Research Article

Bootstrapping Weighted Inequalities for Hankel Transform

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A method of producing new inequalities for Hankel transform from known ones is given using the theory of positive integral operators. The new inequalities produced depend on six parameters, two real indices, two complex-valued measures, and two positive functions. The method may be iterated using the last inequality generated as input to the next stage.

1. Introduction

Weighted inequalities for the integral transforms with the general weights are of great importance in many branches of mathematics (functional analysis, integral equation, interpolation theory, etc.). They provide a tool to solve numerous problems related to the estimation of expressions with given integral transform. One of the most important problems is the characterization problem of the operator theory in function spaces such as the criteria of the continuity, compactness, and other qualitative estimations of a classical and nonclassical transformation. Many authors studied some generalizations of the Hardy inequalities and give some applications of these inequalities, [1–7].

In this work, we are interested in problems related to weighted inequalities for Hankel transform. More precisely, the main goal of this note is that from a given “input” Hankel inequality, a parametrized collection of “output” Hankel inequalities can be deduced. The idea is to exploit the close relationship of the Hankel transform to the operation of Hankel convolution and then to apply techniques from the theory of positive integral operators.

A single application of the theorem will produce new weighted Hankel inequalities from known ones. However, since the output inequalities are of the same form as the input inequality, it becomes possible to “bootstrap” the production of new inequalities by using the output at one stage as the input at the next. The implications of this sort of the iteration are not examined here.

Our investigation is inspired by the idea developed by Sinnamon [8], to the classical Fourier transform.

Throughout the paper, we adhere to conventions that are more common in the study of positive integral operators than in harmonic analysis generally. When integrals of non negative functions are involved, we will not concern ourselves with convergence; if the integral happens to take the value $+\infty$, then its appearance in formulas is to be interpreted according to arithmetic on $[0, \infty]$. In particular, expressions of the form 0∞ , ∞/∞ , $0/0$, 0^0 are all taken to be 0, while $\infty^0 = 1$.

2. Hankel Convolution Structure and Hankel Transform

In the following, we give some basic definitions and some properties of Hankel transform analogous to those of the classical Fourier transform. For more details, see [9, 10].

For fixed $\alpha > -1/2$, we define

$$\begin{aligned} d\mu_\alpha(x) &= \frac{x^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} dx, \\ j_\alpha(x) &= 2^\alpha \Gamma(\alpha+1) x^{-\alpha} J_\alpha(x), \end{aligned} \quad (2.1)$$

where J_α denotes the Bessel function of order α .

We denote that by $L_{\alpha,p}^p$, $1 \leq p < \infty$, the space of all real-valued, measurable functions f defined on $(0, \infty)$ with norm

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p d\mu_\alpha(x) \right)^{1/p} \quad (2.2)$$

is finite, whereas $L_\alpha^\infty = L^\infty$ that does not depend on α denotes the space of those measurable functions defined on $(0, \infty)$ for which

$$\|f\|_\infty = \operatorname{ess\,sup}_{0 < x < \infty} |f(x)| \quad (2.3)$$

is finite.

Let $\Delta(x, y, z)$ be the area of the triangle with sides x, y, z if such a triangle exists.

Set

$$D_\alpha(x, y, z) = \frac{2^{(\alpha-1)} (\Gamma(\alpha+1))^2}{\Gamma(\alpha+1/2) \pi^{1/2}} (xyz)^{-2\alpha} A(x, y, z)^{2\alpha-1} \quad (2.4)$$

if Δ exists and zero otherwise. We note that $D_\alpha(x, y, z) > 0$ and that $D_\alpha(x, y, z)$ is symmetric in x, y, z . Further we have the following basic formula:

$$j_\alpha(xt)j_\alpha(yt) = \int_0^\infty D_\alpha(x, y, z)j_\alpha(zt)d\mu_\alpha(z), \quad (2.5)$$

[11, page 411], from which it follows immediately, on setting $t = 0$, that

$$\int_0^\infty D_\alpha(x, y, z) d\mu_\alpha(z) = 1. \quad (2.6)$$

Using (2.5), we may show that

$$|j_\alpha(x)| \leq 1, \quad \forall x \geq 0, \quad (2.7)$$

see [10, page 310].

For each f in L_α^1 , it is clear, by (2.7), that the integral $\int_0^\infty j_\alpha(xt)f(t)d\mu_\alpha(t)$ exists, so that we may define the Hankel transform $\mathcal{H}_\alpha(f)$ of a function f in L_α^1 by

$$\mathcal{H}_\alpha(f)(x) = \int_0^\infty j_\alpha(xt)f(t)d\mu_\alpha(t). \quad (2.8)$$

For $f \in L_\alpha^1$, $\mathcal{H}_\alpha(f)$ is bounded and continuous for $x \geq 0$, see [9, page 336].

Proposition 2.1 (Haimo [9, page 338]). *Let f be such that f and $\mathcal{H}_\alpha(f) \in L_\alpha^1$. Then*

$$f(x) = \int_0^\infty \mathcal{H}_\alpha(f)(t)j_\alpha(xt)d\mu_\alpha(t) \quad a.e. \quad (2.9)$$

Proposition 2.2 (Trimèche [12]). *The Hankel transform $\mathcal{H}_\alpha$ is an isomorphism from $\mathcal{S}_*(\mathbb{R})$ onto itself, and its inverse denoted $\mathcal{H}_\alpha^{-1} = \mathcal{H}_\alpha$, where $\mathcal{S}_*(\mathbb{R})$ is the space of even infinitely differentiable functions on $\mathbb{R}$, rapidly decreasing together with all their derivatives equipped with its usual topology.*

Proposition 2.3 (Trimèche [12]). *The Hankel transform on $\mathcal{S}'_*(\mathbb{R})$ (the space of even tempered distribution on $\mathbb{R}$), defined by*

$$\langle \mathcal{H}_\alpha(T), \varphi \rangle = \langle T, \mathcal{H}_\alpha(\varphi) \rangle, \quad T \in \mathcal{S}'_*(\mathbb{R}), \quad \varphi \in \mathcal{S}_*(\mathbb{R}), \quad (2.10)$$

is an isomorphism from $\mathcal{S}'_(\mathbb{R})$ onto itself, and $\mathcal{H}_\alpha^{-1} = \mathcal{H}_\alpha$.*

Example 2.4. If we take $T = \delta_0$ where δ_0 is the Dirac measure at zero, then

$$\mathcal{H}_\alpha(\delta_0) = 1. \quad (2.11)$$

Indeed, set $\varphi \in \mathcal{S}_*(\mathbb{R})$

$$\langle \mathcal{H}_\alpha(\delta_0), \varphi \rangle = \langle \delta_0, \mathcal{H}_\alpha(\varphi) \rangle = \mathcal{H}_\alpha(\varphi)(0) = \int_0^\infty j_\alpha(0)\varphi(t)d\mu_\alpha(t). \quad (2.12)$$

Since $j_\alpha(0) = 1$, then we get

$$\langle \mathcal{H}_\alpha(\delta_0), \varphi \rangle = \int_0^\infty \varphi(t) d\mu_\alpha(t) = \langle 1, \varphi \rangle. \quad (2.13)$$

This proves the result.

Haimo [9] and Hirschman [10] investigated a convolution operation and translation operation associated to the Hankel transformation. If $f, g \in L_\alpha^1$, the Hankel convolution $f *_\alpha g$ of f and g is defined by

$$\forall x \geq 0, \quad f *_\alpha g(x) = \int_0^\infty \mathcal{T}_x(f)(y) g(y) d\mu_\alpha(y), \quad (2.14)$$

where the $\mathcal{T}_x$ is the Hankel translation given by

$$\begin{aligned} \mathcal{T}_x(f)(y) &= \int_0^\infty D_\alpha(x, y, z) f(z) d\mu_\alpha(z), \quad \text{for } x > 0, \\ \mathcal{T}_0(f)(y) &= f(y), \quad \forall y \geq 0. \end{aligned} \quad (2.15)$$

From properties of the kernel $D_\alpha(x, y, z)$, we deduce the following properties.

(i)

$$\mathcal{T}_x(f)(y) = \mathcal{T}_y(f)(x). \quad (2.16)$$

(ii) If $f \in L_\alpha^p$, $p \geq 1$; then for all $x \geq 0$ the function $\mathcal{T}_x f$ belongs to L_α^p , $p \geq 1$ and we have

$$\|\mathcal{T}_x f\|_{p,\alpha} \leq \|f\|_{p,\alpha}. \quad (2.17)$$

(iii) Let $f \in \mathcal{S}_*(\mathbb{R})$; then for all $x \geq 0$ the function $\mathcal{T}_x f$ belongs to $\mathcal{S}_*(\mathbb{R})$ and we have

$$\mathcal{H}_\alpha(\mathcal{T}_x f)(y) = j_\alpha(xy) \mathcal{H}_\alpha(f)(y). \quad (2.18)$$

Proposition 2.5. (i) The Hankel convolution $*_\alpha$ is commutative and associative. (ii) Assume that $p, q, r \in [1, \infty]$ satisfies the Young conditions $1/p + 1/q = 1 + 1/r$. Then $(f, g) \rightarrow f *_\alpha g$ extends to a continuous map from $L_\alpha^p \times L_\alpha^q$ to L_α^r and we have

$$\|f *_\alpha g\|_{r,\alpha} \leq \|f\|_{p,\alpha} \|g\|_{q,\alpha}. \quad (2.19)$$

Definition 2.6. Let $f \in \mathcal{S}_*(\mathbb{R})$ and $T \in \mathcal{S}'_*(\mathbb{R})$. The generalized convolution of f and T is defined by the following:

$$T *_\alpha f(s) = \langle T, \mathcal{T}_s f \rangle, \quad (2.20)$$

where $\mathcal{T}_s$ is the Hankel translation, which is given by relation (2.15).

Proposition 2.7. Let σ and $\nu \in \mathcal{S}'_*(\mathbb{R})$; then for all $f \in \mathcal{S}_*(\mathbb{R})$, we have

$$\mathcal{H}_\alpha(f *_\alpha \sigma) = \mathcal{H}_\alpha(f) \cdot \mathcal{H}_\alpha(\sigma), \quad \mathcal{H}_\alpha(f \cdot \mathcal{H}_\alpha(\nu)) = \mathcal{H}_\alpha(f) *_\alpha \nu. \quad (2.21)$$

Proof. For $\sigma \in \mathcal{S}'_*(\mathbb{R})$ and $f \in \mathcal{S}_*(\mathbb{R})$, we have that $\sigma *_\alpha f$ belongs to $\xi_*(\mathbb{R})$ (the space of even infinitely differentiable function on $\mathbb{R}$) and increase slowly.

Thus $\sigma *_\alpha f \in \mathcal{S}'_*(\mathbb{R})$, and we have

$$\begin{aligned} \langle \mathcal{H}_\alpha(\sigma *_\alpha f), \varphi \rangle &= \langle \sigma *_\alpha f, \mathcal{H}_\alpha(\varphi) \rangle = \int_0^\infty \sigma *_\alpha f(x) \mathcal{H}_\alpha(\varphi)(x) d\mu_\alpha(x) \\ &= \int_0^\infty \langle \sigma_y, \mathcal{T}_x f(y) \rangle \mathcal{H}_\alpha(\varphi)(x) d\mu_\alpha(x) \\ &= \left\langle \sigma_y, \int_0^\infty \mathcal{T}_x f(y) \mathcal{H}_\alpha(\varphi)(x) d\mu_\alpha(x) \right\rangle \\ &= \left\langle \sigma_y, \int_0^\infty \mathcal{T}_y f(x) \mathcal{H}_\alpha(\varphi)(x) d\mu_\alpha(x) \right\rangle \\ &= \left\langle \sigma_y, \int_0^\infty \mathcal{H}_\alpha(\mathcal{T}_y f)(x) \varphi(x) d\mu_\alpha(x) \right\rangle \\ &= \left\langle \sigma_y, \int_0^\infty j_\alpha(xy) \mathcal{H}_\alpha f(x) \varphi(x) d\mu_\alpha(x) \right\rangle \\ &= \langle \sigma, \mathcal{H}_\alpha(\varphi \mathcal{H}_\alpha(f)) \rangle = \langle \mathcal{H}_\alpha(\sigma), \mathcal{H}_\alpha(f) \varphi \rangle = \langle \mathcal{H}_\alpha(\sigma) \mathcal{H}_\alpha(f), \varphi \rangle. \end{aligned} \quad (2.22)$$

This proves the relation (2.21) on the left.

On the other hand,

$$\langle \mathcal{H}_\alpha(f \mathcal{H}_\alpha(\sigma)), \varphi \rangle = \langle \mathcal{H}_\alpha(\sigma), f \mathcal{H}_\alpha(\varphi) \rangle = \langle \sigma, \mathcal{H}_\alpha(f \mathcal{H}_\alpha(\varphi)) \rangle = \langle \sigma, \mathcal{H}_\alpha(f) *_\alpha \varphi \rangle. \quad (2.23)$$

We complete the proof of the relation (2.21) on the right by the same way as the relation on the left. $\square$

Remark 2.8. Since the space $\mathcal{S}_*(\mathbb{R})$ is a dense subset of L^1_α and it is easy to verify that $f \rightarrow \mathcal{H}_\alpha(f *_\alpha \sigma)$, $f \rightarrow \mathcal{H}_\alpha(f) \mathcal{H}_\alpha(\sigma)$, $f \rightarrow \mathcal{H}_\alpha(f \mathcal{H}_\alpha(\nu))$, and $f \rightarrow \mathcal{H}_\alpha(f) *_\alpha \nu$ are all continuous maps from L^1_α to L^∞ , thus, the identities in (2.21) extend to be valid for all $f \in L^1_\alpha$.

3. Weighted Inequalities for Hankel Transforms

Let L_α^+ denote the nonnegative, extended real-valued function on the measure space $((0, \infty), d\mu_\alpha)$. We say that a map $T : L_\alpha^+ \rightarrow L_\alpha^+$ has a formal adjoint $T^* : L_\alpha^+ \rightarrow L_\alpha^+$ provided

$$\int_0^\infty T(f)(x)g(x)d\mu_\alpha(x) = \int_0^\infty f(x)T^*(g)(x)d\mu_\alpha(x) \quad (3.1)$$

for all $f, g \in L_\alpha^+$.

Let $1 < q \leq p < \infty$ and a map $T : L_\alpha^+ \rightarrow L_\alpha^+$ have formal adjoint. Fix a weight $u \in L_\alpha^+$. For each $h \in L_\alpha^+$, we define

$$v_h = h^{1-p}T^*(u(Th)^{q-1}), \quad C = \left(\int_0^\infty h^p v_h d\mu_\alpha \right)^{1/q-1/p}. \quad (3.2)$$

Note that by our convention, $C_h = 1$ when $p = q$, even if $\int_0^\infty (h(x))^p v_h(x) d\mu_\alpha(x) = \infty$.

Proposition 3.1. *If $0 < h < \infty$ μ_α , almost everywhere, and $0 < T(h) < \infty$ $u\mu_\alpha$, almost everywhere then*

$$\left(\int_0^\infty (T(f)(x))^q u(x) d\mu_\alpha(x) \right)^{1/q} \leq C \left(\int_0^\infty (f(x))^p v_h(x) d\mu_\alpha(x) \right)^{1/p} \quad (3.3)$$

for all $f \in L_\alpha^+$.

This result is a special case ($n = 1$ and $r = 1$) of Theorem 2.1 in [7].

The next result may be deduced from the last by duality argument. It is also a special case of Theorem 3.1 of [6]. Again note that $C = 1$ when $p = q$.

Proposition 3.2. *Suppose $1 < q \leq p < \infty$, $T : L_\alpha^+ \rightarrow L_\alpha^+$ has a formal adjoint T^* , and $v, h \in L_\alpha^+$ with $0 < h < \infty$. Set*

$$u = h \left(T \left(v^{1-p'} (T^*h)^{p'-1} \right) \right)^{1-q}, \quad C = \left(\int_0^\infty h^{q'} u^{1-q'} \right)^{1/q-1/p}. \quad (3.4)$$

Then

$$\left(\int_0^\infty (T(f)(x))^q u(x) d\mu_\alpha(x) \right)^{1/q} \leq C \left(\int_0^\infty (f(x))^p v(x) d\mu_\alpha(x) \right)^{1/p} \quad (3.5)$$

for all $f \in L_\alpha^+$.

Observe that if $p > q$, then the formulas for the constants C given in these propositions may take useful alternative forms. In the first,

$$\int_0^\infty h^p v d\mu_\alpha = \int_0^\infty h T^* \left(u(Th)^{q-1} \right) d\mu_\alpha = \int_0^\infty (Th) u (Th)^{q-1} = \int_0^\infty (Th)^q u d\mu_\alpha \quad (3.6)$$

so

$$C = \left(\int_0^\infty (Th)^q u d\mu_\alpha \right)^{1/q-1/p}. \quad (3.7)$$

In the second,

$$\int_0^\infty h^{q'} u^{1-q'} = \int_0^\infty hT \left(v^{1-p'} (T^*h)^{p'-1} \right) d\mu_\alpha = \int_0^\infty (T^*h) v^{1-p'} (T^*h)^{p'-1} \quad (3.8)$$

so

$$C = \left(\int_0^\infty (T^*h)^{p'} v^{1-p'} d\mu_\alpha \right)^{1/q-1/p}. \quad (3.9)$$

Let σ be finite complex-valued Borel measure on $(0, \infty)$ and $|\sigma|$ denote its absolute value. Then

$$\int_0^\infty d|\sigma|(x) < \infty. \quad (3.10)$$

The fact that $|j_\alpha(x)| \leq 1$ allows us to get that the Hankel transform of σ defined by

$$\mathcal{H}_\alpha(\sigma)(t) = \int_0^\infty j_\alpha(xt) d\sigma(x) \quad (3.11)$$

is continuous for $[0, \infty[$ and that

$$\|\mathcal{H}_\alpha(\sigma)\|_\infty \leq |||\sigma|||, \quad (3.12)$$

where

$$|||\sigma||| = \int_0^\infty d|\sigma|(x) < \infty. \quad (3.13)$$

Furthermore, the bounded function $\mathcal{H}_\alpha(\sigma)$ has a Hankel transform in the distributional sense and

$$\mathcal{H}_\alpha(\mathcal{H}_\alpha(\sigma)) = \sigma. \quad (3.14)$$

Moreover, if f is real measurable function on $(0, \infty)$ and σ a finite complex-valued Borel measure on $(0, \infty)$, we formally set

$$f *_\alpha \sigma(x) = \int_0^\infty \tau_x f(y) d\sigma(y), \quad (3.15)$$

where $\mathcal{T}_x$ is the Hankel translation given by relation (2.15). If $f \in L_\alpha^1$, then $f *_\alpha \sigma$ is defined in L_α^1 , and if $f \in L^\infty$, the integral (3.15) is defined in L^∞ .

For σ a finite complex-valued Borel measure on $(0, \infty)$, define the positive operator

$$K_\sigma(f) = f *_\alpha |\sigma| \quad (3.16)$$

for all $f \in L_\alpha^+$.

Thus, if $f \in L_\alpha^1 \cup L^\infty$, the convolution operator $f *_\alpha \sigma$ is well defined and we have

$$|f *_\alpha \sigma| \leq K_\sigma(|f|). \quad (3.17)$$

Proposition 3.3. *For σ a finite complex-valued Borel measure on $(0, \infty)$, the positive operator K_σ defined by relation (3.16) is autoadjoint. That is, K_σ has a formal adjoint operator K_σ^* and*

$$K_\sigma^* = K_\sigma. \quad (3.18)$$

Proof. If $f, g \in L_\alpha^+$, then by using properties of the kernel D_α and by applying Fubini Tonelli argument we have

$$\begin{aligned} \int_0^\infty K_\sigma(f)(x)g(x)d\mu_\alpha(x) &= \int_0^\infty \left(\int_0^\infty \mathcal{T}_x f(y)d|\sigma|(y) \right) g(x)d\mu_\alpha(x) \\ &= \int_0^\infty \left(\int_0^\infty g(x) \left(\int_0^\infty D_\alpha(x, y, z)f(z)d\mu_\alpha(z) \right) d\mu_\alpha(x) \right) d|\sigma|(y) \\ &= \int_0^\infty f(z) \left(\int_0^\infty \mathcal{T}_z g(y)d|\sigma|(y) \right) d\mu_\alpha(z) = \int_0^\infty f(z)\mathcal{K}_\sigma g(z)d\mu_\alpha(z). \end{aligned} \quad (3.19)$$

This completes the proof. $\square$

Before introducing any technical details, we give sketch of the argument behind the main following theorem. We suppose that the following Hankel inequality:

$$\left(\int_0^\infty (\mathcal{H}_\alpha(f))^q u d\mu_\alpha \right)^{1/q} \leq \left(\int_0^\infty |f|^p v d\mu_\alpha \right)^{1/p} \quad (3.20)$$

for all $f \in L_\alpha^1 \cap L_\alpha^p(v)$ is known to be valid for some fixed p_0, q_0, u_0 , and v_0 . For each appropriate function g , define $f = (g *_\alpha \sigma) / \mathcal{H}_\alpha(v)$, where σ and v are finite, complexe-valued Borel measures on $[0, \infty[$.

Using Proposition 2.7, we get $\mathcal{H}_\alpha(g) = (\mathcal{H}_\alpha(f) *_\alpha v) / \mathcal{H}_\alpha(\sigma)$. For $p_1 \geq p_0, q_1 \leq q_0$ and arbitrary functions h_σ and h_v we apply Propositions 3.1 and 3.2 to give formulas for u_1 and v_1 so that

$$\begin{aligned} L_\alpha^{p_1}(v_1) &\longrightarrow L_\alpha^{p_0}(v_0) \longrightarrow L_\alpha^{q_0}(u_0) \longrightarrow L_\alpha^{q_1}(u_1), \\ g &\longmapsto f \longmapsto \mathcal{H}_\alpha(f) \longmapsto \mathcal{H}_\alpha(g). \end{aligned} \quad (3.21)$$

The arrow in the middle corresponds to the known “input” Hankel inequality, and the other arrows correspond to the Hankel convolution inequalities for the operators

$$g \longrightarrow \frac{(g *_\alpha \sigma)}{\mathcal{H}_\alpha(\nu)}, \quad \mathcal{H}_\alpha(f) \longrightarrow \frac{(\mathcal{H}_\alpha(f) *_\alpha \nu)}{\mathcal{H}_\alpha(\sigma)}. \quad (3.22)$$

The inequality relating $\mathcal{H}_\alpha(g)$ and g that results from this composition is just the above inequality with new indices p_1 and q_1 and new weights u_1 and v_1 . This is our “output” inequality.

Theorem 3.4. Suppose C_0 is a positive constant, p_0 and q_0 are indices in $(1, \infty)$, and u_0 and v_0 are nonnegative weight functions such that the Hankel inequality

$$\left(\int_0^\infty (\mathcal{H}_\alpha(f)(x))^{q_0} u_0(x) d\mu_\alpha(x) \right)^{1/q_0} \leq C_0 \left(\int_0^\infty |f(x)|^{p_0} v_0(x) d\mu_\alpha(x) \right)^{1/p_0} \quad (3.23)$$

for all $f \in L_\alpha^1$. Let σ and ν be finite complex-valued Borel measure and h_σ and h_ν positive functions on $[0, \infty[$. For p_1 and q_1 satisfying $1 < p_0 \leq p_1 < \infty$ and $1 < q_1 \leq q_0 < \infty$, set

$$\begin{aligned} w_\sigma &= |\mathcal{H}(\nu)|^{-p_0} v_0, & v_1 &= h_\sigma^{1-p_1} K_\sigma(w_\sigma (K_\sigma h_\sigma)^{p_0-1}), \\ w_\nu &= h_\nu \left(K_\nu \left(u_0^{1-q'} (K_\nu h_\nu)^{q'_0-1} \right) \right)^{1-q_1}, & u_1 &= |\mathcal{H}(\sigma)|^{q_1} w_\nu. \end{aligned} \quad (3.24)$$

Also set

$$C_\sigma = \left(\int_0^\infty h_\sigma^{p_1} v_1 d\mu_\alpha \right)^{1/p_0-1/p_1}, \quad C_\nu = \left(\int_0^\infty h_\nu^{q'_1} w_\nu^{1-q'_1} d\mu_\alpha \right)^{1/q_1-1/q_0}. \quad (3.25)$$

If $\mathcal{H}_\alpha(\nu)$ is bounded away from zero, then the Hankel inequality

$$\left(\int_0^\infty (\mathcal{H}_\alpha(g)(x))^{q_1} u_1(x) d\mu_\alpha(x) \right)^{1/q_1} \leq C_1 \left(\int_0^\infty |g(x)|^{p_1} v_1(x) d\mu_\alpha(x) \right)^{1/p_1} \quad (3.26)$$

holds for all $g \in L_\alpha^1$. Here $C_1 = C_\nu C_0 C_\sigma$.

Proof. Let $g \in L_\alpha^1$ and set $f = (g *_\alpha \sigma) / \mathcal{H}_\alpha(\nu)$. Since $\mathcal{H}_\alpha(\nu)$ is bounded away from zero, $1/\mathcal{H}_\alpha(\nu) \in L^\infty$ so $f \in L_\alpha^1$. Taking the Hankel transform of both sides of the equation $g *_\alpha \sigma = f \mathcal{H}_\alpha(\nu)$ and using identities (2.21) yields

$$\mathcal{H}_\alpha(g) \mathcal{H}_\alpha(\sigma) = \mathcal{H}_\alpha(f) *_\alpha \nu. \quad (3.27)$$

Proposition 3.2 shows that

$$\left(\int_0^\infty \mathcal{K}_\nu(\mathcal{H}_\alpha(f))^{q_1} w_\nu d\mu_\alpha \right)^{1/q_1} \leq C_\nu \left(\int_0^\infty |\mathcal{H}_\alpha(f)|^{q_0} u_0 d\mu_\alpha \right)^{1/q_0} \quad (3.28)$$

and the estimate

$$|\mathcal{H}_\alpha(g)\mathcal{H}_\alpha(\sigma)| = |\mathcal{H}_\alpha(f) *_\alpha \nu| \leq \mathcal{K}_\nu |\mathcal{H}_\alpha(f)| \quad (3.29)$$

gives

$$\left(\int_0^\infty |\mathcal{H}_\alpha(g)|^{q_1} u_1 d\mu_\alpha \right)^{1/q_1} \leq C_\nu \left(\int_0^\infty |\mathcal{H}_\alpha(f)|^{q_0} u_0 d\mu_\alpha \right)^{1/q_0}. \quad (3.30)$$

Proposition 3.1 shows that

$$\left(\int_0^\infty (\mathcal{K}_\sigma |g|)^{p_0} w_\sigma d\mu_\alpha \right)^{1/p_0} \leq C_\sigma \left(\int_0^\infty |g|^{p_1} v_1 d\mu_\alpha \right)^{1/p_1} \quad (3.31)$$

and the trivial estimates

$$|f\mathcal{H}_\alpha(\nu)| = |g *_\alpha \sigma| \leq \mathcal{K}_\sigma |g| \quad (3.32)$$

give

$$\left(\int_0^\infty |f|^{p_0} v_0 d\mu_\alpha \right)^{1/p_0} \leq C_\sigma \left(\int_0^\infty |g|^{p_1} v_1 d\mu_\alpha \right)^{1/p_1}. \quad (3.33)$$

The three inequalities (3.30), (3.23), and (3.33) combine to yield (3.26) as required. This completes the proof. $\square$

When all indices are taken to 2 the theorem simplifies substantially.

Corollary 3.5. *Suppose C is a positive constant and u_0 and v_0 are nonnegative weight functions such that the Hankel inequality*

$$\int_0^\infty (\mathcal{H}_\alpha(f)(x))^2 u_0(x) d\mu_\alpha(x) \leq C \int_0^\infty |f(x)|^2 v_0(x) d\mu_\alpha(x) \quad (3.34)$$

holds for all $f \in L_\alpha^1$. Let σ and ν be finite complex-valued Borel measures and h_σ and h_ν positive function on $[0, \infty[$. Set

$$v_1 = \frac{1}{h_\sigma} K_\sigma \left(\frac{v_0(K_\sigma h_\sigma)}{|\mathcal{H}_\alpha(\nu)|^2} \right), \quad u_1 = \frac{|\mathcal{H}_\alpha(\sigma)|^2 h_\nu}{K_\nu(K_\nu h_\nu / u_0)}. \quad (3.35)$$

If $\mathcal{H}_\alpha(\nu)$ is bounded away from zero, then the Hankel inequality

$$\int_0^\infty (\mathcal{H}_\alpha(g)(x))^2 u_1(x) d\mu_\alpha(x) \leq C \int_0^\infty |g(x)|^2 v_1(x) d\mu_\alpha(x) \quad (3.36)$$

holds for all $g \in L_\alpha^1$.

A further simplification yields a following new weighted Hankel inequality.

Corollary 3.6. *If σ is finite positive Borel measure and h is a positive function on $[0, \infty[$ then for $\alpha > -1/2$*

$$\int_0^\infty |\mathcal{H}_\alpha(g)(x)|^2 |\mathcal{H}_\alpha(\sigma)(x)|^2 d\mu_\alpha(x) \leq C \int_0^\infty |g(x)|^2 \frac{(\sigma *_\alpha (\sigma *_\alpha h))(x)}{h(x)} d\mu_\alpha(x) \quad (3.37)$$

holds for all $g \in L_\alpha^1$.

Proof. It is well known that the Hankel transform is continuous from L_α^p into $L_\alpha^{p'}$, where $1/p + 1/p' = 1$, $1 \leq p \leq 2$. This shows that the inequality (3.33) holds with $p = 2$ and $u_0 = 1$ and $v_0 = 1$. Thus if we take $h_\nu = 1$, $h_\sigma = h$, and take ν to be the Dirac measure at zero. Then $\mathcal{K}_\nu = f *_\alpha \delta_0 = f$, which proves that $\mathcal{K}_\nu$ reduce to the identity. Furthermore, since $\mathcal{H}_\alpha(\nu) = 1$, then we get

$$u_1 = \frac{|\mathcal{H}_\alpha(\sigma)|^2 h_\nu}{K_\nu((K_\nu h_\nu)/u_0)} = \frac{|\mathcal{H}_\alpha(\sigma)|^2 h_\nu}{h_\nu/u_0} = |\mathcal{H}_\alpha(\sigma)|^2. \quad (3.38)$$

From the fact that K_σ is just convolution by σ and $\mathcal{H}_\alpha(\nu) = 1$, it follows

$$v_1 = \frac{1}{h_\sigma} K_\sigma \left(\frac{v_0(K_\sigma h_\sigma)}{|\mathcal{H}_\alpha(\nu)|^2} \right) = \frac{1}{h_\sigma} \mathcal{K}_\sigma(v_0(\mathcal{K}_\sigma h_\sigma)). \quad (3.39)$$

Then if we take $h_\sigma = h$, we get the result. $\square$

If we take σ and h as Gaussian functions, that is, $\sigma_t = C_1 e^{-tx^2}$ and $h_s = C_2 e^{-sx^2}$, then we obtain the following new particular case of weighted Hardy-type inequality of Hankel transform.

Corollary 3.7. *For $\alpha > -1/2$ and $s, t > 0$, we have*

$$\int_0^\infty |\mathcal{H}_\alpha(g)(x)|^2 e^{-(1/2t)x^2} d\mu_\alpha(x) \leq C_{\alpha,t,s} \int_0^\infty |g(x)|^2 e^{(2s^2/(2s+t))x^2} d\mu_\alpha(x). \quad (3.40)$$

Proof. The result is obtained by using Propositions 2.1 and 2.7 and the fact that

$$\mathcal{H}_\alpha(\sigma_t)(x) = C_1 \frac{e^{-x^2/4t}}{(2t)^{\alpha+1}}, \quad \mathcal{H}_\alpha(h_s)(x) = C_2 \frac{e^{-x^2/4s}}{(2s)^{\alpha+1}}. \quad (3.41)$$

$\square$

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