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Research Article

Bifurcation from Infinity and Resonance Results at High Eigenvalues in Dimension One

José L. Gámez[1] and Juan F. Ruiz-Hidalgo[2]

[1] *Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain*
[2] *Departamento de Didáctica de la Matemática, Facultad de Ciencias de la Educación, Campus de Cartuja, Universidad de Granada, 18071 Granada, Spain*

Correspondence should be addressed to Juan F. Ruiz-Hidalgo, jfruiz@ugr.es

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This paper is devoted to two different but related tags: firstly, the side of the bifurcation from infinity at every eigenvalue of the problem $-u''(t) = \lambda u(t) + g(t, u(t))$, $u \in H_0^1(0, \pi)$, secondly, the solutions of the associated resonant problem at any eigenvalue. From the global shape of the nonlinearity g we obtain computable integral values which will decide the behavior of the bifurcations and, consequently, the possibility of finding solutions of the resonant problems.

1. Introduction

Considering the problem

$$\begin{aligned} -u''(t) &= \lambda u(t) + g(t, u(t)), \quad t \in (0, \pi) \\ u(0) &= u(\pi) = 0, \end{aligned} \tag{1.1}$$

we are interested in two different tags: firstly, the local behavior of bifurcations from infinity of (1.1) at every eigenvalue, σ_k, and, secondly how this behavior can help us to find solutions of the resonant problem

$$\begin{aligned} -u''(t) &= \sigma_k u(t) + g(t, u(t)), \quad t \in (0, \pi) \\ u(0) &= u(\pi) = 0. \end{aligned} \tag{1.2}$$

The first objective is the behavior of bifurcation. In Section 3, we determine if the branches are either subcritical or supercritical, that is, the parameters λ of the connected set

of solutions (λ, u) of the problem (1.1) lie either to the left of the eigenvalue or to the right. This question has just been studied by the authors, as in ([1]), where only the behaviour at first eigenvalue, σ_1, was treated. In particular, considering the problem

$$\begin{gathered} -u''(t) = \lambda u(t) + g(u(t)), \quad t \in (0, \pi) \\ u(0) = u(\pi) = 0 \end{gathered} \tag{1.3}$$

the authors showed that the side of the bifurcation from infinity at σ_1 is determined by an integral value involving the nonlinearity g, concretely

$$\int_0^{+\infty} g(s)s\,ds, \tag{1.4}$$

for positive solutions and

$$\int_{-\infty}^{0} g(s)s\,ds, \tag{1.5}$$

for negative ones. This result is consequence of a sharp estimate on the first eigenfunction values on a neighborhood of the boundary of the domain.

An extension of this result to the Laplacian operator in a bounded domain $\Omega \subset \mathbb{R}^N$ can be found in [2], where the interior of the domain Ω loses importance and the boundary joint with the integral of $g(y, s)s$ is enough to decide the side of the bifurcation. The importance of the boundary lies in that the set of zeroes of the first eigenfunction coincides with the boundary. This is a first obstacle in order to adjust the result to higher eigenvalues. In fact, the zeroes of the eigenfunctions in dimensions greater than one form the called nodal lines (see, e.g., [3] or [4]), which does not allow us formulate a similar conclusion for dimensions $N > 1$. In this way, a paper by Fleckinger et al. [5] could give new clues to make generalizations. A second hurdle is the lack of positivity of all eigenfunctions but the first one.

The second objective of this paper, in Section 4, is the solution of the resonant problem (1.2) as consequence of the study of laterality in (1.1). By using ideas put forward by Hess [6] and used by Arcoya and Gámez [7] among other authors, and in a similar way of the case of the first eigenvalue σ_1 (see [1]), if the bifurcations from $+\infty$ and $-\infty$ are both subcritical (resp. supercritical), then problem (1.2) has, at least, one solution.

The resonant problems have been studied by several authors. Chronologically, we can emphasize Dancer [8], Solimini [9], Ward [10], Mawhin and Schmitt [11], Schaff and Schmitt [12–14], Habets et al. [15], Habets et al. [16] and Cañada and Ruiz [17]. Concretely, in [8, 14] asymptotics methods are used. Papers [9, 10], which are considered as classics in search of solution of resonant problems, are dedicated to periodic nonlinearities and use variational techniques. In paper [13], closer to this paper, authors use bifurcation to solve resonant problems with periodic nonlinearities in dimension one. Finally, in [17] variational techniques are managed, also with periodic nonlinearities.

The main contribution here is twofold: on the one hand, consider any eigenvalue not only the first one. On the other hand, periodic nonlinearities do not need to get solutions of (1.2).

The last section presents two examples which cannot be characterized in some of the papers cited.

2. Preliminaries

This section is devoted to present the hypotheses needed to use bifurcation tools and also to rewrite the definition of bifurcation from $+\infty$ and $-\infty$.

In order to ensure that bifurcation occurs usually the hypotheses assumed on $g : (0,\pi) \times \mathbb{R} \to \mathbb{R}$ are

(H)

(i) $g : [0,\pi] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function (i.e., continuous in $s \in \mathbb{R}$ for a.e. $t \in (0,\pi)$ and measurable in $t \in (0,\pi), \forall s \in \mathbb{R}$),

(ii) there exists $r > 1$ and $C \in L^r(0,\pi)$ such that $|g(t,s)| \leq C(t)(1+|s|)$, for all $(t,s) \in (0,\pi) \times \mathbb{R}$,

(iii) $\lim_{|s|\to\infty} g(t,s)/s = 0$ uniformly in $t \in [0,\pi]$

Considering the linearized problem,

$$\begin{gathered} -u''(t) = \lambda u(t), \quad t \in (0,\pi) \\ u(0) = u(\pi) = 0 \end{gathered} \tag{2.1}$$

let k be a fixed positive integer and denote by $\|\cdot\|$ the usual norm in $H_0^1(0,\pi)$, that is $\|u\|^2 = \int_0^\pi (u'(t))^2 dt$. Under (H), every eigenvualue, σ_k of (2.1) is a bifurcation point from infinity due to the fact that each σ_k has algebraic and geometric multiplicity 1 (see [18]). That is, there exists a sequence (λ_n, u_n) of solutions of (1.1) such that $\lambda_n \to \sigma_k$ and $\|u_n\| \to +\infty$. Since the weak solutions of (1.1) lie in the space $W^{2,r}(0,\pi)$ continuously embedded in $C^1([0,\pi])$ $(r > 1)$, $C^1([0,\pi])$ will be the space to work.

Furthermore, the number of zeroes of the eigenfunctions, φ_k, is finite. Concretely, $\sigma_k = k^2$ and the normalized eigenfunctions are as follows:

$$\varphi_k(t) = \sqrt{\frac{2}{k^2\pi}} \sin(kt), \tag{2.2}$$

and $\varphi_k'(0) = \sqrt{2/\pi} > 0, \forall k \in \mathbb{N}$.

It is well known that for any bifurcating sequence $(\lambda_n, u_n) \to (\sigma_k, \infty)$ there exists a subsequence (denoted as the sequence) (λ_n, u_n) such that

$$\frac{u_n}{\|u_n\|} \longrightarrow \varphi_k \quad \text{in } C^1([0,\pi]), \tag{2.3}$$

where φ_k is a eigenfunction associated to σ_k with $\|\varphi_k\| = 1$. In the particular case of bifurcation at the principal eigenvalue σ_1, both φ_1 and $-\varphi_1$ have associated sequences as above. Since φ_1 lies in the interior of the C^1-cone of positive functions, we refer to such bifurcations as "bifurcation from $(\sigma_1, +\infty)$," and "bifurcations from $(\sigma_1, -\infty)$" respectively. One can also deduce from the above convergence that, near the bifurcation point, the solutions have constant sign. At higher eigenvalues the main difficulty revolves around the changes of sign of the eigenfunction. We overcome this trouble taking into account the existence of two branches of solutions (see [19]) bifurcating from infinity. We mean by "bifurcation from

$(\sigma_k, +\infty)$" to be the sequence of solutions (λ_n, u_n) of (1.1) satisfying $u_n/\|u_n\| \to \varphi_k$ (C^1-convergence), where $\varphi_k'(0) > 0$. In a similar way, we mean by "bifurcation from $(\sigma_k, -\infty)$" the sequence of solutions (λ_n, u_n) of (1.1) with $u_n/\|u_n\| \to -\varphi_k$.

3. Laterality of the Bifurcation from Infinity at All Eigenvalues

For the sake of simplicity we firstly point out our attention on the autonomous problem (1.3) and on the suitable resonant problem

$$\begin{gathered} -u''(t) = \sigma_k u(t) + g(u(t)), \quad t \in (0, \pi) \\ u(0) = u(\pi) = 0. \end{gathered} \tag{3.1}$$

Next hypothesis restricts the considered nonlinearities to a class of "small" functions with some technical properties in the boundary.

(G)

(i) there exists $f \in L^1(\mathbb{R})$ with $\lim_{|s| \to +\infty} f(s)s = 0$ such that $|g(s)s| < f(s)$, for all $s \in \mathbb{R}$,

(ii) $g(s)$ is continuous in $\mathbb{R}$.

Observe that (G) is more restricted than (H).

For any function g satisfying (G) and for every eigenfunction φ_k, we define L_k^+ and L_k^- as follows:

(i) for even k

$$L_k^+ = L_k^- := k\sqrt{\frac{\pi}{2}}\left(\int_0^{+\infty} g(s)s\,ds + \int_{-\infty}^0 g(s)s\,ds\right) = k\sqrt{\frac{\pi}{2}}\int_{-\infty}^{+\infty} g(s)s\,ds, \tag{3.2}$$

(ii) for odd k

$$\begin{aligned} L_k^+ &:= \sqrt{\frac{\pi}{2}}\left[(k+1)\int_0^{+\infty} g(s)s\,ds + (k-1)\int_{-\infty}^0 g(s)s\,ds\right], \\ L_k^- &:= \sqrt{\frac{\pi}{2}}\left[(k-1)\int_0^{+\infty} g(s)s\,ds + (k+1)\int_{-\infty}^0 g(s)s\,ds\right]. \end{aligned} \tag{3.3}$$

Observe that in previous expressions the term $\int_0^{+\infty} g(s)s\,ds$ appears twice the number of positive pieces φ_k. Conversely, the term $\int_{-\infty}^0 g(s)s\,ds$ appears twice the number of negative pieces of φ_k.

Theorem 3.1 (Asume (G)). *Is true, (λ_n, u_n) is a sequence of solutions of* (1.3) *bifurcating from $(\sigma_k, +\infty)$, then*

$$\lim_{n \to +\infty} \|u_n\|^3(\sigma_k - \lambda_n) = \sigma_k L_k^+ = k^2 L_k^+. \tag{3.4}$$

If (λ_n, u_n) *is a sequence of solutions of* (1.3) *bifurcating from* $(\sigma_k, -\infty)$, *then*

$$\lim_{n\to+\infty} \|u_n\|^3(\sigma_k - \lambda_n) = \sigma_k L_k^- = k^2 L_k^-. \tag{3.5}$$

Proof. We consider the bifurcation from $+\infty$. The bifurcation from $-\infty$ can be proved by using similar steps. Firstly, we remark that the eigenfunction associated to (2.1) φ_k has, exactly, $k-1$ zeroes in the interval $(0,\pi)$. These zeroes coincide with the points $i\pi/k$, where $i = 1,\ldots,k-1$.

Taking a sequence (λ_n, u_n) of solutions of (1.3) bifurcating from $(\sigma_k, +\infty)$, for any $n \in \mathbb{N}$, there exist $k+1$ zeroes of u_n, named $z_{n,i}$, such that $u_n(z_{n,i}) = 0$, $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} z_{n,i} = i\pi/k$. Observe that $z_{n,0} = 0$ and $z_{n,k} = \pi$ for all $n \in \mathbb{N}$.

For every k, let φ_k be a function test in the problem (1.3) obtaining

$$(\sigma_k - \lambda_n)\int_0^\pi u_n\varphi_k = \int_0^\pi g(u_n)\varphi_k. \tag{3.6}$$

Taking into account that

$$\int_0^\pi \varphi_k^2 = \frac{1}{\sigma_k}\int_0^\pi \left|\nabla\varphi_k\right|^2 = \frac{1}{\sigma_k} = \frac{1}{k^2} > 0, \tag{3.7}$$

and for n large enough

$$\int_0^\pi u_n\varphi_k = \|u_n\|\int_0^\pi \frac{u_n}{\|u_n\|}\varphi_k \begin{cases} > 0, & \text{if } \dfrac{u_n}{\|u_n\|} \longrightarrow \varphi_k, \\ < 0, & \text{if } \dfrac{u_n}{\|u_n\|} \longrightarrow -\varphi_k, \end{cases} \tag{3.8}$$

and, following from the equality (3.6), the sign of

$$\lim_{n\to+\infty}\int_0^\pi g(u_n)\varphi_k \tag{3.9}$$

will decide the side of the bifurcation.

Therefore, the question is reduced to prove that

$$L_k^+ = \lim_{n\to+\infty}\|u_n\|^2\int_0^\pi g(u_n)\varphi_k. \tag{3.10}$$

We divide the integral $\int_0^\pi g(u_n)\varphi_k$ in $3k$ integrals (see Figure 1),

$$\int_0^\pi g(u_n)\varphi_k = \sum_{i=0}^{k-1}\left[\int_{z_{n,i}}^{z_{n,i}+t_0} + \int_{z_{n,i}+t_0}^{z_{n,i+1}-t_0} + \int_{z_{n,i+1}-t_0}^{z_{n,i+1}}\right] g(u_n(t))\varphi_k(t)dt = \sum_{i=0}^{k-1}[I_{1,i} + I_{2,i} + I_{3,i}], \tag{3.11}$$

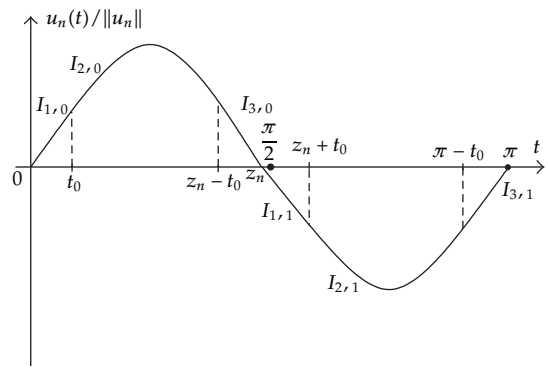

Figure 1: Solution of (1.3) near to the bifurcation point $(\sigma_2, +\infty)$.

where $t_0 \in (0, \pi/2k)$ is "small enough" (see Lemma 1 in [1]). By using the hypothesis (G) and the same arguments as in [1],

$$\lim_{n\to\infty} \|u_n\|^2 I_{2,i} = 0. \tag{3.12}$$

The rest of the integrals $I_{1,i}$ and $I_{3,i}$ should be considered depending on the value of $u'(z_{n,i})$, which depend on the parity of i. By changing variables $u_n(t) = s$, with $dt = ds/u_n'(u_n^{-1}(s))$, one can see, as in Lemma 1 in [1],

(i) for i even, $\|u_n\|/u_n'(u_n^{-1}(s)) \to -1/\varphi_k'(i\pi/k) = \sqrt{\pi/2}$,

(ii) for i odd, $\|u_n\|/u_n'(u_n^{-1}(s)) \to 1/\varphi_k'(i\pi/k) = -\sqrt{\pi/2}$.

Moreover, taking the suitable limits and by using the convergence theorem as in cited lemma, one can see that if $i = 0, \ldots, k-1$,

$$\lim_{n\to\infty} \|u_n\|^2 (I_{3,i} + I_{1,i+1}) = \sqrt{\frac{\pi}{2}} \int_{-\infty}^{+\infty} g(s)s\,ds \tag{3.13}$$

and also that

$$I_{1,0} = \sqrt{\frac{\pi}{2}} \int_0^{+\infty} g(s)s\,ds,$$

$$I_{3,k} = \begin{cases} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{0} g(s)s\,ds, & \text{when } k \text{ is even,} \\ \sqrt{\frac{\pi}{2}} \int_{0}^{+\infty} g(s)s\,ds, & \text{when } k \text{ is odd.} \end{cases} \tag{3.14}$$

Therefore, the value of

$$\lim_{n\to\infty} \|u_n\|^2 \int_0^{\pi} g(u_n)\varphi_k \tag{3.15}$$

depends on the parity of k. Concretely, when k is even,

$$\begin{aligned}\lim_{n\to\infty}\|u_n\|^2\int_0^{\pi} g(u_n)\varphi_k &= k\sqrt{\frac{\pi}{2}}\int_0^{+\infty} g(s)s\,ds + k\sqrt{\frac{\pi}{2}}\int_{-\infty}^{0} g(s)s\,ds \\ &= k\sqrt{\frac{\pi}{2}}\int_{-\infty}^{+\infty} g(s)s\,ds\end{aligned} \tag{3.16}$$

and when k is odd,

$$\lim_{n\to\infty}\|u_n\|^2\int_0^{\pi} g(u_n)\varphi_k = (k+1)\sqrt{\frac{\pi}{2}}\int_0^{+\infty} g(s)s\,ds + (k-1)\sqrt{\frac{\pi}{2}}\int_{-\infty}^{0} g(s)s\,ds. \tag{3.17}$$

This completes the proof of Theorem 3.1. □

Previous theorem enables us to describe the side of bifurcations in terms of $\int_{-\infty}^{0} g(s)s\,ds$ and $\int_0^{+\infty} g(s)s\,ds$ as follows.

Corollary 3.2. *Under hypothesis (G),*

(a) *if k is even and $\int_{-\infty}^{+\infty} g(s)s\ ds > 0$ (resp., < 0), the bifurcations from $(\sigma_k, +\infty)$ and from $(\sigma_k, -\infty)$ of solutions of* (1.3) *are both subcritical (resp., supercritical),*

(b.1) *if k is odd and $(k-1)\int_{-\infty}^{0} g(s)s\ ds + (k+1)\int_0^{+\infty} g(s)s\ ds > 0$ (resp., < 0), the bifurcation from $(\sigma_k, +\infty)$ of solutions of* (1.3) *is subcritical (resp., supercritical),*

(b.2) *If k is odd and $(k+1)\int_{-\infty}^{0} g(s)s\ ds + (k-1)\int_0^{+\infty} g(s)s\ ds > 0$ (resp., < 0), the bifurcation from $(\sigma_k, -\infty)$ of solutions of* (1.3) *is subcritical (resp., supercritical).*

Remark 3.3. The nonautonomous case. By including the dependence on $t \in [0, \pi]$ and under hypothesis (G')

(G')

(i) there exists $f \in L^1(\mathbb{R})$ with $\lim_{|s|\to+\infty} f(s)s = 0$ such that $|g(t,s)s| < f(s)$, for $t \in (0,\pi)$ and for all $s \in \mathbb{R}$,

(ii) $g(t,s)$ is continuous in $[0,\pi] \times \mathbb{R}$.

one can alter the integrals L_k, given by (3.2) y (3.3), as follows.

(i) If k is even,

$$\begin{aligned}L_k^+ &= \sqrt{\frac{\pi}{2}}\sum_{i=0}^{k-1}\left[\int_0^{+\infty} g\left(\frac{i\pi}{k}, s\right)s\,ds + \int_{-\infty}^{0} g\left(\frac{(i+1)\pi}{k}, s\right)s\,ds\right], \\ L_k^- &= \sqrt{\frac{\pi}{2}}\sum_{i=0}^{k-1}\left[\int_0^{+\infty} g\left(\frac{(i+1)\pi}{k}, s\right)s\,ds + \int_{-\infty}^{0} g\left(\frac{i\pi}{k}, s\right)s\,ds\right].\end{aligned} \tag{3.18}$$

(ii) If k is odd,

$$L_k^+ = \sqrt{\frac{\pi}{2}}\left[\sum_{i=0}^{k}\int_0^{+\infty} g\left(\frac{i\pi}{k}, s\right)s\ ds + \sum_{i=0}^{k-2}\int_{-\infty}^{0} g\left(\frac{(i+1)\pi}{k}, s\right)s\ ds\right],$$
$$L_k^- = \sqrt{\frac{\pi}{2}}\left[\sum_{i=0}^{k-2}\int_0^{+\infty} g\left(\frac{(i+1)\pi}{k}, s\right)s\ ds + \sum_{i=0}^{k}\int_{-\infty}^{0} g\left(\frac{i\pi}{k}, s\right)s\ ds\right]. \tag{3.19}$$

Theorem 3.1 can be rewritten. Assume (G') holds, if (λ_n, u_n) is a sequence of solutions of (1.1) bifurcating from $(\sigma_k, +\infty)$, then

$$\lim_{n\to+\infty}\|u_n\|^3(\sigma_k - \lambda_n) = \sigma_k L_k^+ = k^2 L_k^+. \tag{3.20}$$

If (λ_n, u_n) is a sequence of solutions of (1.1) bifurcating from $(\sigma_k, -\infty)$, then

$$\lim_{n\to+\infty}\|u_n\|^3(\sigma_k - \lambda_n) = \sigma_k L_k^- = k^2 L_k^-. \tag{3.21}$$

4. The "Strongly" Resonant Problems at High Eigenvalues

Arguments used in [1, 7] or [6], under hypotheses either (G) or (G'), ensure that if the bifurcation from $(\sigma_k, +\infty)$ and bifurcation from $(\sigma_k, -\infty)$ are either both subcritical or both supercritical, then the problem (1.2) has, at least, one solution. Consequently, by using Corollary 3.2 one can determine the laterality of bifurcations from $\pm\infty$ and deduce the existence of solutions of the resonant problems (3.1) and (1.2).

Corollary 4.1. *Under (G) and for any k, if* $\operatorname{sign}(L_k^+) = \operatorname{sign}(L_k^-)$, *then the resonant problem* (1.2) *has, at least, one solution. More concretely,*

(a) *for any even k, if $\int_{-\infty}^{+\infty} g(s)s\,ds \neq 0$, resonant problem* (3.1) *has, at least, one solution (see Corollary 3.2(a)),*

(b) *for odd k, if $\int_{-\infty}^{+\infty} g(s)s\,ds \neq 0$, there exists $k_0 \in \mathbb{N}$ such that every resonant problem* (3.1) *has, at least, one solution $\forall k \geq k_0$ (see Corollary 3.2(b.1) and (b.2)).*

5. Examples

Two examples are given. The first one requires Theorem 3.1 and the second one add a the dependence on $t \in [0, \pi]$.

5.1. Autonomous Example

Consider problem (1.3) with $g(s) = e^{-(s-0.15)^2}$. In this case

$$\int_0^{+\infty} g(s)s\,ds \approx 0.664, \qquad \int_{-\infty}^{0} g(s)s\,ds \approx -0.378, \qquad \int_{-\infty}^{+\infty} g(s)s\,ds \approx 0.286 \tag{5.1}$$

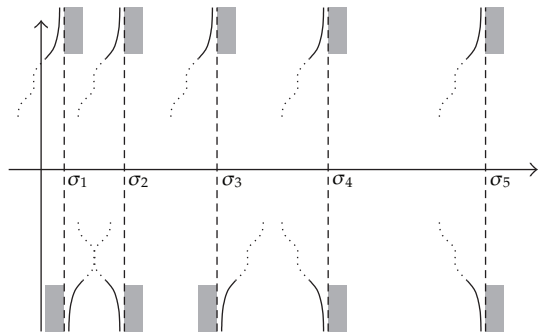

Figure 2: Bifurcation diagram for $g(s) = e^{-(s-0.15)^2}$.

and then we have the following.

(i) For even k, $L_k^+ = L_k^- > 0$ and the bifurcations from $+\infty$ and from $-\infty$ are both subcritical.

(ii) For every odd k, $L_k^+ > 0$. Moreover, $L_1^- < 0$, $L_3^- < 0$, and $L_k^- > 0$ for all $k \geq 5$, and the bifurcations have variety of behaviors (see Figure 2).

Consequently, we can ensure that if either $k = 2$ or $k \geq 4$, the resonant problem (3.1) with $g(s) = e^{-(s-0.15)^2}$ has, at least, one solution.

5.2. Nonautonomous Example

Consider a function $g(x, s) = \sin(7x)g_1(s)$, $\forall x \in (0, \pi)$ where g_1 satisfies

$$\int_0^{+\infty} g_1(s)s\,ds = \int_{-\infty}^0 g_1(s)s\,ds = 1. \tag{5.2}$$

In this case, the values L_k have easily computable forms (see Remark 3.3).

Since $L_k^+ = L_k^-$ and by watching carefully the L_k values represented in Figure 3, one can see that

(i) bifurcation from $(\sigma_1, \pm\infty)$: no conclusion can be extracted by using Remark 3.3,

(ii) bifurcations from $(\sigma_2, \pm\infty)$ are both supercritical,

(iii) bifurcations from $(\sigma_3, \pm\infty)$ are both subcritical,

(iv) bifurcations from $(\sigma_k, \pm\infty)$ for $k = 4, 5, 6$ are supercritical,

(v) bifurcations from $(\sigma_7, \pm\infty)$: nothing can be concluded,

(vi) bifurcations from $(\sigma_k, \pm\infty)$ for $k \geq 8$ are subcritical.

Furthermore, for $k = 2, 3, 4, 5, 6$ and $k \geq 8$, every resonant problem (1.2) has, at least, one solution.

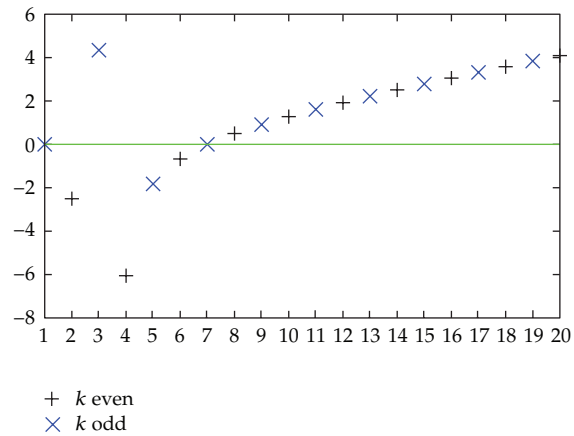

Figure 3: Values $L_k^+ = L_k^-$ for $h(x) = \sin(7x)$.

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