligned} I = & \frac{|w-z|^{2n}}{\pi} \int_C \frac{|\lambda|^{2n}}{\bar{\lambda}(\lambda-1)} E_{\rho/2}^{(n)} \left(\sigma^{2/\rho} \left(|z|^2 + |w-z| \bar{\lambda} \cdot a \right); \mu \right) \\ & \times e^{-\sigma(|w-z|^2 \cdot |\lambda + (a/|w-z|)|^2 + \delta)^{\rho/2}} \cdot \left(|w-z|^2 \cdot \left| \lambda + \frac{a}{|w-z|} \right|^2 + \delta \right)^{\gamma/2} dm(\lambda) \stackrel{|w-z|\lambda+a \rightarrow \lambda}{=} \\ = & \frac{1}{\pi} \int_C \frac{(\lambda-a)^n \cdot (\bar{\lambda}-\bar{a})^{n-1}}{\lambda-c} E_{\rho/2}^{(n)} \left(\sigma^{2/\rho} \left(a \cdot \bar{\lambda} + \delta \right); \mu \right) \cdot e^{-\sigma(|\lambda|^2 + \delta)^{\rho/2}} \left(|\lambda|^2 + \delta \right)^{\gamma/2} dm(\lambda). \end{aligned} \quad (6.13)$$

In view of notations (6.3) and (6.5), we have

$$I = \frac{1}{\pi} \int_C \frac{(\lambda - a)^n}{\lambda - c} \cdot \psi(\bar{\lambda}) \cdot \phi(|\lambda|^2) dm(\lambda). \quad (6.14)$$

In order to simplify the last integral, let us consider the following auxiliary integrals:

$$\begin{aligned} A_\nu &= \frac{1}{\pi} \int_C \lambda^\nu \cdot \psi(\bar{\lambda}) \cdot \phi(|\lambda|^2) dm(\lambda), \quad \nu = 0, 1, 2, 3, \dots, \\ B &= \frac{1}{\pi} \int_C \frac{\psi(\bar{\lambda})}{\lambda - c} \cdot \phi(|\lambda|^2) dm(\lambda). \end{aligned} \quad (6.15)$$

Using the Maclaurin expansion $\psi(\bar{\lambda}) = \sum_{k=0}^{\infty} (\psi^{(k)}(0)/\Gamma(k+1)) \cdot \bar{\lambda}^k$ and (6.10), we obtain

$$\begin{aligned} A_\nu &= \frac{1}{\pi} \cdot \frac{\psi^{(\nu)}(0)}{\Gamma(\nu+1)} \cdot \int_C |\lambda|^{2\nu} \cdot \phi(|\lambda|^2) dm(\lambda) \\ &= \frac{\psi^{(\nu)}(0)}{\Gamma(\nu+1)} \cdot \int_0^\infty t^\nu \phi(t) dt = \psi^{(\nu)}(0) \cdot \Phi_{\nu+1}(0). \end{aligned} \quad (6.16)$$

Further,

$$B = \frac{1}{\pi} \int_{|\lambda| < |c|} \frac{\psi(\bar{\lambda})}{\lambda - c} \cdot \phi(|\lambda|^2) dm(\lambda) + \frac{1}{\pi} \int_{|\lambda| > |c|} \frac{\psi(\bar{\lambda})}{\lambda - c} \cdot \phi(|\lambda|^2) dm(\lambda) \equiv B^{(+)} + B^{(-)}. \quad (6.17)$$

If $c = 0$, then, naturally, $B^{(+)} = 0$, and

$$\begin{aligned} B^{(-)} &= \frac{1}{\pi} \int_{|\lambda| > 0} \frac{\psi(\bar{\lambda})}{\lambda} \cdot \phi(|\lambda|^2) dm(\lambda) \\ &= \frac{1}{\pi} \int_{|\lambda| > 0} \bar{\lambda} \cdot \psi(\bar{\lambda}) \cdot \frac{\phi(|\lambda|^2)}{|\lambda|^2} dm(\lambda) = 0, \end{aligned} \quad (6.18)$$

that is, $B = 0$. If $c \neq 0$, then

$$\frac{1}{\lambda - c} = \begin{cases} -\frac{1}{c} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{c^k}, & |\lambda| < |c|, \\ \frac{1}{\lambda} \cdot \sum_{k=0}^{\infty} \frac{c^k}{\lambda^k}, & |\lambda| > |c|. \end{cases} \quad (6.19)$$

Consequently,

$$\begin{aligned} B^{(-)} &= \frac{1}{\pi} \sum_{k=0}^{\infty} c^k \cdot \int_{|\lambda| > |c|} \frac{\psi(\bar{\lambda})}{\lambda^{k+1}} \cdot \phi(|\lambda|^2) dm(\lambda) \\ &= \frac{1}{\pi} \sum_{k=0}^{\infty} c^k \cdot \int_{|\lambda| > |c|} \bar{\lambda}^{k+1} \cdot \psi(\bar{\lambda}) \cdot \frac{\phi(|\lambda|^2)}{|\lambda|^{2k+2}} dm(\lambda) = 0. \end{aligned} \quad (6.20)$$

Hence

$$\begin{aligned} B = B^{(+)} &= -\frac{1}{\pi c} \sum_{k=0}^{\infty} \int_{|\lambda| < |c|} \frac{\lambda^k}{c^k} \cdot \psi(\bar{\lambda}) \cdot \phi(|\lambda|^2) dm(\lambda) \\ &= -\frac{1}{\pi c} \sum_{k=0}^{\infty} \int_{|\lambda| < |c|} \frac{\psi^{(k)}(0)}{\Gamma(k+1) \cdot c^k} \cdot |\lambda|^{2k} \cdot \phi(|\lambda|^2) dm(\lambda) \\ &= -\frac{1}{c} \sum_{k=0}^{\infty} \int_0^{|c|^2} \frac{\psi^{(k)}(0)}{\Gamma(k+1) \cdot c^k} \cdot t^k \phi(t) dm(t) \\ &= -\frac{1}{c} \int_0^{|c|^2} \sum_{k=0}^{\infty} \frac{\psi^{(k)}(0)}{\Gamma(k+1)} \left(\frac{t}{c}\right)^k \phi(t) dm(t) \\ &= -\frac{1}{c} \int_0^{|c|^2} \psi\left(\frac{t}{c}\right) \phi(t) dm(t) \stackrel{t \rightarrow |c|^2 t}{=} -\bar{c} \cdot \int_0^1 \psi(\bar{c} \cdot t) \phi(|c|^2 \cdot t) dm(t). \end{aligned} \quad (6.21)$$

Using the binomial expansion of $(\lambda - a)^n$ and combining (6.14)–(6.21), we obtain

$$\begin{aligned} I &= \sum_{k=0}^n C_n^k (-a)^{n-k} \cdot \frac{1}{\pi} \int_C \frac{\lambda^k}{\lambda - c} \cdot \psi(\bar{\lambda}) \cdot \phi(|\lambda|^2) dm(\lambda) \\ &= \sum_{k=1}^n C_n^k (-a)^{n-k} \cdot \frac{1}{\pi} \int_C \frac{(\lambda^k - c^k) + c^k}{\lambda - c} \cdot \psi(\bar{\lambda}) \cdot \phi(|\lambda|^2) dm(\lambda) + (-a)^n \cdot B \\ &= \sum_{k=1}^n C_n^k (-a)^{n-k} \cdot \frac{1}{\pi} \int_C \frac{(\lambda - c) \cdot \sum_{v=0}^{k-1} c^{k-1-v} \lambda^v}{\lambda - c} \cdot \psi(\bar{\lambda}) \cdot \phi(|\lambda|^2) dm(\lambda) \\ &\quad + \sum_{k=1}^n C_n^k (-a)^{n-k} \cdot c^k \cdot B + (-a)^n \cdot B \\ &= \sum_{k=1}^n C_n^k (-a)^{n-k} \cdot \sum_{v=0}^{k-1} c^{k-1-v} \cdot A_v + (c-a)^n \cdot B \stackrel{k \rightarrow k+1}{=} \\ &= \sum_{k=0}^{n-1} C_n^{k+1} (-a)^{n-1-k} \cdot \sum_{v=0}^k c^{k-v} \cdot \psi^{(v)}(0) \cdot \Phi_{v+1}(0) - \bar{c}(c-a)^n \cdot \int_0^1 \psi(\bar{c} \cdot t) \phi(|c|^2 \cdot t) dm(t) \\ &= \sum_{v=0}^{n-1} \psi^{(v)}(0) \cdot \Phi_{v+1}(0) \cdot \sum_{k=0}^{n-1-v} C_n^{k+1+v} (-a)^{n-1-v-k} c^k - \bar{c}(c-a)^n \cdot \int_0^1 \psi(\bar{c} \cdot t) \phi(|c|^2 \cdot t) dm(t). \end{aligned} \quad (6.22)$$

Finally, note that (6.12) and (6.22) imply (6.11) and the proof is complete. $\square$

7. Important Special Case: $\rho = 2, \gamma = 0$

In this section, we analyze the special case $\rho = 2, \gamma = 0$, when the formulas become more simple and more explicit.

First of all, $\mu = (\gamma + 2n)/\rho = n$. Hence, the coefficient $\rho\sigma^\mu/2\Gamma(n)$ transforms into $\sigma^n/\Gamma(n)$. Next,

$$E_{\rho/2}^{(n)}(\eta; \mu) = \sum_{k=0}^{\infty} \frac{\Gamma(k+n)}{\Gamma(k+1)} \cdot \frac{\eta^k}{\Gamma(\mu + 2k/\rho)} = \sum_{k=0}^{\infty} \frac{\eta^k}{\Gamma(k+1)} \equiv e^\eta, \quad \eta \in \mathbb{C}. \quad (7.1)$$

As a consequence, the formula (2.15) takes the following form:

$$\Phi(z; w) = e^{-\sigma|w|^2} \cdot e^{\sigma\langle z, w \rangle} \cdot \sum_{v=0}^{n-1} \frac{\sigma^v}{v!} |w - z|^{2v}, \quad \forall z \in \mathbb{C}^n, \forall w \in \mathbb{C}^n \setminus \{z\}, \quad (7.2)$$

or, equivalently,

$$\Phi(z; z + w) = e^{-\sigma|w|^2} \cdot e^{-\sigma\langle w, z \rangle} \cdot \sum_{v=0}^{n-1} \frac{\sigma^v}{v!} |w|^{2v}, \quad \forall z \in \mathbb{C}^n, \forall w \in \mathbb{C}^n \setminus \{0\}. \quad (7.3)$$

Indeed, we have

$$\begin{aligned} \Phi(z; w) &= \frac{\sigma^n}{\pi\Gamma(n)} |w - z|^{2n} \cdot \int_{\mathbb{C}} \frac{|\lambda|^{2n}}{\lambda(\lambda - 1)} e^{\sigma\langle z, z + \lambda(w - z) \rangle} \cdot e^{-\sigma|z + \lambda(w - z)|^2} dm(\lambda) \\ &= \frac{\sigma^n}{\pi\Gamma(n)} |w - z|^{2n} \cdot \int_{\mathbb{C}} \frac{|\lambda|^{2n}}{\lambda(\lambda - 1)} e^{-\sigma\langle \lambda(w - z), z + \lambda(w - z) \rangle} dm(\lambda) \\ &= \frac{\sigma^n}{\pi\Gamma(n)} |w - z|^{2n} \cdot \int_{\mathbb{C}} |\lambda|^{2n-2} \cdot e^{-\sigma|w - z|^2|\lambda|^2} \cdot \frac{\lambda \cdot e^{-\sigma\lambda\langle w - z, z \rangle}}{\lambda - 1} dm(\lambda) \\ &= \frac{2\sigma^n}{\Gamma(n)} |w - z|^{2n} \cdot \int_0^{+\infty} r^{2n-1} \cdot e^{-\sigma|w - z|^2 r^2} \cdot \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{e^{-\sigma\zeta\langle w - z, z \rangle}}{\zeta - 1} d\zeta dr \\ &= \frac{2\sigma^n}{\Gamma(n)} |w - z|^{2n} \cdot \int_0^{+\infty} r^{2n-1} \cdot e^{-\sigma|w - z|^2 r^2} \times \begin{cases} 0, & 0 < r < 1 \\ e^{-\sigma\langle w - z, z \rangle}, & 1 < r < \infty \end{cases} \times dr \\ &= \frac{\sigma^n}{\Gamma(n)} |w - z|^{2n} \cdot e^{-\sigma\langle w - z, z \rangle} \cdot \int_1^{+\infty} t^{n-1} \cdot e^{-\sigma|w - z|^2 t} dt \\ &= \frac{\sigma^n}{\Gamma(n)} |w - z|^{2n} \cdot e^{-\sigma\langle w - z, z \rangle} \cdot e^{-\sigma|w - z|^2} \cdot \int_0^{+\infty} (t + 1)^{n-1} \cdot e^{-\sigma|w - z|^2 t} dt \\ &= \frac{\sigma^n}{\Gamma(n)} |w - z|^{2n} \cdot e^{-\sigma|w|^2} \cdot e^{\sigma\langle z, w \rangle} \cdot \sum_{v=0}^{n-1} C_{n-1}^v \cdot \frac{\Gamma(n - v)}{\sigma^{n-v} |w - z|^{2n-2v}} \\ &= e^{-\sigma|w|^2} \cdot e^{\sigma\langle z, w \rangle} \cdot \sum_{v=0}^{n-1} \frac{\sigma^v}{v!} |w - z|^{2v}. \end{aligned} \quad (7.4)$$

It is interesting to note (see Remark 4.4) that

$$\lim_{w \rightarrow 0} \Phi(z; z + w) = \Phi(z; z) = 1. \quad (7.5)$$

Moreover (compare with Proposition 4.7), for arbitrary $R > 0$ and for (arbitrary small) $\varepsilon > 0$, there exists a positive constant $c = c(n, \sigma, \varepsilon, R)$ such that

$$|\Phi(z; z + w)| \leq c \cdot e^{-(\sigma - \varepsilon)|w|^2}, \quad \forall w \in C^n, \quad (7.6)$$

uniformly in z with $|z| < R$. At last, the main Theorem 5.1 takes the following form

Theorem 7.1. *Assume that $n \geq 1$, $1 < p < +\infty$ and $\sigma > 0$, then the integral representation of the form*

$$\begin{aligned} f(z) &= \frac{\sigma^n}{\pi^n} \int_{C^n} f(w) \cdot e^{\sigma \langle z, w \rangle} \cdot e^{-\sigma |w|^2} dm(w) \\ &\quad - \frac{\Gamma(n)}{\pi^n} \int_{C^n} \frac{\langle (\partial f / \partial \bar{w})(w), w - z \rangle}{|w - z|^{2n}} \cdot e^{\sigma \langle z, w \rangle} \\ &\quad \times \sum_{v=0}^{n-1} \frac{\sigma^v}{v!} |w - z|^{2v} \cdot e^{-\sigma |w|^2} dm(w), \quad z \in C^n, \end{aligned} \quad (7.7)$$

is valid for each function $f \in C^1(C^n)$ satisfying the following conditions:

- (1) $\int_{C^n} |f(w)|^p \cdot e^{-\sigma |w|^2} dm(w) < +\infty$;
- (2) for any fixed $z \in C^n$,

$$\int_{C^n} \frac{|(\partial f / \partial \bar{w})(w)|}{|w - z|^{2n-1}} \cdot \left| e^{\sigma \langle z, w \rangle} \right| \cdot \sum_{v=0}^{n-1} \frac{\sigma^v}{v!} |w - z|^{2v} \cdot e^{-\sigma |w|^2} dm(w) < +\infty, \quad (7.8)$$

or, equivalently,

$$\int_{C^n} \frac{|(\partial f / \partial \bar{w})(w)|}{|w - z|} \cdot \left| e^{\sigma \langle z, w \rangle} \right| \cdot e^{-\sigma |w|^2} dm(w) < +\infty, \quad (7.9)$$

or, not equivalently (but more conveniently),

$$\int_{C^n} \left| \frac{\partial f}{\partial \bar{w}}(w) \right| \cdot e^{-(\sigma - \varepsilon)|w|^2} dm(w) < +\infty, \quad (7.10)$$

where ε is arbitrary small.

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