Research Article

Kadec-Klee Properties of Calderón-Lozanovskii Function Spaces

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Received 31 January 2011; Accepted 2 February 2011

Academic Editor: Henryk Hudzik

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We study Kadec-Klee properties with respect to global (local) convergence in measure. First, we present some results concerning Köthe spaces and Orlicz functions. Next, we shall give full criteria for Kadec-Klee properties with respect to global (local) convergence in measure in Calderón-Lozanovskii function spaces. In particular, we obtain the full characterizations of Kadec-Klee properties in Orlicz-Lorentz function spaces, which have not been presented until now.

1. Introduction

The Kadec-Klee properties are fundamental notions in the theory of Banach function spaces (see [1, 2]). On the other hand, the Calderón-Lozanovskii spaces are important class of Banach lattices, especially because of their role in the interpolation theory. Geometry of Calderón-Lozanovskii spaces has been intensively developed during the last decades (see, e.g., [3–9]). The complete characterization of Kadec-Klee properties with respect to local H_l (global H_g) convergence in measure for Orlicz function spaces L_φ has been presented in [10]. The Orlicz spaces L_φ are special case of Calderón-Lozanovskii spaces E_φ . Thus, it is natural to investigate properties H_l and H_g in the spaces E_φ generated by arbitrary Köthe spaces E . Since $L^1 \in H_l$ and $E_\varphi = L_\varphi$ when $E = L^1$, so finding full criteria of H_l and H_g in Orlicz spaces is simpler than in general Calderón-Lozanovskii spaces. The sufficient conditions of H_l in Calderón-Lozanovskii function spaces E_φ have been proved in [4, 6]. Note that the implication $E_\varphi \in (H_l) \Rightarrow \varphi \in \Delta_2^E$ follows from the fact that property H_l implies order continuity in Köthe spaces and from results concerning copy of l^∞ in E_φ (see [5, 10]). The implication $E_\varphi \in (H_l) \Rightarrow E \in (H_l)$ has not been discussed at all. Moreover, property H_g in E_φ has not been considered early. We present full criteria of properties H_l and H_g in Calderón-Lozanovskii function spaces E_φ . Furthermore, in the case of property H_g , we prove some general results concerning Köthe spaces which will be essential in studying this property in

spaces E_φ . The crucial difficulty in the proofs concerning property H_g in spaces E_φ is the fact that property H_g needs not imply order continuity in general. Note that if $E \in (H_l)$, then $E \in (OC)$ (see [10]), so looking for complete characterizations in the spaces E_φ , the case of property H_l is easier than H_g . As an application, we get the criteria of properties H_l and H_g in Orlicz-Lorentz function spaces $(\Lambda_\omega)_\varphi$. A sufficient condition for property H_l of $(\Lambda_\omega)_\varphi$ over a finite, atomless measure space has been presented in [6, Theorem 1]. Note also that the case of Kadec-Klee properties with respect to pointwise (uniform) convergence in Calderón-Lozanovskii sequence spaces has been solved in [11].

2. Preliminaries

Let $\mathbb{R}, \mathbb{R}_+, \mathbb{N}$ be the sets of real, nonnegative real, and positive integer numbers, respectively. As usual, $S(X)$ (resp., $B(X)$) stands for the unit sphere (resp., the closed unit ball) of a real Banach space $(X, \|\cdot\|_X)$.

Let (T, Σ, μ) be a σ -finite and complete measure space. By $L^0 = L^0(T)$ we mean the set of all μ -equivalence classes of real-valued measurable functions defined on T .

A Banach space $E = (E, \|\cdot\|_E)$ is said to be a *Köthe space* if E is a linear subspace of L^0 and

- (i) if $x \in E, y \in L^0$, and $|y| \leq |x|$ μ -a.e., then $y \in E$ and $\|y\|_E \leq \|x\|_E$;
- (ii) there exists a function x in E that is positive on the whole T (see [2, 12]).

Every *Köthe space* is a Banach lattice under the natural partial order ($x \geq 0$ if $x(t) \geq 0$ for μ -a.e. $t \in T$). In particular, if we consider the space E over the nonatomic measure μ , then we will say that E is a *Köthe function space*. If we replace the measure space (T, Σ, μ) by the counting measure space $(\mathbb{N}, 2^\mathbb{N}, m)$, then we will say that E is a *Köthe sequence space*.

The set $E_+ = \{x \in E : x \geq 0\}$ is called the *positive cone* of E . For any subset $A \subset E$, define $A_+ = A \cap E_+$.

By the *symmetric Köthe function space* E (symmetric Banach function space) on I , where $I = [0, 1]$ or $I = [0, \infty)$ with the Lebesgue measure m , we mean a Köthe space $E = (E, \|\cdot\|_E)$ with the additional property that for any two equimeasurable functions $x \sim y$, $x, y \in L^0(I)$ (i.e., they have the same distribution functions $d_x = d_y$, where $d_x(\lambda) = m(\{t \in I : |x(t)| > \lambda\})$, $\lambda \geq 0$) and $x \in E$ we have $y \in E$ and $\|x\|_E = \|y\|_E$. In particular, $\|x\|_E = \|x^*\|_E$, where

$$x^*(t) = \inf\{\lambda > 0 : d_x(\lambda) \leq t\}, \quad t > 0. \quad (2.1)$$

For basic properties of symmetric spaces and rearrangements, we refer to [13, 14]. In the sequel, considering symmetric Köthe function space E over measure space (T, Σ, μ) we mean $T = [0, 1]$ or $T = [0, \infty)$ with the Lebesgue measure μ .

A point $x \in E$ is said to have *order continuous norm* if for any sequence (x_m) in E such that $0 \leq x_m \leq |x|$ and $x_m \rightarrow 0$ μ -a.e. we have $\|x_m\|_E \rightarrow 0$. A Köthe space E is called *order continuous* ($E \in (OC)$) if every element of E has an order continuous norm (see [2, 12, 15]). As usual, E_a stands for the subspace of order continuous elements of E . It is known that $x \in E_a$ if and only if $\|x\chi_{A_n}\|_E \downarrow 0$ for any sequence $\{A_n\}$ satisfying $A_n \downarrow \emptyset$ (i.e., $A_n \supset A_{n+1}$ and $\mu(\bigcap_{n=1}^\infty A_n) = 0$). Clearly, $\mu(\bigcap_{n=1}^\infty A_n) = 0$ if and only if $\chi_{A_n} \rightarrow 0$ μ -a.e. in T .

Recall that E is said to have Kadec-Klee property ($E \in (KK)$ for short) whenever $\|x_n - x\| \rightarrow 0$ for any x and $\{x_n\}$ in E satisfying $x_n \rightarrow x$ in the weak topology $\sigma(E, E^*)$

and $\|x_n\| \rightarrow \|x\|$ (see [2]). This property is sometimes called the Radon-Riesz property or property H . It has been considered in many classes of Banach spaces (see [1, 4, 16–18]). If we replace the weak convergence $\sigma(E, E^*)$ by the convergence in measure $(x_n \xrightarrow{\mu} x)$, by the convergence in measure on every set of finite measure $(x_n \xrightarrow{\mu} x \text{ locally})$, or by the uniform convergence $(x_n \rightrightarrows x)$, then we say that E has the Kadec-Klee property with respect to convergence in measure, local convergence in measure, or uniform convergence, respectively (we will write $E \in (H_g)$, $E \in (H_l)$, and $E \in (H_u)$). Clearly, $E \in (H_l) \Rightarrow E \in (H_g) \Rightarrow E \in (H_u)$. Moreover, the converse of any of these implications is not true in general (see Sections 3.1 and 3.2 below).

Given a property P , we will say that a Köthe space E satisfies property P_+ ($E \in (P)_+$) if E satisfies property P restricted only to nonnegative elements.

In the whole paper, φ denotes an *Orlicz function*, that is, $\varphi : \mathbb{R} \rightarrow [0, \infty]$, it is convex, even, vanishing, and continuous at zero, left continuous on $(0, \infty)$ and not identically equal to zero. Denote

$$a_\varphi = \sup\{u \geq 0 : \varphi(u) = 0\}, \quad b_\varphi = \sup\{u \geq 0 : \varphi(u) < \infty\}. \quad (2.2)$$

We write $\varphi > 0$ when $a_\varphi = 0$ and $\varphi < \infty$ if $b_\varphi = \infty$. Let $\varphi_r = \varphi \chi_{G_\varphi}$, where

$$G_\varphi = \begin{cases} [a_\varphi, b_\varphi] & \text{if } \varphi(b_\varphi) < \infty, \\ [a_\varphi, b_\varphi) & \text{otherwise.} \end{cases} \quad (2.3)$$

Define on L^0 a convex semimodular I_φ by

$$I_\varphi(x) = \begin{cases} \|\varphi \circ x\|_E & \text{if } \varphi \circ x \in E, \\ \infty & \text{otherwise,} \end{cases} \quad (2.4)$$

where $(\varphi \circ x)(t) = \varphi(x(t))$, $t \in T$. By the Calderón-Lozanovskiĭ space E_φ we mean

$$E_\varphi = \left\{ x \in L^0 : I_\varphi(cx) < \infty \text{ for some } c > 0 \right\}, \quad (2.5)$$

equipped with so called *Luxemburg norm* defined by

$$\|x\|_\varphi = \inf \left\{ \lambda > 0 : I_\varphi\left(\frac{x}{\lambda}\right) \leq 1 \right\}. \quad (2.6)$$

We generally assume that if $b_\varphi < \infty$, then $a_\varphi < b_\varphi$, because when $0 < a_\varphi = b_\varphi$, then $E_\varphi = L^\infty$ and $\|x\|_\varphi = (1/b_\varphi)\|x\|_\infty$.

If $E = L^1$ ($e = l^1$), then E_φ (e_φ) is the Orlicz function (sequence) space equipped with the Luxemburg norm. If $E = \Lambda_\omega$ —the Lorentz function space ($e = \lambda_\omega$ —the Lorentz sequence space), then E_φ (e_φ) is the corresponding Orlicz-Lorentz function (sequence) space denoted by $(\Lambda_\omega)_\varphi$ ($(\lambda_\omega)_\varphi$) and equipped with the Luxemburg norm (see [5, 8]).

We will assume in the whole paper that E has the *Fatou property*, that is, if $0 \leq x_n \uparrow x \in L^0$ with $(x_n)_{n=1}^\infty$ in E and $\sup_n \|x_n\|_E < \infty$, then $x \in E$ and $\|x\|_E = \lim_n \|x_n\|_E$. Since E has the Fatou property, E_φ has also this property, whence E_φ is a Banach space.

We say an Orlicz function φ satisfies *condition* $\Delta_2(0)$ (resp., $\Delta_2(\infty)$) if there exist $K > 0$ and $u_0 > 0$ such that $\varphi(u_0) > 0$ (resp., $\varphi(u_0) < \infty$), and the inequality $\varphi(2u) \leq K\varphi(u)$ holds for all $u \in [0, u_0]$ (resp., $u \in [u_0, \infty)$). If there exists $K > 0$ such that $\varphi(2u) \leq K\varphi(u)$ for all $u \geq 0$, then we say that φ satisfies *condition* $\Delta_2(\mathbb{R}_+)$. We write for short $\varphi \in \Delta_2(0)$, $\varphi \in \Delta_2(\infty)$, and $\varphi \in \Delta_2(\mathbb{R}_+)$, respectively.

For a Köthe space E and an Orlicz function φ , we say that φ satisfies *condition* Δ_2^E ($\varphi \in \Delta_2^E$ for short) if

- (1) $\varphi \in \Delta_2(0)$ whenever $E \hookrightarrow L^\infty$,
- (2) $\varphi \in \Delta_2(\infty)$ whenever $L^\infty \hookrightarrow E$,
- (3) $\varphi \in \Delta_2(\mathbb{R}_+)$ whenever neither $L^\infty \hookrightarrow E$ nor $E \hookrightarrow L^\infty$ (see [5]),

where the symbol $E \hookrightarrow F$ stands for the continuous embedding of the space E into the space F .

Relationships between the modular I_φ and the norm $\|\cdot\|_\varphi$ are collected in [8].

3. Results

3.1. Köthe Spaces and Orlicz Functions

Given a Köthe function space E , we set

$$m_E = \sup_{(B, \chi_B \in E)} \sup_{[(A_n) \subset B, \mu(A_n) \rightarrow 0]} \inf_n \|\chi_{A_n}\|_E. \quad (3.1)$$

Remark 3.1. It is easy to see that if E is a symmetric Köthe function space, then

$$m_E = \inf_{A \in \Sigma, \mu(A) > 0} \|\chi_A\|_E. \quad (3.2)$$

Recall that if $x \in L^0$, then $\text{supp } x = \{t \in T : x(t) \neq 0\}$. For a Köthe space $E \subset L^0(T, \Sigma, \mu)$, let $F \subset E$ be a subset. By the support of F , we mean the set $\text{supp } F \in \Sigma$ satisfying

- (i) for each $x \in F$, there is $A \in \Sigma$, with $\mu(A) = 0$, such that $\text{supp } x \subset A \cup \text{supp } F$,
- (ii) there is $x \in F$ with $\mu(\text{supp } F \setminus \text{supp } x) = 0$.

Lemma 3.2. *Suppose that E is a Köthe function space. Consider the following statements:*

- (i) $E \in (OC)$,
- (ii) $E \in (H_g)_+$,
- (iii) For any $u \in E$ and each sequence $\{A_n\}$ in Σ the implication

$$\mu(A_n) \rightarrow 0 \implies \|u\chi_{A_n}\| \rightarrow 0 \text{ holds.} \quad (3.3)$$

- (iv) $m_E = 0$,
- (v) $\text{supp } E_a = T$,
- (vi) $E_a \neq \{0\}$.

Then (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) and (ii) $\Rightarrow$ (iii). For a symmetric Köthe function space we have additionally (vi) $\Rightarrow$ (iv). Furthermore, for a nonsymmetric Köthe function space, the implication (vi) $\Rightarrow$ (iv) is not true in general. Finally, the implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) cannot be reversed in general.

Proof. The implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (vi) are obvious.

(iv) $\Rightarrow$ (v). If $\text{supp } E_a \neq T$, then for each $x \in E$ with $\text{supp } x \subset T \setminus \text{supp } E_a$ we have $x \notin E_a$. Let $T_0 \subset T \setminus \text{supp } E_a$ be such that $0 < \mu(T_0) < \infty$ and $\chi_{T_0} \in E$. Then $x_1 = \chi_{T_0} \notin E_a$ and consequently there is a sequence (A_n) of measurable pairwise disjoint sets with $\bigcup_{n=1}^{\infty} A_n = T_0$ and a number $\delta > 0$ satisfying $\|\chi_{A_n}\| \geq \delta$ for each n (see [19]). Since $\mu(A_n) \rightarrow 0$, we get $m_E \geq \delta > 0$.

(ii) $\Rightarrow$ (iii). Assume that (iii) does not hold, then we find an element $u \in E$, a number $\delta > 0$, and a sequence (A_i) with $\mu(A_i) \rightarrow 0$ such that $\inf_i \|u\chi_{A_i}\|_E \geq \delta$. Without loss of generality, we may assume that $\sum_{i=1}^{\infty} \mu(A_i) < \infty$. Set

$$x_k = \sup\{u\chi_{A_i} : i \geq k\} \in L^0. \quad (3.4)$$

Let $A = \bigcup_i A_i$. Clearly, $x_k \leq u\chi_A \in E$. Put $y = u\chi_A$ and $y_k = y - x_k$. Since $\{x_k\}$ is decreasing, so $\{y_k\}$ is increasing. Next, $x_k \rightarrow 0$ in measure, because, for each $\varepsilon > 0$,

$$\mu\{t \in T : x_k(t) > \varepsilon\} \leq \mu\left\{\bigcup_{i=k}^{\infty} A_i\right\} \leq \sum_{i=k}^{\infty} \mu(A_i) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (3.5)$$

In particular, by $E \in (FP)$, $\|y_k\| \rightarrow \|y\|$. Finally, $\|y_k - y\| = \|x_k\| \geq \|u\chi_{A_k}\| \geq \delta$. Thus, $E \notin (H_g)_+$.

Assume that E is symmetric. We prove that (vi) $\Rightarrow$ (iv). If $m_E > 0$, then we find a set B with $\chi_B \in E$ and a sequence $\{A_n\} \subset B$, $\mu(A_n) \rightarrow 0$ such that $\inf_n \|\chi_{A_n}\|_E \geq m_E/2$. Let $x \in E \setminus \{0\}$. We prove that $x \in E \setminus E_a$. There is a number $\delta > 0$ such that the set $D = \{t \in T : |x(t)| \geq \delta\}$ has positive measure. We find a sequence $\{C_k\}$ in D of pairwise disjoint sets and a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ satisfying $\mu(C_k) \leq (1/2^k)\mu(D)$ and $\mu(C_k) \geq \mu(A_{n_k})$. Then, by the symmetry of E ,

$$\|x\chi_{C_k}\|_E \geq \|\delta\chi_{C_k}\|_E \geq \|\delta\chi_{A_{n_k}}\|_E \geq \frac{\delta m_E}{2}, \quad (3.6)$$

for any $k \in \mathbb{N}$, whence $x \notin E_a$.

Now we will prove that the implication (vi) $\Rightarrow$ (iv) is not true in general in nonsymmetric case. Take

$$\begin{aligned} E &= \left(L^1[0, 1] \oplus L^\infty[1, 2]\right)_1 \\ &= \left\{x \in L^0[0, 2] : x = u + w, \ u \in L^1[0, 1], \ w \in L^\infty[1, 2], \ \|x\| = \|u\|_1 + \|w\|_\infty\right\}. \end{aligned} \quad (3.7)$$

Obviously, $\text{supp } E_a = [0, 1]$ and $m_E = 1$.

Now we will see that implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) cannot be reversed in general. If $E = \Lambda_\omega$ —the Lorentz space over $(0, \infty)$ with Lebesgue measure and $\int_0^\infty \omega < \infty$, then $E \in (H_g)$ and $E \notin (OC)$ (see [1, 8]), whence (iii) $\nRightarrow$ (i) in general.

Finally, let $E = L_\varphi[0, 1]$ —the Orlicz space with $\varphi < \infty$ and $\varphi \notin \Delta_2(\infty)$. By the proof of Theorem 1 from [5], we find an element $u \in E$ and a sequence $\{A_n\}$ with $\mu(A_n) \rightarrow 0$ and $\|u\chi_{A_n}\|_\varphi = 1$. On the other hand, for each $B \in \Sigma$ with $u = \chi_B \in E$, we claim that $\chi_B \in E_a$, whence $m_E = 0$. Indeed, for any sequence $\{A_n\}$ satisfying $A_n \downarrow \emptyset$ and for each $\lambda > 0$, we have

$$I_\varphi(\lambda u\chi_{A_n}) = \varphi(\lambda) \int_0^1 \chi_{A_n} d\mu = \varphi(\lambda)\mu(A_n) \rightarrow 0, \quad (3.8)$$

whence $\|u\chi_{A_n}\|_\varphi \rightarrow 0$. Thus, the implication (iii) $\Rightarrow$ (iv) cannot be reversed in general. $\square$

Recall that if $E \in (H_l)_+$, then $E \in (OC)$ (see [10]) and, as we have seen above, property H_g needs not imply OC in general. However, from the above result, it follows that if $E \in (H_g)_+$, then $m_E = 0$ and consequently $\text{supp } E_a = T$. Note also that the implication (vi) $\Rightarrow$ (v) for a symmetric Köthe function space has been proved in [8, Proposition 2.2].

Remark 3.3. It may happen that $\mu(A) < \infty$ and $\chi_A \notin E$ only in a nonsymmetric Köthe space. However, if $\mu(A) < \infty$ and $\chi_A \notin E$, then there is an increasing sequence (A_n) of subsets of A satisfying $\bigcup_n A_n = A$ and $\chi_{A_n} \in E$ for each n .

Proof. Let x_0 be a weak unit of E , that is, $x_0(t) > 0$ for each $t \in T$. It is enough to take $A_n = \{t \in A : x_0(t) > 1/n\}$ for each n . $\square$

Remark 3.4. Let E be a Köthe space. If $E \in (H_u)_+$, then $\|x\chi_{B_k}\| \rightarrow 0$ for each $x \in E_+$, where $B_k = \{t \in T : x(t) < 1/k\}$.

Proof. Let $x \in E_+$. Clearly, $x\chi_{T \setminus B_k} \uparrow x$ and $\sup_k \|x\chi_{T \setminus B_k}\| < \infty$. By $E \in (FP)$, we get $\|x\chi_{T \setminus B_k}\| \rightarrow \|x\|$. Moreover, $x - x\chi_{T \setminus B_k} \rightarrow 0$ uniformly. Then $\|x\chi_{T \setminus B_k} - x\| \rightarrow 0$, because $E \in (H_u)_+$.

It is easy to see that the above is not true if $E \notin (H_u)_+$. It is enough to take

$$E = L_\omega^\infty(0, \infty) = \left\{ x \in L^0 : \|x\| = \sup_{t \geq 0} \omega(t)|x(t)| < \infty \right\}, \quad (3.9)$$

where $\omega(t) = t$ and $x(t) = 1/t$. Then $\|x\chi_{B_k}\| = 1$ for each k . On the other hand, taking $x_n = x\chi_{[0, n]}$, we get $\|x\| = \|x_n\| = 1$, $x_n \rightarrow x$ uniformly and $\|x_n - x\| = 1$, whence $E \notin (H_u)_+$. $\square$

It is known that $E \in (H_l)$ if and only if $E \in (H_l)_+$ whenever $E \in (OC)$ (see [6]). Since $(H_l)_+ \Rightarrow (OC)$ (see [10]), so $E \in (H_l)$ if and only if $E \in (H_l)_+$. The same proof as in [6] shows that if $E \in (OC)$, then $E \in (H_g)$ if and only if $E \in (H_g)_+$. However, the assumption $E \in (OC)$ is now too strong in this case because there are Köthe spaces satisfying $E \in (H_g)$ and $E \notin (OC)$. Hence, the following result is more natural and precise.

Lemma 3.5. *Suppose E is a Köthe space. Then $E \in (H_g)$ if and only if $E \in (H_g)_+$.*

Proof. The necessity is obvious. We prove the sufficiency. Let $E \in (H_g)_+$. Assume that E is a Köthe function space. Let $\|x_n\| \rightarrow \|x\|$ and $x_n \rightarrow x$ in measure. For each $\eta > 0$, we get

$$\mu\{t \in T : \|x_n(t)\| - |x(t)| \geq \eta\} \leq \mu\{t \in T : |x_n(t) - x(t)| \geq \eta\} \rightarrow 0, \quad (3.10)$$

whence $|x_n| \rightarrow |x|$ in measure. By the assumption $E \in (H_g)_+$, we conclude that

$$\| |x_n| - |x| \| \rightarrow 0. \quad (3.11)$$

Then we find a subsequence $(|x_{n_m}|)_m$ of $(|x_n|)_n$, an element $y \in E_+$, and a sequence $0 \leq \beta_m \rightarrow 0$ such that $\| |x_{n_m}| - |x| \| \leq \beta_m y$ for each m (see [12]). Denote $(|x_{n_m}|)_m$ still by $(|x_n|)_n$ and (β_m) by (β_n) . Thus, we may assume that $\| |x_n| - |x| \| \leq y$ for all n . Set $u = |x| + y$. Let $\varepsilon > 0$. Applying Remark 3.4, with respective sets

$$B_k = \left\{ t : u(t) < \frac{1}{k} \right\}, \quad (3.12)$$

take k big enough to satisfy $\| u \chi_{B_k} \| < \varepsilon/8$. Moreover, $\| |x_n| - |x| \| < \varepsilon/8$ for sufficiently large n . Setting

$$C_n = \{ t \in T : 0 \leq \operatorname{sgn}[x(t)x_n(t)] \leq 1 \}, \quad D_n = \{ t \in T : \operatorname{sgn}[x(t)x_n(t)] = -1 \}, \quad (3.13)$$

we get

$$\begin{aligned} \| x - x_n \| &\leq \| (|x| - |x_n|) \chi_{C_n} \| + \| (x - x_n) \chi_{D_n \cap B_k} \| + \| (x - x_n) \chi_{D_n \setminus B_k} \| \\ &\leq \frac{5\varepsilon}{8} + \| (x - x_n) \chi_{D_n \setminus B_k} \|. \end{aligned} \quad (3.14)$$

Take

$$0 < \delta < \frac{\varepsilon}{16k} \min \left\{ \frac{1}{\|y\|}, \frac{1}{\|x\|} \right\}. \quad (3.15)$$

Denote

$$\begin{aligned} F_n^k &= \{ t \in D_n \setminus B_k : |x_n(t) - x(t)| > \delta \}, \\ G_n^k &= \{ t \in D_n \setminus B_k : |x_n(t) - x(t)| \leq \delta \}. \end{aligned} \quad (3.16)$$

Note that $x_n \xrightarrow{\mu} x$ and consequently $\mu(F_n^k) \rightarrow 0$ as $n \rightarrow \infty$. Then, by Lemma 3.2 (iii),

$$\begin{aligned} \| (x - x_n) \chi_{D_n \setminus B_k} \| &\leq \| (x - x_n) \chi_{F_n^k} \| + \| (x - x_n) \chi_{G_n^k} \| \\ &\leq \| 2u \chi_{F_n^k} \| + \| (x - x_n) \chi_{G_n^k} \| < \frac{\varepsilon}{8} + \| (x - x_n) \chi_{G_n^k} \|, \end{aligned} \quad (3.17)$$

for sufficiently large n . Finally, divide set G_n^k into two subsets

$$H_n^k = \left\{ t \in G_n^k : |x(t)| \geq \frac{1}{2k} \right\}, \quad I_n^k = \left\{ t \in G_n^k : |x(t)| < \frac{1}{2k} \right\}. \quad (3.18)$$

Clearly, $H_n^k \cup I_n^k = G_n^k$ because

$$G_n^k \subset T \setminus B_k = \left\{ t : |x(t)| + y(t) \geq \frac{1}{k} \right\}. \quad (3.19)$$

Then, by (3.15),

$$\begin{aligned} \|(x - x_n)\chi_{G_n^k}\| &\leq \delta \left(\|\chi_{H_n^k}\| + \|\chi_{I_n^k}\| \right) \leq \delta \left(\|2k|x|\chi_{H_n^k}\| + \|2ky\chi_{I_n^k}\| \right) \\ &\leq 2k\delta \left(\|x|\chi_{H_n^k}\| + \|y\chi_{I_n^k}\| \right) < \frac{\varepsilon}{4}. \end{aligned} \quad (3.20)$$

Combining (3.14), (3.17), and (3.20), we get $\|x - x_n\| \leq \varepsilon$.

If E is a Köthe sequence space, the proof is the same, because for a counting measure space $(\mathbb{N}, 2^{\mathbb{N}}, m)$ for each $u \in E$ and any sequence (A_n) in $2^{\mathbb{N}}$ with $m(A_n) \rightarrow 0$, we have $\|u\chi_{A_n}\| \rightarrow 0$. $\square$

The straightforward calculation shows that we may equivalently take x and $\{x_n\}$ in $S(E)$ in the definition of property H_l (see also the proof below). However, the analogous problem for property H_g is a little more complicated. The crucial point is described in the following lemma.

Lemma 3.6. *Suppose that E is a symmetric Köthe function space. If $x_n \rightarrow x$ in measure and $\|x_n\| \rightarrow \|x\| \neq 0$, then $x_n/\|x_n\| \rightarrow x/\|x\|$ in measure. Moreover, the above implication is not true in general if E is a nonsymmetric Köthe function space.*

Proof. Having our assumptions, for each $\varepsilon, \eta > 0$, we get

$$\begin{aligned} \mu \left\{ t : \left| \frac{x_n(t)}{\|x_n\|} - \frac{x(t)}{\|x\|} \right| > \varepsilon \right\} &= \mu \left\{ t : \left| \frac{x_n(t)}{\|x_n\|} - \frac{x(t)}{\|x_n\|} + \frac{x(t)}{\|x_n\|} - \frac{x(t)}{\|x\|} \right| > \varepsilon \right\} \\ &\leq \mu \left\{ t : \frac{1}{\|x_n\|} |x_n(t) - x(t)| + \left| \frac{1}{\|x_n\|} - \frac{1}{\|x\|} \right| |x(t)| > \varepsilon \right\} \\ &\leq \mu \left\{ t : \frac{1}{\|x_n\|} |x_n(t) - x(t)| > \frac{\varepsilon}{2} \right\} + \mu \left\{ t : \left| \frac{1}{\|x_n\|} - \frac{1}{\|x\|} \right| |x(t)| > \frac{\varepsilon}{2} \right\}. \end{aligned} \quad (3.21)$$

Since $\|x_n\| \rightarrow \|x\|$, so $\|x_n\| > (1/2)\|x\|$ for $n \geq N_0$, whence

$$\mu \left\{ t : \frac{1}{\|x_n\|} |x_n(t) - x(t)| > \frac{\varepsilon}{2} \right\} \leq \mu \left\{ t : |x_n(t) - x(t)| > \frac{\varepsilon\|x\|}{4} \right\} < \frac{\eta}{2}, \quad (3.22)$$

for $n \geq N_1 > N_0$. Moreover, applying the symmetry of E , we get

$$\mu \left\{ t : \left| \frac{1}{\|x_n\|} - \frac{1}{\|x\|} \right| |x(t)| > \frac{\varepsilon}{2} \right\} < \frac{\eta}{2} \quad \text{for } n \geq N_2. \quad (3.23)$$

Thus,

$$\mu \left\{ t : \left| \frac{x_n(t)}{\|x_n\|} - \frac{x(t)}{\|x\|} \right| > \varepsilon \right\} < \eta, \quad (3.24)$$

for $n \geq \max\{N_1, N_2\}$.

Considering the second assertion, take

$$E = L^1_{1/t^3}(1, \infty) = \left\{ x \in L^0((1, \infty), \Sigma, m) : \|x\| = \int_1^\infty |x(t)| \frac{1}{t^3} dt < \infty \right\}. \quad (3.25)$$

Set $x(t) = t$ and $x_n(t) = t + (1/n)$, then $\|x\| = 1$, $\|x_n\| = 1 + (1/2n)$, and $x_n \rightarrow x$ in measure. However,

$$\mu \left\{ t : \left| \frac{x_n(t)}{\|x_n\|} - \frac{x(t)}{\|x\|} \right| > 1 \right\} = \mu \left\{ t : \left| \frac{t-2}{2n+1} \right| > 1 \right\} = \infty, \quad (3.26)$$

for each n , whence $(x_n/\|x_n\|) \not\rightarrow (x/\|x\|)$ in measure. $\square$

Corollary 3.7. *Assume that E is a symmetric Köthe function space. Then $E \in (H_g)$ if and only if $\|x_n - x\| \rightarrow 0$ for any $x \in S(E)$ and (x_n) in $S(E)$ satisfying $x_n \rightarrow x$ in measure.*

Proof. The necessity is obvious. We prove the sufficiency. Let $x_n \rightarrow x$ in measure and $\|x_n\| \rightarrow \|x\| \neq 0$. Taking $y_n = x_n/\|x_n\|$ and $y = x/\|x\|$, we get $y_n \rightarrow y$ in measure, by Lemma 3.6. By the assumption, we conclude that $\|y_n - y\| \rightarrow 0$. Notice that

$$\begin{aligned} & \left| \frac{1}{\|x_n\|} \|x_n - x\| - \|x\| \right| \left| \frac{1}{\|x_n\|} - \frac{1}{\|x\|} \right| \\ & \leq \left\| \frac{x_n}{\|x_n\|} - \frac{x}{\|x_n\|} - \left(-\frac{x}{\|x_n\|} + \frac{x}{\|x\|} \right) \right\| = \left\| \frac{x_n}{\|x_n\|} - \frac{x}{\|x\|} \right\| \rightarrow 0. \end{aligned} \quad (3.27)$$

Since $\|x_n\| \rightarrow \|x\| \neq 0$, so $\|x_n - x\| \rightarrow 0$ as desired. $\square$

We will need in the sequel the following.

Lemma 3.8. *If $\varphi \in \Delta_2(\infty)$, then for any $a > 0$, there are $\sigma \in (0, 1]$ and $u_1 > a_\varphi + a$ such that*

$$\inf_{u_0 \in (0, a)} \inf_{u \geq u_1} \frac{\varphi(u - u_0)}{\varphi(u) - \varphi(u_0)} \geq \sigma. \quad (3.28)$$

Proof. Suppose conversely that there is $a > 0$ such that for any $\sigma \in (0, 1]$ and each $u_1 > a_\varphi + a$, there holds

$$\inf_{u_0 \in (0, a)} \inf_{u \geq u_1} \frac{\varphi(u - u_0)}{\varphi(u) - \varphi(u_0)} < \sigma. \quad (3.29)$$

Taking $\sigma_n = 1/2^n$ and $u_n \rightarrow \infty$, we get $\inf_{u_0 \in (0, a)} \inf_{u \geq u_n} \varphi(u - u_0) / (\varphi(u) - \varphi(u_0)) < 1/2^n$. We may assume that $u_n > 2a$ for each n . Hence, for each n , one may choose $v_n \geq u_n$ and $u_0 \in (0, a)$ with $\varphi(v_n - u_0) / (\varphi(v_n) - \varphi(u_0)) \leq 1/2^n$. Consequently, we get

$$2^n \varphi\left(\frac{1}{2}v_n\right) < 2^n \varphi(v_n - u_0) < \varphi(v_n) - \varphi(u_0) \leq \varphi(v_n), \quad (3.30)$$

for each n . Taking $b_n = (1/2)v_n$, one has $2^n \varphi(b_n) < \varphi(2b_n)$, whence $\varphi \notin \Delta_2(\infty)$. $\square$

Lemma 3.9. (i) Suppose that E is a symmetric Köthe space. If $x_n \rightarrow x$ in measure and $\varphi \circ x_n, \varphi \circ x \in E$ are finitely valued a.e., then $\varphi \circ x_n \rightarrow \varphi \circ x$ in measure. Furthermore, for a nonsymmetric Köthe space, the above implication is not true in general.

(ii) If $x_n \rightarrow x$ in measure, $x_n, x \in E_+$ and $\varphi_r^{-1} \circ x_n, \varphi_r^{-1} \circ x$ are well-defined functions, then $\varphi_r^{-1} \circ x_n \rightarrow \varphi_r^{-1} \circ x$ in measure.

Proof. (i) Let $x_n \xrightarrow{\mu} x$.

(A) Suppose that $b_\varphi = \infty$. Set

$$A_k = \{t \in T : |x(t)| > k\}. \quad (3.31)$$

Then, by symmetry of E , $\mu(A_k) \rightarrow 0$. Let $\eta > 0$ and $\varepsilon > 0$. Take k_0 such that $\mu(A_{k_0}) < \eta/3$. We have

$$\begin{aligned} & \mu\{t \in T : |\varphi(x_n(t)) - \varphi(x(t))| \geq \varepsilon\} \\ & \leq \mu\{t \in T \setminus A_{k_0} : |\varphi(x_n(t)) - \varphi(x(t))| \geq \varepsilon\} + \frac{\eta}{3}. \end{aligned} \quad (3.32)$$

Since φ is uniformly continuous on the interval $[-k_0, k_0]$, so for each $\gamma > 0$ there is $\delta = \delta(\gamma, k_0) > 0$ such that the implication $|u - s| < \delta \Rightarrow |\varphi(u) - \varphi(s)| < \gamma$ is true for all $u, s \in [-k_0, k_0]$. Set $\delta = \delta(\varepsilon, 2k_0)$ and

$$A_n^1 = \{t \in T \setminus A_{k_0} : |x_n(t)| > 2k_0\}, \quad A_n^2 = \{t \in T \setminus A_{k_0} : |x_n(t)| \leq 2k_0\}. \quad (3.33)$$

Consequently

$$\begin{aligned} & \mu\{t \in T : |\varphi(x_n(t)) - \varphi(x(t))| \geq \varepsilon\} \\ & \leq \mu(A_n^1) + \mu\{t \in A_n^2 : |\varphi(x_n(t)) - \varphi(x(t))| \geq \varepsilon\} + \frac{\eta}{3} \\ & \leq \mu\{t \in A_n^2 : |x_n(t) - x(t)| \geq \delta\} + \frac{2\eta}{3} \leq \eta \end{aligned} \quad (3.34)$$

for sufficiently large n , because $A_n^1 = \{t \in T : |x_n(t) - x(t)| \geq k_0\}$. Thus $\varphi \circ x_n \xrightarrow{\mu} \varphi \circ x$.

(B) Assume $\varphi(b_\varphi) < \infty$. The proof is simpler than above because $x_n(t) \leq b_\varphi$ and $x(t) \leq b_\varphi$ for μ -a.e. $t \in T$ and φ is uniformly continuous on the interval $[-b_\varphi, b_\varphi]$.

(C) Suppose that $b_\varphi < \infty$, $\varphi(b_\varphi) = \infty$. Then $x_n(t) < b_\varphi$ and $x(t) < b_\varphi$ for μ -a.e. $t \in T$. Set

$$A_k = \left\{ t \in T : |x(t)| > b_\varphi - \frac{1}{k} \right\}. \quad (3.35)$$

Then $\mu(A_k) \rightarrow 0$, because E is symmetric. Let $\eta > 0$ and $\varepsilon > 0$. Take k_0 such that $\mu(A_{k_0}) < \eta/3$. We have

$$\begin{aligned} & \mu\{t \in T : |\varphi(x_n(t)) - \varphi(x(t))| \geq \varepsilon\} \\ & \leq \mu\{t \in T \setminus A_{k_0} : |\varphi(x_n(t)) - \varphi(x(t))| \geq \varepsilon\} + \mu(A_{k_0}). \end{aligned} \quad (3.36)$$

Since φ is uniformly continuous on the interval $[-u_0, u_0]$ for each $u_0 < b_\varphi$, so for each $u_0 < b_\varphi$ and $\gamma > 0$ there is $\delta = \delta(\gamma, u_0) > 0$ such that the implication $|u - s| < \delta \Rightarrow |\varphi(u) - \varphi(s)| < \gamma$ is true for all $u, s \in [-u_0, u_0]$. Put $\delta = \delta(\varepsilon, b_\varphi - 1/2k_0)$ and

$$\begin{aligned} A_n^1 &= \left\{ t \in T \setminus A_{k_0} : |x_n(t)| > b_\varphi - \frac{1}{2k_0} \right\}, \\ A_n^2 &= \left\{ t \in T \setminus A_{k_0} : |x_n(t)| \leq b_\varphi - \frac{1}{2k_0} \right\}. \end{aligned} \quad (3.37)$$

Consequently

$$\begin{aligned} & \mu\{t \in T : |\varphi(x_n(t)) - \varphi(x(t))| \geq \varepsilon\} \\ & \leq \mu(A_n^1) + \mu\{t \in A_n^2 : |\varphi(x_n(t)) - \varphi(x(t))| \geq \varepsilon\} + \frac{\eta}{3} \\ & \leq \mu\{t \in A_n^2 : |x_n(t) - x(t)| \geq \delta\} + \frac{2\eta}{3} \leq \eta \end{aligned} \quad (3.38)$$

for sufficiently large n , because $A_n^1 = \{t \in T : |x_n(t) - x(t)| \geq 1/2k_0\}$. Thus $\varphi \circ x_n \xrightarrow{\mu} \varphi \circ x$.

Now we will show that in nonsymmetric case the above implication is not true in general. Consider the Lebesgue measure space $((0, \infty), \Sigma, m)$. Take

$$\varphi(u) = u^2, \quad x = \sum_n n \chi_{[n, n+1]}, \quad x_k = \sum_n \left(n + \frac{1}{k}\right) \chi_{[n, n+1]}. \quad (3.39)$$

Then $x_k \rightarrow x$ in measure. On the other hand, $|\varphi \circ x_k - \varphi \circ x| = \sum_n ((1/k^2) + (2n/k)) \chi_{[n, n+1]}$, whence $\varphi \circ x_k \not\rightarrow \varphi \circ x$ in measure. Clearly, such element $\varphi \circ x$ can be well defined only in some nonsymmetric Köthe space E .

(ii) First, we claim that for each $\varepsilon > 0$, there is $\delta > 0$ such that the implication

$$|\varphi_r^{-1}(u) - \varphi_r^{-1}(v)| > \varepsilon \implies |u - v| > \delta \quad (3.40)$$

is true for all $u, v \geq 0$. First, notice that for each $\varepsilon > 0$, there is $\delta > 0$ such that the implication

$$|\varphi_r^{-1}(u) - \varphi_r^{-1}(0)| > \varepsilon \implies |u| > \delta \quad (3.41)$$

is true for all $u \geq 0$, because φ_r^{-1} is continuous at zero. Next, defining $g(u) = \varphi_r^{-1}(u) - \varphi_r^{-1}(0)$, we see that g is subadditive on $\mathbb{R}_+$, because $g = \varphi^{-1}$, where φ is an Orlicz function with $\varphi > 0$. Consequently,

$$\varphi_r^{-1}(u - v) - \varphi_r^{-1}(0) = g(u - v) \geq g(u) - g(v) = \varphi_r^{-1}(u) - \varphi_r^{-1}(v). \quad (3.42)$$

Hence, if $|\varphi_r^{-1}(u) - \varphi_r^{-1}(v)| > \varepsilon$, then $|\varphi_r^{-1}(u - v) - \varphi_r^{-1}(0)| > \varepsilon$, whence $|u - v| > \delta$, by (3.41). Thus, the claim is proved. Assume that $x_n, x \in E_+$ and $\varphi_r^{-1} \circ x_n, \varphi_r^{-1} \circ x$ are well-defined functions and $x_n \rightarrow x$ in measure, then, applying (3.40), for each $\varepsilon > 0$, we get

$$\mu\left\{t \in T : \left|(\varphi_r^{-1} \circ x_n - \varphi_r^{-1} \circ x)(t)\right| > \varepsilon\right\} \leq \mu\{t \in T : |(x_n - x)(t)| > \delta\} \rightarrow 0. \quad (3.43)$$

□

3.2. Calderón-Lozanovskii Function Spaces

We still assume below that E is a Köthe function space.

It is known that $E_\varphi \in (OC)$ if and only if $E \in (OC)$ and $\varphi \in \Delta_2^E$ (see Corollary 12 from [9]). Considering some weaker property than OC , we get the following.

Lemma 3.10. *Suppose that $b_\varphi = \infty$, then $E_a \neq \{0\}$ if and only if $(E_\varphi)_a \neq \{0\}$.*

Proof. Necessity

Let $z \in E_a \setminus \{0\}$. We find a number $m > 0$ such that $\mu(A) = \mu\{t \in T : |z(t)| > 1/m\} > 0$. Taking $w = \varphi^{-1}(1)\chi_A$, we get $\varphi \circ w \in E_a$. Moreover, since $b_\varphi = \infty$, φ satisfies the local Δ_2^E condition with respect to the element w (see [9]). Consequently, $w \in (E_\varphi)_a$, by Theorem 11 from [9].

Sufficiency

We apply Theorem 11 from [9]. Let $x \in (E_\varphi)_a \setminus \{0\}$, then $\varphi \circ x \in E_a$. If $\mu\{t : |x(t)| > a_\varphi\} > 0$, then $\mu(A_m) > 0$ for some $m \in \mathbb{N}$, where $A_m = \{t \in T : |x(t)| \geq (1 + (1/m))a_\varphi\}$. Thus, $\chi_{A_m} \in E_a$, and we are done, if $\mu\{t : |x(t)| \leq a_\varphi\} = \mu(A) > 0$ then $\chi_A \in E_a \setminus \{0\}$, by Theorem 11 from [9]. □

Theorem 3.11. *$E_\varphi \in (H_l)$ if and only if*

- (a) $E \in (H_l)$,
- (b) $\varphi \in \Delta_2(\infty)$ whenever $L^\infty \hookrightarrow E$ and $\varphi \in \Delta_2(\mathbb{R}_+)$ provided $L^\infty \not\hookrightarrow E$.

Proof. Since the sufficiency has been proved in [4, Theorem 11] (we need also to apply [7, Lemma 2]), we need only to prove the necessity. Since $E_\varphi \in (H_l)$, so $E_\varphi \in (OC)$ (see [10, Proposition 2.1]), and consequently, $E \in (OC)$, $\varphi < \infty$, and $\varphi \in \Delta_2^E$ (see [9, Corollary 12]). Notice that $E \in (OC)$ implies that $E \not\hookrightarrow L^\infty$. Indeed, if $E \hookrightarrow L^\infty$, then there is a decreasing sequence (A_n) in Σ with $0 < \mu(A_n) \rightarrow 0$ and $\chi_{A_n} \in E$, whence $1 = \|\chi_{A_n}\|_{L^\infty} \leq M\|\chi_{A_n}\|_E$, so $E \notin (OC)$. Thus, (b) is proved. Suppose that $E \notin (H_l)$. Recall that property H_l can be equivalently considered only on the positive cone E_+ [6, Proposition 1]. Consequently, we find x and $\{x_n\}$ in $(S(E))_+$ with $x_n \xrightarrow{\mu} x$ locally in measure and $\|x_n - x\|_E \geq \varepsilon$. Set $y_n = \varphi_r^{-1} \circ x_n$

and $y = \varphi_r^{-1} \circ x$, then $I_\varphi(y) = I_\varphi(y_n) = 1$, whence $y, y_n \in S(E_\varphi)$. Moreover, $y_n \xrightarrow{\mu} y$ locally in measure, because, by σ finiteness of the measure space (T, Σ, μ) , the local convergence in measure can be equivalently replaced by convergence μ -a.e. (also Lemma 3.9(ii) can be applied). It is enough to prove that there is $\eta > 0$ with

$$\|y_n - y\|_\varphi \geq \eta, \quad (3.44)$$

for infinitely many n . The similar condition has been proved in [8, Theorem 2.11], but under additional assumption that $\varphi > 0$. Here, the lack of this assumption requires new techniques in comparison with the respective proof in [8, Theorem 2.11]. We have

$$\|\varphi \circ y_n - \varphi \circ y\|_E \geq \varepsilon. \quad (3.45)$$

Denote

$$A_n = \{t \in T : \varphi(y_n(t)) \geq \varphi(y(t))\}, \quad B_n = T \setminus A_n. \quad (3.46)$$

By (3.45), we get

$$\max\{\|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n}\|_E, \|(\varphi \circ y_n - \varphi \circ y)\chi_{B_n}\|_E\} \geq \frac{\varepsilon}{2}, \quad (3.47)$$

for each n . We assume that $\|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n}\|_E \geq \varepsilon/2$ for infinitely many $n \in \mathbb{N}$, because otherwise the proof is analogous. We divide the proof into two cases.

(I) Suppose that $L^\infty \hookrightarrow E$ and $\varphi \in \Delta_2(\infty)$. Let $c = \varphi_r^{-1}(\varepsilon/32\|\chi_T\|_E) > a_\varphi$. Denoting

$$A_n^1 = \{t \in A_n : y_n(t) < c\}, \quad A_n^2 = \{t \in A_n : y_n(t) \geq c\}, \quad (3.48)$$

we get

$$\frac{\varepsilon}{2} \leq \|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n}\|_E \leq \|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n^2}\|_E + \frac{\varepsilon}{4}. \quad (3.49)$$

Thus,

$$\|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n^2}\|_E \geq \frac{\varepsilon}{4}. \quad (3.50)$$

Set

$$c_1 = \frac{c + a_\varphi}{2}, \quad B_n^1 = \{t \in A_n^2 : y(t) < c_1\}, \quad B_n^2 = \{t \in A_n^2 : y(t) \geq c_1\}. \quad (3.51)$$

The remaining proof of the case (I) we divide into two parts.

- (1) Suppose that $\|(\varphi \circ y_n - \varphi \circ y)\chi_{B_n^1}\|_E \geq \varepsilon/8$. By Lemma 3.8, for $a = c_1$, there are $\sigma > 0$ and $u_1 > 0$ such that

$$\inf_{u_0 \in (0, c_1)} \inf_{u \geq u_1} \frac{\varphi(u - u_0)}{\varphi(u) - \varphi(u_0)} \geq \sigma. \quad (3.52)$$

Set

$$C_n^1 = \{t \in B_n^1 : y_n(t) < u_1\}, \quad C_n^2 = \{t \in B_n^1 : y_n(t) \geq u_1\}. \quad (3.53)$$

- (a) Assume that $\|(\varphi \circ y_n - \varphi \circ y)\chi_{C_n^1}\|_E \geq \varepsilon/16$. Taking $0 < \alpha < (c - c_1)/u_1$, we get

$$\begin{aligned} \left\| \left(\varphi \circ \left(\frac{y_n - y}{\alpha} \right) \right) \chi_{C_n^1} \right\|_E &\geq \left\| \left(\varphi \left(\frac{c - c_1}{\alpha} \right) \right) \chi_{C_n^1} \right\|_E \geq \|\varphi(u_1)\chi_{C_n^1}\|_E \\ &\geq \|(\varphi \circ y_n - \varphi \circ y)\chi_{C_n^1}\|_E \geq \frac{\varepsilon}{16}. \end{aligned} \quad (3.54)$$

Thus, $\|y_n - y\|_\varphi \geq \eta_1 = \min\{1, \alpha\varepsilon/16\}$.

- (b) Let $\|(\varphi \circ y_n - \varphi \circ y)\chi_{C_n^2}\|_E \geq \varepsilon/16$. Thus, by Lemma 3.8,

$$\|\varphi \circ (y_n - y)\chi_{C_n^2}\|_E \geq \frac{\sigma}{2} \|(\varphi \circ y_n - \varphi \circ y)\chi_{C_n^2}\|_E \geq \frac{\sigma\varepsilon}{32}, \quad (3.55)$$

and consequently $\|y_n - y\|_\varphi \geq \eta_2 = \min\{1, \sigma\varepsilon/32\}$ (cf. [8, Lemma 1.1]).

- (2) Assume that $\|(\varphi \circ y_n - \varphi \circ y)\chi_{B_n^1}\|_E < \varepsilon/8$, then, by (3.50),

$$\|(\varphi \circ y_n - \varphi \circ y)\chi_{B_n^2}\|_E \geq \frac{\varepsilon}{8}. \quad (3.56)$$

Since $\varphi \in \Delta_2(\infty)$, for $l = 1 + \varepsilon/32$ and $u_1 = c_1$, there is $a = a(l, u_1) \in (0, 1)$ such that

$$\varphi((1 + a)u) \leq l\varphi(u), \quad (3.57)$$

for every $u \geq c_1$ (see [20, Theorem 1.13(4)]). Moreover, we can choose $a > 0$ satisfying

$$\frac{a}{1 + a} \varphi(c_1) \|\chi_T\|_E < \frac{\varepsilon}{32}. \quad (3.58)$$

Let

$$\begin{aligned} B_n^{21} &= \left\{ t \in B_n^2 : (y_n - y)(t) < \frac{ac_1}{1 + a} \right\}, \\ B_n^{22} &= \left\{ t \in B_n^2 : (y_n - y)(t) \geq \frac{ac_1}{1 + a} \right\}, \end{aligned} \quad (3.59)$$

then, by (3.57) and (3.58), using convexity of φ , we get

$$\begin{aligned}
 |\varphi \circ y_n - \varphi \circ y|_{\chi_{B_n^{21}}} &= (\varphi \circ (y_n - y + y) - \varphi \circ y)_{\chi_{B_n^{21}}} \\
 &\leq \left(\frac{a}{1+a} \varphi \circ \left(\frac{1+a}{a} (y_n - y) \right) + \frac{1}{1+a} \varphi \circ ((1+a)y) - \varphi \circ y \right)_{\chi_{B_n^{21}}} \\
 &\leq \left(\frac{a}{1+a} \varphi \circ \left(\frac{1+a}{a} (y_n - y) \right) + \frac{1}{1+a} l \varphi \circ y - \varphi \circ y \right)_{\chi_{B_n^{21}}} \\
 &= \left(\frac{a}{1+a} \varphi \circ \left(\frac{1+a}{a} (y_n - y) \right) + \left(\frac{\varepsilon/32 - a}{1+a} \right) \varphi \circ y \right)_{\chi_{B_n^{21}}}.
 \end{aligned} \tag{3.60}$$

Note that $f(a) = (\varepsilon/32 - a)/(1+a)$ is a decreasing function of $a > 0$. Moreover, $a \leq \varepsilon/32$ by convexity of φ and inequality (3.57). Hence, by (3.58), $\|(\varphi \circ y_n - \varphi \circ y)_{\chi_{B_n^{21}}}\|_E < \varepsilon/16$. Then, by (3.56),

$$\|(\varphi \circ y_n - \varphi \circ y)_{\chi_{B_n^{22}}}\|_E \geq \frac{\varepsilon}{16}. \tag{3.61}$$

Since $\varphi \in \Delta_2(\infty)$, for $u_3 = c_1$ and $\beta = (1+a)/a$, there is $k_2 > 0$ such that $\varphi(\beta u) \leq k_2 \varphi(u)$ for each $u \geq u_3$ (see [20]). Taking $0 < \gamma < a/(1+a)$ and applying (3.57), we get

$$\begin{aligned}
 |\varphi \circ y_n - \varphi \circ y|_{\chi_{B_n^{22}}} &= (\varphi \circ (y_n - y + y) - \varphi \circ y)_{\chi_{B_n^{22}}} \\
 &\leq \left(\varphi \circ \left(\frac{y_n - y}{\gamma} + y \right) - \varphi \circ y \right)_{\chi_{B_n^{22}}} \\
 &\leq \left(\frac{a}{1+a} \varphi \circ \left(\frac{1+a}{a} \left(\frac{y_n - y}{\gamma} \right) \right) + \frac{1}{1+a} \varphi \circ ((1+a)y) - \varphi \circ y \right)_{\chi_{B_n^{22}}} \\
 &\leq \left(\frac{a}{1+a} k_2 \varphi \circ \left(\frac{y_n - y}{\gamma} \right) + \frac{1}{1+a} l \varphi \circ y - \varphi \circ y \right)_{\chi_{B_n^{22}}} \\
 &= \left(\frac{a}{1+a} k_2 \varphi \circ \left(\frac{y_n - y}{\gamma} \right) + \left(\frac{\varepsilon/32 - a}{1+a} \right) \varphi \circ y \right)_{\chi_{B_n^{22}}}.
 \end{aligned} \tag{3.62}$$

Then $\|\varphi \circ ((y_n - y)/\gamma)\|_E \geq \varepsilon(1+a)/32ak_2$, and consequently $\|y_n - y\|_\varphi \geq \eta_3 = \min\{1, \gamma\varepsilon(1+a)/32ak_2\}$ (see [8, Lemma 1.1]).

Combining cases (1) and (2), we get (3.44) with $\eta = \min\{\eta_1, \eta_2, \eta_3\}$.

- (II) Suppose that $L^\infty \not\prec E$. Since $\varphi \in \Delta_2(\mathbb{R}_+)$, so for every $l > 1$, there is $a = a(l) \in (0, 1)$ such that $\varphi((1+a)u) \leq l\varphi(u)$ for every $u \geq 0$ (see [20, Theorem 1.13(4)]). Moreover, for every $\beta > 0$, there is $k > 0$ such that $\varphi(\beta u) \leq k\varphi(u)$ for each $u \geq 0$. Then the proof is analogous as in case (I) (it is simpler and shorter). $\square$

Now we will consider the Kadec-Klee property with respect to global convergence in measure. Theorem 3.19 below presents the full characterization of property H_g . We divide the whole proof into several lemmas for readers convenience.

Lemma 3.12. *If $E_\varphi \in (H_g)_+$, then $m_E = 0$.*

Proof. If $m_E > 0$, then there are $\delta > 0$, a set B , $\mu(B) > 0$, $\chi_B \in E$, and a sequence $(A_n) \subset B$ with $\mu(A_n) \rightarrow 0$ and $\inf_n \|\chi_{A_n}\|_E \geq \delta$. Without loss of generality, we may assume that $\sum_{i=1}^{\infty} \mu(A_i) < \infty$. Set

$$z_n = \sup\{\chi_{A_i} : i \geq n\} \in L^0. \quad (3.63)$$

Let $A = \bigcup_i A_i$. Clearly, $z_n \leq \chi_A \leq \chi_B \in E$. Put $y = \chi_A$ and $y_n = y - z_n$. Choose $a > 0$ such that

$$\begin{aligned} a &= b_\varphi & \text{if } \varphi(b_\varphi) \|\chi_A\|_E < 1, \\ a &\in (a_\varphi, b_\varphi] & \text{if } \varphi(b_\varphi) < \infty, \varphi(b_\varphi) \|\chi_A\|_E \geq 1, \\ a &\in (a_\varphi, b_\varphi) & \text{if } \varphi(b_\varphi) = \infty, \end{aligned} \quad (3.64)$$

and $\varphi(a) \|\chi_A\|_E = 1$ in the last two cases. Take $x = ay$ and $x_n = ay_n$, then $\|x_n\|_\varphi \leq \|x\|_\varphi = 1$. It is obvious that $\|x_n\|_\varphi = 1$ when $a = b_\varphi$. We show that $\|x_n\|_\varphi \rightarrow 1$ in the case $a < b_\varphi$. Note that $z_n \rightarrow 0$ in measure (see the proof of Lemma 3.2), $z_n \downarrow$, and consequently $\|y_n\|_E \rightarrow \|y\|_E$, because $E \in (FP)$. If $\|x_n\|_\varphi \leq 1 - \alpha$ for some $\alpha > 0$ and infinitely many n , then we get

$$1 = \varphi(a) \|\chi_A\|_E < \varphi\left(\frac{a}{1-\alpha}\right) \|\chi_A\|_E \leftarrow \varphi\left(\frac{a}{1-\alpha}\right) \|y_n\|_E = I_\varphi\left(\frac{x_n}{1-\alpha}\right) \leq 1, \quad (3.65)$$

which is a contradiction. Moreover, $x_n \rightarrow x$ in measure. Since

$$\|x - x_n\|_\varphi \geq I_\varphi(x - x_n) \geq \varphi(a) \|\chi_{A_n}\|_E \geq \varphi(a) \delta, \quad (3.66)$$

we conclude that $E_\varphi \notin (H_g)_+$. □

Remark 3.13. If $E_\varphi \in (H_g)_+$, then $E \not\asymp L^\infty$.

Proof. It is immediate, since if $E \hookrightarrow L^\infty$, then $m_E > 0$. By Lemma 3.12, we get $E_\varphi \notin (H_g)_+$. □

From Lemmas 3.12 and 3.2, we get immediately the following.

Corollary 3.14. *If $E_\varphi \in (H_g)_+$, then $\text{supp } E_a = T$.*

Lemma 3.15. *If $E_\varphi \in (H_g)$, then $\varphi(b_\varphi) = \infty$.*

Proof. Suppose that $E_\varphi \in (H_g)$ and $\varphi(b_\varphi) < \infty$. By Lemma 3.12, we get $m_E = 0$. Let $A \in \Sigma$, $\mu(A) > 0$, and $\mu(T \setminus A) > 0$ be such that $\chi_A \in E$ and $\varphi(b_\varphi) \|\chi_A\|_E \leq 1/2$. Let $(A_n) \subset T \setminus A$ be such that $0 < \mu(A_n) \rightarrow 0$ and $\varphi(b_\varphi) \|\chi_{A_n}\|_E \leq 1/2$. Taking $x = b_\varphi \chi_A$ and $x_n = b_\varphi \chi_{A \cup A_n}$, we get $\|x\|_\varphi = \|x_n\|_\varphi = 1$, $x_n \rightarrow x$ in measure and $\|x_n - x\|_\varphi = 1$, whence $E_\varphi \notin (H_g)$. □

Lemma 3.16. *Suppose that $(b_\varphi < \infty \text{ and } \varphi(b_\varphi) = \infty)$ or $(b_\varphi = \infty \text{ and } \varphi \notin \Delta_2(\infty))$. If $E_a \neq \{0\}$, then $E_\varphi \notin (H_g)$.*

Proof. It is enough to apply order isometry $P : l^\infty \rightarrow E_\varphi$ constructed in the proof of [5, Theorem 1] and take $z = \chi_{\mathbb{N}}$ and $z_n = z - e_n$, where e_n is the n th unit vector in l^∞ . If $y = Pz$ and $y_n = Pz_n$, then $y_n \rightarrow y$ in measure, because, without loss of generality, one can take in

that proof a sequence $\{A_n\}$ satisfying $\min\{\mu(A_n), \|\chi_{A_n}\|_E\} \leq 2^{-n}$. Moreover, $\|y_n\|_\varphi = \|y\|_\varphi = \|y_n - y\|_\varphi = 1$. $\square$

Lemma 3.17. (i) If $L^\infty \not\prec E$, $a_\varphi > 0$, $\varphi(b_\varphi) = \infty$, then $E_\varphi \notin (H_g)$.

(ii) If $L^\infty \not\prec E$, $\text{supp } E_a = T$, $a_\varphi = 0$, $b_\varphi = \infty$ and $\varphi \notin \Delta_2(0)$, then $E_\varphi \notin (H_g)$.

Proof. (i) Suppose that $L^\infty \not\prec E$, $\varphi(b_\varphi) = \infty$, and $a_\varphi > 0$. If $T_1 \cup T_2 = T$, $T_1 \cap T_2 = \emptyset$, $\mu(T_1) > 0$, and $\mu(T_2) > 0$, then $\chi_{T_1} \notin E$ or $\chi_{T_2} \notin E$. Assume that $\chi_{T_1} \notin E$. Setting $x = a_\varphi \chi_{T_1}$ and $x_n = x + a_n \chi_{A_n}$, where $(A_n) \subset T_2$, $\chi_{A_n} \in E$, $\mu(A_n) \rightarrow 0$, and $\varphi(a_n) \|\chi_{A_n}\|_E = 1$, we get $\|x\|_\varphi = \|x_n\|_\varphi = 1$, $x_n \rightarrow x$ in measure and $\|x_n - x\|_\varphi = 1$, whence $E_\varphi \notin (H_g)$.

(ii) The proof is analogous to that of [10, Proposition 2.3]. In fact, it shows even more that in this case, $E_\varphi \notin (H_u)$. By Lemma 3.10, we get $(E_\varphi)_a \neq \{0\}$. Let $x \in (E_\varphi)_a$, $x \geq 0$, and $\|x\|_\varphi = 1$.

First, we prove that $\varphi \circ (2x) \in E_a$. Since $x \in S((E_\varphi)_a)$, so $\varphi \circ x \in E_a$ and φ satisfies the local Δ_2^E condition with respect to the element x by Theorem 11 from [9] (notice $a_\varphi = 0$, whence $\Delta_2^E(x \chi_{T \setminus A}) = \Delta_2^E(x)$ in Theorem 11 from [9]). Furthermore, by Lemma 9 from [9], we conclude that $\varphi \circ (2x) \in E_a$, because $b_\varphi = \infty$.

We claim that there is a sequence $\{A_n\}$ in Σ satisfying $\chi_{A_n} \notin E$ and $I_\varphi(2x \chi_{A_n}) \leq 2^{-n}$ for each n . Since the measure space (T, Σ, μ) is σ -finite, there is an increasing sequence $\{C_n\}$ with $0 < \mu(C_n) < \infty$ for any $n \in \mathbb{N}$ and $\bigcup_n C_n = T$. Applying Remark 3.3, without loss of generality, we may assume that $\chi_{C_n} \in E$ for each n (see also [12]). Consequently, $\|\varphi \circ (2x) \chi_{T \setminus C_n}\|_E \rightarrow 0$. Moreover, $\chi_{T \setminus C_n} \notin E$, because $\chi_T \notin E$ and $\chi_{C_n} \in E$. Thus, the claim is proved with $\{A_n\}$ being the respective subsequence of $\{T \setminus C_n\}$. Since $\varphi \notin \Delta_2(0)$, there is a sequence $\{u_n\}$ in $\mathbb{R}_+$ with $u_n \rightarrow 0$ and

$$\varphi(2u_n) > 2^n \varphi(u_n), \quad (3.67)$$

for any $n \in \mathbb{N}$. Notice that if $\chi_A \in E_a$, then the submeasure $\nu_A(B) = \|\chi_B\|_E$, $B \in \Sigma \cap A$ has the Darboux property in the interval $[0, \|\chi_A\|_E]$ (see [21]). We claim that for any n , there is a set $B_n \subset A_n$ with

$$\varphi(u_n) \|\chi_{B_n}\|_E = 2^{-n}. \quad (3.68)$$

Let $n \in \mathbb{N}$. First, by $E \in (FP)$, there is a set $D_n \subset A_n$ with $\mu(D_n) < \infty$, $\chi_{D_n} \in E$, and $\varphi(u_n) \|\chi_{D_n}\|_E > 2 \cdot 2^{-n}$. Next, since $\text{supp } E_a = T$ and $E \in (FP)$, there is a set $E_n \subset D_n$ with $\chi_{E_n} \in E_a$ and $\varphi(u_n) \|\chi_{E_n}\|_E > (3/2) \cdot 2^{-n}$. Finally, there is a set $B_n \subset E_n$ with $\varphi(u_n) \|\chi_{B_n}\|_E = 2^{-n}$. Taking

$$x_n = x + \frac{u_n}{2} \chi_{B_n} = x \chi_{T \setminus B_n} + \left(x + \frac{u_n}{2}\right) \chi_{B_n}, \quad (3.69)$$

we can complete arguments as in the proof of [10, Proposition 2.3]. $\square$

Lemma 3.18. Suppose that E is a symmetric Köthe function space.

(i) If $L^\infty \hookrightarrow E$, $E \in (H_g)$ and $\varphi \in \Delta_2(\infty)$, then $E_\varphi \in (H_g)$.

(ii) If $L^\infty \not\prec E$, $E \in (H_g)$ and $\varphi \in \Delta_2(\mathbb{R}_+)$, then $E_\varphi \in (H_g)$.

Proof. (i) Applying Lemma 3.5 and Corollary 3.7, it is enough to take $x, x_n \in (S(E_\varphi))_+, n \in \mathbb{N}$, and $x_n \rightarrow x$ in measure. Then $\|\varphi \circ x\|_E = \|\varphi \circ x_n\|_E = 1$, by $\varphi \in \Delta_2(\infty) = \Delta_2^E$. Moreover, $\varphi \circ x_n \xrightarrow{\mu} \varphi \circ x$ by Lemma 3.9(i). Consequently, $\|\varphi \circ x_n - \varphi \circ x\|_E \rightarrow 0$, because $E \in (H_g)$. Applying superadditivity φ on $\mathbb{R}_+$, we get

$$\|\varphi \circ (x_n - x)\|_E \leq \|\varphi \circ x_n - \varphi \circ x\|_E \rightarrow 0. \quad (3.70)$$

Take $\lambda > 1$. We need to show that

$$\|\varphi \circ (\lambda(x_n - x))\|_E \rightarrow 0. \quad (3.71)$$

Let $\varepsilon > 0$. The remaining part of the proof we divide into two parts.

(A) Suppose that $a_\varphi > 0$. Take $\delta > 0$ such that $\varphi((1 + \delta)a_\varphi)\|\chi_T\|_E < \varepsilon/3$. Put

$$\begin{aligned} A_n &= \{t \in T : \lambda|x_n(t) - x(t)| > (1 + \delta)a_\varphi\}, \\ A_n^1 &= \{t \in A_n : |x_n(t) - x(t)| > (1 + \delta)a_\varphi\}, \\ A_n^2 &= \{t \in A_n : |x_n(t) - x(t)| \leq (1 + \delta)a_\varphi\}. \end{aligned} \quad (3.72)$$

By $\varphi \in \Delta_2(\infty)$, there is $K(\lambda, (1 + \delta)a_\varphi) = K > 0$ such that $\varphi(\lambda u) \leq K\varphi(u)$ for each $u \geq (1 + \delta)a_\varphi$. Consequently, by (3.70),

$$\begin{aligned} &\|\varphi \circ (\lambda(x_n - x))\|_E \\ &\leq \|\varphi \circ (\lambda(x_n - x))\chi_{A_n}\|_E + \|\varphi \circ (\lambda(x_n - x))\chi_{T \setminus A_n}\|_E \\ &\leq \|\varphi \circ (\lambda(x_n - x))\chi_{A_n^1}\|_E + \|\varphi \circ (\lambda(x_n - x))\chi_{A_n^2}\|_E + \varphi((1 + \delta)a_\varphi)\|\chi_{T \setminus A_n}\|_E \\ &\leq K\|\varphi \circ (x_n - x)\chi_{A_n^1}\|_E + \varphi(\lambda(1 + \delta)a_\varphi)\|\chi_{A_n^2}\|_E + \frac{\varepsilon}{3} < \frac{2}{3}\varepsilon + \varphi(\lambda(1 + \delta)a_\varphi)\|\chi_{A_n^2}\|_E, \end{aligned} \quad (3.73)$$

for sufficiently large n . Recall that $m_E = 0$, because $E \in (H_g)$ (see Lemma 3.2). Moreover, $A_n^2 \subset A_n$, whence $\mu(A_n^2) \leq \mu(A_n) \rightarrow 0$, because $x_n \rightarrow x$ in measure. Consequently, $\|\chi_{A_n^2}\|_E \rightarrow 0$. Thus, $\|\varphi \circ (\lambda(x_n - x))\|_E \leq \varepsilon$ for sufficiently large n .

(B) If $a_\varphi = 0$, then we follow as in (A) but with the number $u_0 > 0$ in place of $(1 + \delta)a_\varphi$ satisfying $\varphi(\lambda u_0)\|\chi_T\|_E < \varepsilon/2$. We also set $A_n = \{t \in T : |x_n(t) - x(t)| > u_0\}$.

(ii) The proof is analogous and simpler than in case (i). $\square$

Theorem 3.19. *Suppose that E is a symmetric Köthe function space. The Calderón-Lozanovskii function space E_φ has the Kadec-Klee property with respect to convergence in measure if and only if:*

- (a) $E \in (H_g)$,
- (b) $\varphi \in \Delta_2(\infty)$ whenever $L^\infty \hookrightarrow E$ and $\varphi \in \Delta_2(\mathbb{R}_+)$ provided $L^\infty \not\hookrightarrow E$.

Proof. The necessity By Lemma 3.12, we conclude that $m_E = 0$, whence $\text{supp } E_a = T$. We also have $E \not\prec L^\infty$ (see Remark 3.13). By Lemmas 3.15 and 3.16, we get $b_\varphi = \infty$ and $\varphi \in \Delta_2(\infty)$. We consider two cases.

- (A) Assume that $L^\infty \hookrightarrow E$. To prove $E \in (H_g)$, we follow analogously as in the proof of condition $E \in (H_l)$ in Theorem 3.11 (case I). We apply also Lemma 3.5, Corollary 3.7, and Lemma 3.9(ii).
- (B) Assume that $L^\infty \not\hookrightarrow E$, then $\varphi \in \Delta_2(0)$, by Lemma 3.17. Consequently, $\varphi \in \Delta_2(\mathbb{R}_+)$, and to prove that $E \in (H_g)$, see the proof of Theorem 3.11 (case II).

The sufficiency is done by Lemma 3.18. $\square$

Remark 3.20. Property H_u does not imply H_g in general. If $L^\infty \hookrightarrow E$, then $\chi_T \in E_\varphi$, because there is $\lambda > 0$ with $\varphi(\lambda) < \infty$, whence $\varphi \circ (\lambda\chi_T) \in E$. Consequently, taking $x_n \rightarrow x$ uniformly, we get $\|x_n - x\|_\varphi \leq \varepsilon \|\chi_T\|_\varphi$ for sufficiently large n . Hence, $E_\varphi \in (H_u)$. However, if $\varphi \notin \Delta_2(\infty)$, then $E_\varphi \notin (H_g)$.

Denote by (I, Σ, m) the Lebesgue measure space with $I = [0, 1]$ or $I = [0, \infty)$. Recall that the function $\omega : [0, \gamma) \rightarrow \mathbb{R}_+$ with $\gamma = m(I)$ is said to be the *weight function*, if it is nonnegative, nonincreasing and locally integrable function with the respect to the Lebesgue measure m . Then the Lorentz function space Λ_ω consists of all functions $x \in L^0(I, \Sigma, m)$ such that $\|x\| = \int_0^\gamma x^*(t)\omega(t)dt < \infty$.

Applying Theorems 3.11 and 3.19 with $E = \Lambda_\omega$, we get the criteria for Orlicz-Lorentz function spaces. Note that the implication (iii) $\Rightarrow$ (i) in the case (a) has been proved directly in [6, Theorem 1]. The sufficient condition for case (b)(ii) has been proved in [22, Theorem 6] but under additional assumption.

Corollary 3.21. *Let $(\Lambda_\omega)_\varphi$ be the Orlicz-Lorentz function space.*

(a) *If $I = [0, 1]$, then the following assertions are equivalent:*

- (i) $(\Lambda_\omega)_\varphi \in (H_l)$,
- (ii) $(\Lambda_\omega)_\varphi \in (H_g)$,
- (iii) $\varphi \in \Delta_2(\infty)$.

(b) *Suppose that $I = [0, \infty)$.*

- (i) $(\Lambda_\omega)_\varphi \in (H_l)$ if and only if $\int_0^\infty \omega = \infty$ and $\varphi \in \Delta_2(\mathbb{R}_+)$,
- (ii) $(\Lambda_\omega)_\varphi \in (H_g)$ if and only if $\varphi \in \Delta_2(\mathbb{R}_+)$.

Proof. Recall that $\Lambda_\omega \in (H_g)$ for any weight ω [1, Corollary 1.3]. Thus, (a) and (b) (ii) follow immediately. Clearly, each symmetric Köthe space is separable if and only if it is order continuous (see [13]). Consequently, from [1, Theorem 3.3], [8, Lemma 3.2], it follows that $\Lambda_\omega \in (H_l)$ if and only if $\int_0^\infty \omega = \infty$. Thus, (b) (i) follows. $\square$

Obviously, since $L^1 \in (H_l)$, from Theorems 3.11 and 3.19 with $E = L^1$, we obtain the characterization of H_l and H_g for Orlicz spaces proved directly in [10, Theorem 2.1].

4. Open Questions

(1) We have seen that Lemma 3.9(i) is crucial in the proof of sufficiency in the main Theorem 3.19. Moreover, Lemma 3.9(i) is not true for a nonsymmetric Köthe space. Consequently, natural questions are as follows

- (a) are conditions presented in Theorem 3.19 sufficient also in a nonsymmetric case (different proof)? or, if the answer is no,
- (b) how the necessary and sufficient conditions for H_g are in general?

Looking for full criteria in nonsymmetric case, we are not allowed to apply Corollary 3.7 in the proof of sufficiency of Theorem 3.19.

Note also that the symmetry of E is applied in the proof of necessity of Theorem 3.19, by using Corollary 3.7. However, it seems that these parts go also if we drop the symmetry of E (after minor changes).

(2) We say that E has property H_g^s provided for any $x \in S(E)$ and (x_n) in $S(E)$ with $x_n \rightarrow x$ in measure we have $\|x_n - x\| \rightarrow 0$. Clearly, by Corollary 3.7, $E \in (H_g^s)$ if and only if $E \in (H_g)$ whenever E is a symmetric Köthe function space. Are properties H_g^s and H_g different in nonsymmetric case?

Acknowledgment

This work was supported partially by the State Committee for Scientific Research, Poland, Grant no. N N201 362236.

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