Research Article

The Functional-Analytic Properties of the Limit q -Bernstein Operator

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The limit q -Bernstein operator B_q , $0 < q < 1$, emerges naturally as a modification of the Szász-Mirakyan operator related to the Euler distribution. The latter is used in the q -boson theory to describe the energy distribution in a q -analogue of the coherent state. Lately, the limit q -Bernstein operator has been widely under scrutiny, and it has been shown that B_q is a positive shape-preserving linear operator on $C[0, 1]$ with $\|B_q\| = 1$. Its approximation properties, probabilistic interpretation, eigenstructure, and impact on the smoothness of a function have been examined. In this paper, the functional-analytic properties of B_q are studied. Our main result states that there exists an infinite-dimensional subspace M of $C[0, 1]$ such that the restriction $B_q|_M$ is an isomorphic embedding. Also we show that each such subspace M contains an isomorphic copy of the Banach space c_0 .

1. Introduction

The limit q -Bernstein operator comes out naturally as an analogue of the Szász-Mirakyan operator, which is related to the Euler probability distribution—also referred to as the “ q -deformed Poisson distribution” (see [1, 2]). The latter is used in the q -boson theory, which is a q -deformation of the quantum harmonic oscillator formalism [3]. Namely, the q -deformed Poisson distribution describes the energy distribution in a q -analogue of the coherent state [3, 4]. The q -analogue of the boson operator calculus has proved to be a powerful tool in theoretical physics by providing explicit expressions for the representations of the quantum group $SU_q(2)$, which is by now known to play a profound role in a variety of different problems, such as integrable models in the field theory, exactly solvable lattice models of statistical mechanics, and conformal field theory among others. Therefore, properties of the q -deformed Poisson distribution and its related linear operators are of significant interest for applications.

In the sequel, the following notations and definitions are employed (cf., e.g., [5]).
Let $q > 0$. For any $k \in \mathbb{Z}_+$, the q -integer $[k]_q$ is defined by

$$[k]_q := 1 + q + \cdots + q^{k-1} \quad (k \in \mathbb{N}), \quad [0]_q := 0 \quad (1.1)$$

and the q -factorial $[k]_q!$ by

$$[k]_q! := [1]_q [2]_q \cdots [k]_q \quad (k \in \mathbb{N}), \quad [0]_q! := 1. \quad (1.2)$$

Besides, $(x - a)_q^k$ denotes the q -analogue of $(x - a)^k$, that is,

$$(x - a)_q^k := (x - a)(x - qa) \cdots (x - q^{k-1}a) \quad (k \in \mathbb{N}), \quad (x - a)_q^0 := 1, \quad (1.3)$$

while

$$(x - a)_q^\infty := \prod_{j=0}^{\infty} (x - q^j a). \quad (1.4)$$

For $0 < q < 1$, the q -analogues of the exponential function are given by (see [5], formulae (9.7) and (9.10))

$$\begin{aligned} e_q(x) &:= \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}, \quad |x| < \frac{1}{1-q}, \\ E_q(x) &= \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} x^k}{[k]_q!}. \end{aligned} \quad (1.5)$$

By Euler's Identities (cf., e.g., [5], formulae (9.3) and (9.4)),

$$\begin{aligned} e_q(x) &= \frac{1}{(1 - (1-q)x)_q^\infty}, \quad |x| < \frac{1}{1-q}, \\ E_q(x) &= (1 + (1-q)x)_q^\infty, \end{aligned} \quad (1.6)$$

whence

$$e_q(x)E_q(-x) = 1. \quad (1.7)$$

Clearly, for $q = 1$, we have

$$[k]_1 = k, \quad [k]_1! = k!, \quad e_1(x) = E_1(x) = e^x. \quad (1.8)$$

Now, for $b, q > 0$, let $X_{b,q}$ be a random variable possessing a discrete distribution with the probability mass function:

$$p(b[k]_q) = E_q\left(-\frac{x}{b}\right) \frac{x^k}{b^k [k]_q!}, \quad k \in \mathbb{Z}_+. \quad (1.9)$$

When $b = q = 1$, we recover the classical Poisson distribution with parameter x . If f is a function defined on $\{b[k]_q\}_{k=0}^\infty$, then the mathematical expectation of X_b equals

$$\mathbb{E}[f(X_{b,q})] = E_q\left(-\frac{x}{b}\right) \sum_{k=0}^{\infty} \frac{f(b[k]_q) x^k}{b^k [k]_q!} =: (A_{b,q}f)(x). \quad (1.10)$$

We notice that in the case $q = 1$, $b = 1/n$, operator $A_{b,q}$ is the classical Szász-Mirakyan operator. Taking $q \in (0, 1)$ and $b = 1 - q$, we arrive at the definition of the *limit q -Bernstein operator*, which, therefore, may be regarded as an analogue of the Szász-Mirakyan operator.

Definition 1.1 (see [6]). Given $q \in (0, 1)$, $f \in C[0, 1]$, the *limit q -Bernstein operator* is defined by $f \mapsto B_q f$, where

$$(B_q f)(x) = B_q(f; x) := \begin{cases} E_q\left(-\frac{x}{1-q}\right) \cdot \sum_{k=0}^{\infty} \frac{f(1-q^k) x^k}{(1-q)^k [k]_q!} & \text{if } x \in [0, 1), \\ f(1) & \text{if } x = 1. \end{cases} \quad (1.11)$$

Remark 1.2. It has been proved in [6] that, for any $f \in C[0, 1]$, the function $B_q(f; x)$ is continuous on $[0, 1]$ and admits an analytic continuation $B_q(f; z)$ into the open unit disc $\{z : |z| < 1\}$.

Alternatively, the limit q -Bernstein operator emerges as a limit for a sequence of the q -Bernstein polynomials in the case $0 < q < 1$ (see [6–8]). Recently, Wang has shown in [9] that the same operator is the limit for a sequence of q -Meyer-König and Zeller operators. The latter operators have been introduced by Trif in [10].

The limit q -Bernstein operator has been studied from different perspectives by Charalambides, Il'inskii, Ostrovska, Videnskii, and Wang. It has been shown in [6, 11] that B_q is a positive shape-preserving linear operator on $C[0, 1]$ with $\|B_q\| = 1$, which possesses the end-point interpolation property, leaves invariant linear functions, and maps a polynomial of degree m to a polynomial of degree m . To be more specific, it takes binomial $(1 - x)^m$ to the corresponding q -binomial—that is,

$$B_q(1 - x)^m = (1 - x)_q^m, \quad m \in \mathbb{Z}_+. \quad (1.12)$$

The approximation with the help of B_q has been studied in [12], while the properties of its range have been presented in [13]. The probabilistic approaches have been developed in [1, 2]. The investigation of the impact of B_q on the smoothness of a function conducted in [14] has revealed the following remarkable phenomenon: in general, the limit q -Bernstein operator

improves the analytic properties of a function, provided the function is neither “very good” (a polynomial) nor “very bad” (without a certain regularity condition).

In this paper, we study the functional-analytic properties of the limit q -Bernstein operator. Our main result is that there exists an infinite-dimensional subspace M of $C[0, 1]$ such that the restriction $B_q|_M$ is an isomorphic embedding. Also we show that each such subspace M contains an isomorphic copy of the Banach space c_0 .

2. Functional-Analytic Properties of B_q

Let us recall that the *range* of an operator $T : X \rightarrow Y$ is defined as the set $\{y \in Y : \exists x \in X, Tx = y\}$. We say that an operator $T : X \rightarrow Y$ is *bounded below on a subspace* $L \subset X$ if there exists a constant $c > 0$ such that $\|Tx\| \geq c\|x\|$ for each $x \in L$. We say that $T : X \rightarrow Y$ is *bounded below* if it is bounded below on X . The space consisting of all convergent to zero sequences with the maximum modulus norm is denoted by c_0 . Other relevant terminology can be found in [15, 16] or [17].

Proposition 2.1. (i) *The range of the limit q -Bernstein operator $B_q : C[0, 1] \rightarrow C[0, 1]$ is nonclosed.*

(ii) *Let L be the subspace of $C[0, 1]$ consisting of functions f , which are linear on the intervals $[1 - q^{k-1}, 1 - q^k]$ for $k \in \mathbb{N}$. Then the restriction of the limit q -Bernstein operator $B_q : C[0, 1] \rightarrow C[0, 1]$ to L is injective but is not bounded below. There are subspaces of L such that the corresponding restrictions of B_q are compact and nuclear, respectively.*

(iii) *The restriction of B_q to any subspace of $C[0, 1]$, which does not contain a subspace isomorphic to c_0 , is strictly singular and thus is not bounded below.*

Proof. (i) It can be readily seen from (1.12) that all polynomials are in the range of B_q . Thus, by the Weierstrass theorem, the range of B_q is dense in $C[0, 1]$. If it had been closed, it would have coincided with the whole space $C[0, 1]$, which contradicts Remark 1.2.

(ii) Injectivity of the restriction of B_q to L follows immediately from formula (1.11). The same formula implies that the range of the restriction of B_q to L coincides with the range of B_q , whence it is nonclosed. On the other hand, it is known (see [16, Prop. 2.c.4]) that the condition that the range is nonclosed implies that the operator is not bounded below and that there are subspaces, restrictions to which are compact (even nuclear) operators.

(iii) To begin with, we observe that B_q factors through c_0 . This observation is an immediate consequence of the formula (1.11) and the following two observations: (1) the set of restrictions of functions $f \in C[0, 1]$ to the sequence $\{1 - q^k\}_{k=1}^\infty$ is the space of all convergent sequences; (2) this space with the norm $\sup_k |f(1 - q^k)|$ is isomorphic to the space c_0 . Applying the well-known results on Banach space geometry (see [15, Chapter 2], [16, Chapter 2], [17]), we derive the statement. $\square$

Combining Proposition 2.1(iii) with the classical Banach-Mazur theorem [18, Ch. XI, §8] on the universality of $C[0, 1]$, we conclude that there are many different subspaces of $C[0, 1]$ on which the operator B_q is not bounded below.

The main result of the present paper states that for subspaces containing subspaces isomorphic to c_0 the situation can be different.

Theorem 2.2. *There exists a subspace of $C[0, 1]$ isomorphic to c_0 such that the restriction of B_q to this subspace is an isomorphic embedding.*

Proof. For each finite subset $I \subset \mathbb{N}$, we introduce a function $\ell_I \in C[0,1]$ satisfying the following conditions:

- (i) $\|\ell_I\| = 1$,
- (ii) $\ell_I(x) = \begin{cases} 1 & \text{if } x=1-q^k, k \in I, \\ 0 & \text{if } x=1-q^k, k \in \mathbb{N} \setminus I. \end{cases}$

It is clear that we may assume that supports of ℓ_{I_1} and ℓ_{I_2} are disjoint whenever I_1 and I_2 are. Therefore, for each disjoint sequence $\{I_j\}_{j=1}^\infty$, $I_j \subset \mathbb{N}$, the space spanned by $\{\ell_{I_j}\}_{j=1}^\infty$ in $C[0,1]$ is isometric to c_0 .

Our purpose is to show that we can select subsets $\{I_j\}_{j=1}^\infty$ in such a way that the sequence $\{B_q(\ell_{I_j})\}$ is also equivalent to the unit vector basis of c_0 . It is clear that

$$\left\| \sum_{j=1}^\infty a_j B_q(\ell_{I_j}) \right\| \leq \|B_q\| \cdot \left\| \sum_{j=1}^\infty a_j \ell_{I_j} \right\| \leq \sup_j |a_j|, \quad (2.1)$$

as $\|B_q\| = 1$. It suffices, therefore, to prove the following estimate:

$$\left\| \sum_{j=1}^\infty a_j B_q(\ell_{I_j}) \right\| \geq C_q \sup_j |a_j|, \quad (2.2)$$

where $C_q > 0$ is a constant that depends only on q . To prove the estimate, we need to show that $\{I_j\}$ can be chosen in such a way that the functions $\{B_q(\ell_{I_j})\}_{j=1}^\infty$ are almost-disjointly supported with norms bounded below by a positive constant depending only on q . To show this, we observe that since $E_q(-x/(1-q)) = (1-x)_q^\infty = (1-x)(1-qx)_q^\infty$, the following equality holds:

$$(B_q \ell_I)(x) = \sum_{k \in I} x^k (1-x) \cdot \frac{(1-qx)_q^\infty}{(1-q)_q^k} =: \sum_{k \in I} x^k (1-x) \cdot R_k(x), \quad (2.3)$$

where

$$0 < (1-q)_q^\infty \leq \frac{(1-q)_q^\infty}{(1-q)_q^k} \leq R_k(x) \leq \frac{1}{(1-q)_q^k} \leq \frac{1}{(1-q)_q^\infty} < \infty \quad (2.4)$$

for all $k \in \mathbb{N}$, $x \in [0,1]$.

We are going to consider only finite subsets I consisting of consecutive integers, that is, subsets of the form $I = \{m, m+1, \dots, m+d-1\}$, where m is the least element of I and d is the number of elements in I . For such I , the value $(B_q(\ell_I))(x)$, $x \in [0,1]$ is between

$$\begin{aligned} (1-q)_q^\infty (1-x) \sum_{k \in I} x^k &=: m_q x^m (1-x^d), \\ \frac{1}{(1-q)_q^\infty} (1-x) \sum_{k \in I} x^k &=: M_q x^m (1-x^d). \end{aligned} \quad (2.5)$$

To prove inequality (2.2) for suitably chosen finite subsets $\{I_j\}_{j=1}^\infty$, we use the following simple assertions.

- (a) If we fix m and let $d \rightarrow \infty$, the maxima of the functions $x^m(1-x^d)$ on $[0, 1]$ approach 1.
- (b) For each interval of the form $(0, a)$, $a < 1$, and each $\varepsilon > 0$, for sufficiently large $m \in \mathbb{N}$ and an arbitrary $d \in \mathbb{N}$, we have $x^m(1-x^d) < \varepsilon$ on $(0, a)$.

Statement (a) can be verified by straightforward calculations. Indeed,

$$\left(x^k(1-x^d)\right)' = 0 \quad (\text{if } k \geq 2) \iff x = 0 \text{ or } x = x_0 := \sqrt[d]{\frac{k}{k+d}}. \quad (2.6)$$

The value of the function at x_0 is

$$\left(\frac{k}{k+d}\right)^{k/d} \cdot \left(\frac{d}{k+d}\right). \quad (2.7)$$

Since the limit of this expression as $d \rightarrow \infty$ equals 1, (a), has been proved.

The statement (b) is obvious.

Now we complete the proof of Theorem 2.2 as follows. Combining the claim about the restriction of $B_q(\ell_I)$ to $[0, 1]$ with the statement (a), we get that there exists $b_q > 0$ depending only on q such that for each $m \in \mathbb{N}$ there is d satisfying the condition

$$\|B_q(\ell_I)\| > b_q \quad \text{for } I = \{m, m+1, \dots, m+d-1\}. \quad (2.8)$$

We use this statement with $m_1 = 1$ and get $I_1 = \{1, \dots, d_1\}$ such that $\|B_q(\ell_{I_1})\| > b_q$. Since, as it can be readily seen, $(B_q(\ell_{I_1}))(1) = 0$, there is $0 < a_1 < 1$ such that $|(B_q(\ell_{I_1}))(x)| < b_q/2^2$ for $x \in [a_1, 1]$.

Using (b) we establish the existence of m_2 such that, for each finite set I_2 of the form $\{m_2, m_2+1, \dots, M\}$, the condition $|B_q(\ell_{I_2})(x)| < b_q/2^3$ holds for each $x \in (0, a_1)$.

After that we use (a) and pick d_2 so that for $I_2 = \{m_2, m_2+1, \dots, m_2+d_2-1\}$ we have

$$\|B_q(\ell_{I_2})\| > b_q. \quad (2.9)$$

Since $B_q(\ell_{I_2})(1) = 0$, there exists $a_2 \in (a_1, 1)$ such that $|B_q(\ell_{I_2})(x)| < b_q/2^3$ for each $x \in [0, 1] \setminus (a_1, a_2)$.

Proceeding in an obvious way, we construct a sequence $\{I_j\}_{j=1}^\infty$ and an increasing sequence $\{a_j\}_{j=1}^\infty$ ($a_0 = 0$, $0 < a_j < 1$ for $j \in \mathbb{N}$) so that

$$\begin{aligned} |B_q(\ell_{I_j})(x)| &< \frac{b_q}{2^{j+1}} \quad \text{for } x \in [0, 1] \setminus (a_{j-1}, a_j), \\ \max_{x \in [0, 1]} |B_q(\ell_{I_j})(x)| &> b_q. \end{aligned} \quad (2.10)$$

Now, straightforward calculations show that, for each $\{a_j\}_{j=1}^\infty \in c_0$, we have

$$\left\| \sum_{j=1}^{\infty} a_j B_q(\ell_{I_j}) \right\|_{C[0,1]} \geq \sup_j |a_j| \left(b_q - b_q \cdot \frac{1}{2^2} - b_q \cdot \frac{1}{2^3} - \cdots \right) = \sup_j |a_j| \cdot \frac{b_q}{2}. \quad (2.11)$$

□

Since the space c_0 is infinite dimensional and nonreflexive, we get the following corollary.

Corollary 2.3. *The operator B_q is neither weakly compact nor strictly singular (and thus is noncompact).*

Let us denote by M the subspace of $C[0,1]$ constructed in Theorem 2.2. Since it is mapped isomorphically by B_q , the range $B_q(M) \subset C[0,1]$ is also isomorphic to c_0 . By the well-known result of Sobczyk [19] (see also [16, p.106], and [20]), $B_q(M)$ is a complemented subspace of $C[0,1]$.

Corollary 2.4. *There exists an operator $Z : C[0,1] \rightarrow C[0,1]$ such that ZB_q is the identity on M . So there exists a “stable with respect to small errors” procedure of reconstruction of a function in M from its B_q -image, but there are no such procedures for any subspace of $C[0,1]$ containing no subspaces isomorphic to c_0 .*

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