Research Article

A Remark on Wavelet Bases in Weighted L_p Spaces

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The paper deals with unconditional wavelet bases in weighted L_p spaces. Inhomogeneous wavelets of Daubechies type are considered. Necessary and sufficient conditions for weights are found for which the wavelet system is an unconditional basis in weighted L_p spaces in dependence on p .

1. Introduction

Wavelet systems in weighted L_p spaces were investigated by several authors. Lemarié-Rieusset considered one-dimensional case, compare [1]. He proved that the homogeneous wavelet system of Daubechies type is an unconditional basis in $L_p(\mathbb{R}^n, d\mu)$, $1 < p < \infty$, if and only if $d\mu = w dx$, where w is a weight belonging to the Muckenhoupt class $\mathcal{A}_p$. He also found a sufficient and necessary condition for inhomogeneous systems to be unconditional bases. Other one-dimensional systems were investigated by Kazarian [2] and García-Cuerva and Kazarian [3]. Multidimensional homogeneous wavelet systems were considered by Aimar et al., compare [4]. They proved that a homogeneous wavelet system satisfying certain regularity conditions is an unconditional basis in $L_p(d\mu)$ if and only if $d\mu = w dx$ with $w \in \mathcal{A}_p$.

The aim of the paper is to prove the counterpart of Lemarié-Rieusset result for multidimensional inhomogeneous wavelet systems. To formulate the necessary and sufficient condition we use the class of local Muckenhoupt weights $\mathcal{A}_\infty^{\text{loc}}$ introduced by Rychkov in 2001, compare [5]. In his work Rychkov developed the theory of weighted Besov and Triebel-Lizorkin spaces with local Muckenhoupt weight. Wavelets characterizations of the function spaces with so-called admissible weights was given by Haroske and Triebel in [6]. Later Haroske and Skrzypczak proved the characterization for the spaces with Muckenhoupt weights, compare [7]. Recently Izuki and Sawano have proved the result for the function spaces with weights from class $\mathcal{A}_\infty^{\text{loc}}$, compare [8], and another approach can be found in [9].

The main Theorem of the paper asserts that the inhomogeneous wavelet system of Daubechies type is an unconditional basis in $L_p(d\mu)$ if and only if $d\mu = w dx$ with $w \in \mathcal{A}_p^{\text{loc}}$.

2. Classes of Weights

Let w be a nonnegative and locally integrable function on $\mathbb{R}^n$. These functions are called weights, and, for measurable set E , $w(E)$ denotes $\int_E w(x) dx$. Let $L_p^w(\mathbb{R}^n)$ denote a space of p -integrable functions on $\mathbb{R}^n$ with respect to the measure $w dx$.

2.1. Muckenhoupt Weights

Let us recall the definition of Muckenhoupt weights, compare [10].

Weight $w \in \mathcal{A}_p$, $1 < p < \infty$ if

$$A_p(w) := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|^p} \int_Q w(x) dx \left(\int_Q w^{1-p'}(x) dx \right)^{p-1} < \infty, \quad (2.1)$$

and $w \in \mathcal{A}_1$ if

$$A_1(w) := \sup_{Q \subset \mathbb{R}^n} \frac{w(Q)}{|Q|} \|w^{-1}\|_{L_\infty(Q)} < \infty, \quad (2.2)$$

where supremum is taken over all cubes $Q \in \mathbb{R}^n$.

As an example we can take

$$w(x) = |x|^\alpha \in \mathcal{A}_p \quad \text{for} \begin{cases} 1 < p < \infty & \text{if } -n < \alpha < n(p-1), \\ p = 1 & \text{if } -n < \alpha \leq 0 \end{cases} \quad (2.3)$$

or weights with logarithmic part

$$v(x) = |x|^\alpha \log^{-\beta}(2 + |x|). \quad (2.4)$$

Then

$$\begin{aligned} v \in \mathcal{A}_1 & \quad \text{if} \begin{cases} \beta \in \mathbb{R} & \text{and } -n < \alpha < 0, \\ \beta \geq 0 & \text{and } \alpha = 0, \end{cases} \\ v \in \mathcal{A}_p, 1 < p < \infty & \quad \text{if } -n < \alpha < n(p-1), \beta \in \mathbb{R}. \end{aligned} \quad (2.5)$$

2.2. Local Muckenhoupt Weights

Definition 2.1 (see [5]). We define the class of weight $\mathcal{A}_p^{\text{loc}}$ ($1 < p < \infty$) to consist of all non-negative locally integrable functions w defined on $\mathbb{R}^n$ for which

$$A_p^{\text{loc}}(w) := \sup_{|Q| \leq 1} \frac{1}{|Q|^p} \int_Q w(x) dx \left(\int_Q w^{1-p'}(x) dx \right)^{p-1} < \infty \quad (2.6)$$

and $\mathcal{A}_1^{\text{loc}}$

$$A_1^{\text{loc}}(w) := \sup_{|Q| \leq 1} \frac{w(Q)}{|Q|} \|w^{-1}\|_{L_\infty(Q)} < \infty. \quad (2.7)$$

It follows directly from definitions that $\mathcal{A}_p \subset \mathcal{A}_p^{\text{loc}}$ and $A_p^{\text{loc}}(w) \leq A_p(w)$ for any $w \in A_p$, $1 \leq p < \infty$.

Definition 2.2. We say that $w \in \mathcal{A}_\infty^{\text{loc}}$ if for any $\alpha \in (0, 1)$

$$\sup_{|Q| \leq 1} \left(\sup_{F \subset Q, |F| \geq \alpha|Q|} \frac{w(Q)}{w(F)} \right) < \infty, \quad (2.8)$$

where F is taken over all measurable sets in $\mathbb{R}^n$.

Remark 2.3. Any Muckenhoupt weight of the class $\mathcal{A}_p$ belongs to the class $\mathcal{A}_p^{\text{loc}}$. But local Muckenhoupt weights cover also so-called admissible weights and locally regular weights, compare [5, 6, 11].

2.3. Properties of Classes A_p^{loc}

We would like to mention some important properties of classes $\mathcal{A}_p^{\text{loc}}$.

Lemma 2.4 (see [5]). *Let $1 < p_1 < p_2 < \infty$. Then $\mathcal{A}_{p_1}^{\text{loc}} \subset \mathcal{A}_{p_2}^{\text{loc}} \subset \mathcal{A}_\infty^{\text{loc}}$.*

Conversely, if $w \in \mathcal{A}_\infty^{\text{loc}}$, then $w \in \mathcal{A}_p^{\text{loc}}$ for some $p < \infty$.

The last lemma implies that $\mathcal{A}_\infty^{\text{loc}} = \bigcup_{p \geq 1} \mathcal{A}_p^{\text{loc}}$. In consequence we can define for $w \in \mathcal{A}_\infty^{\text{loc}}$ a positive number

$$r_w = \inf \left\{ 1 \leq p < \infty : w \in \mathcal{A}_p^{\text{loc}} \right\}. \quad (2.9)$$

Next lemma shows us an important relation between $\mathcal{A}_p$ and $\mathcal{A}_p^{\text{loc}}$ weights.

Lemma 2.5 (see [5]). *Let $1 \leq p < \infty$, $w \in \mathcal{A}_p^{\text{loc}}$ and Q be a unit cube, that is, $|Q| = 1$. Then there exists a $\bar{w} \in \mathcal{A}_p$, such that $\bar{w} = w$ on Q and*

$$A_p(\bar{w}) \leq c A_p^{\text{loc}}(w), \quad (2.10)$$

where constant c is independent of Q .

Definition 2.6. Let f be locally integrable. Operator

$$M^{\text{loc}} f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy, \quad (2.11)$$

where supremum is taken over all cubes in $\mathbb{R}^n$ for which $|Q| \leq 1$, is called *local maximal function*.

The Fefferman-Stein maximal inequality holds for the operator M^{loc} and local Muckenhoupt weights.

Lemma 2.7 (see [5]). *Let $1 < p < \infty$, $1 < q \leq \infty$, and $w \in \mathcal{A}_p^{\text{loc}}$. Then for any sequence of measurable functions $\{f^j\}$, we have*

$$\left\| \left\{ M^{\text{loc}} f^j \right\} \mid L_p^w(I_q) \right\| \leq c \left\| \left\{ f^j \right\} \mid L_p^w(I_q) \right\|. \quad (2.12)$$

Lemma 2.8 (see [5]). *Let $w \in \mathcal{A}_p^{\text{loc}}$ and $1 < p < \infty$. Then*

$$w(tQ) \leq \exp(c_w t) w(Q) \quad t \geq 1, \quad |Q| = 1. \quad (2.13)$$

It follows from the above lemma that classes $\mathcal{A}_p^{\text{loc}}$ are independent of the upper bound for the cube size used in their definition; that is, for any $C > 0$, we could have replaced $|Q| \leq 1$ by $|Q| \leq C$ in Definition 2.1.

3. Weighted Function Spaces

Following Rychkov we define Triebel-Lizorkin spaces with local Muckenhoupt weights, compare [5]. Because the class of tempered distributions $\mathcal{S}'$ is too narrow for this purpose we introduce a class $\mathcal{S}'_e$, which is a topological dual to the following space:

$$\mathcal{S}_e := \{ \psi \in C^\infty(\mathbb{R}^n) : q_N(\psi) < \infty \quad \forall N \in \mathbb{N} \}, \quad (3.1)$$

where the seminorms q_N are given by

$$q_N(\psi) := \sup_{\alpha \in \mathbb{N}_0, |\alpha| \leq N} \left(\sup_{x \in \mathbb{R}^n} e^{N|x|} |D^\alpha \psi(x)| \right). \quad (3.2)$$

We can identify the class $\mathcal{S}'_e$ with the set of these distributions $f \in \mathfrak{D}'$ for which the estimate

$$|\langle f, \psi \rangle| \leq A \sup \{ |D^\alpha \psi(x)| \exp(N|x|) : x \in \mathbb{R}^n, |\alpha| \leq N \} \quad \forall \psi \in C_0^\infty(\mathbb{R}^n), \quad (3.3)$$

is valid with some constants A, N depend on f . Such a distribution f can be extended to a continuous functional on $\mathcal{S}_e$.

We take a function $\varphi_0 \in \mathfrak{D}$ such that $\int_{\mathbb{R}^n} \varphi_0(x) dx \neq 0$ and $\int_{\mathbb{R}^n} x^\beta \varphi_0(x) dx = 0$ for some $\beta \in \mathbb{N}$, $0 < |\beta| \leq B$. We put $\varphi(x) = \varphi_0(x) - 2^{-n} \varphi_0(x/2)$ and $\varphi_j(x) = 2^{(j-1)n} \varphi(2^{j-1}x)$ for $j = 1, 2, \dots$. Then $\int_{\mathbb{R}^n} \varphi_j(x) x^\beta dx = 0$ if $|\beta| \leq B$. We will write $B = -1$ if no vanishing moment holds.

Definition 3.1 (see [5]). Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, and $w \in \mathcal{A}_\infty^{\text{loc}}$. Let function $\varphi_0 \in \mathfrak{D}(\mathbb{R}^n)$ satisfies

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi_0(x) dx &\neq 0, \\ \int_{\mathbb{R}^n} x^\beta \varphi_0(x) dx &= 0, \quad 0 < |\beta| < B, \end{aligned} \quad (3.4)$$

where $B \geq [s]$. We define the weighted Triebel-Lizorkin space $F_{pq}^{s,w}(\mathbb{R}^n)$ to be the set of all $f \in S'_e$ for which the following quasinorm:

$$\|f\|_{F_{pq}^{s,w}(\mathbb{R}^n)} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\varphi_j * f|^q \right)^{1/q} \right\|_{L_p^w(\mathbb{R}^n)} \quad (3.5)$$

is finite.

Remark 3.2. The definition of the above spaces is independent of choice of the function φ_0 , up to the equivalence of quasinorms. The spaces are quasi-Banach, Banach spaces if $p \geq 1$ and $q \geq 1$.

4. Characterization by Wavelets

We are going to deal with Daubechies wavelets on $\mathbb{R}^n$. Let $\psi^F \in C^k(\mathbb{R})$ be a Daubechies scaling function and $\psi^M \in C^k(\mathbb{R})$ a Daubechies wavelet with $\int_{\mathbb{R}} \psi(x) x^v dx = 0$, $k \in \mathbb{N}$, $v \in \mathbb{N}_0$, and $v < k$. We extend these wavelets from $\mathbb{R}$ to $\mathbb{R}^n$ by the usual tensor product procedure

$$\Psi_{jm}^G = 2^{jn/2} \prod_{r=1}^n \psi^{G_r}(2^j x_r - m_r), \quad (4.1)$$

where $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $G = (G_1, \dots, G_n) \in G^j$, and $G^0 = \{F, M\}^n$ and for $j > 0$ $G^j = \{F, M\}^{n*}$, where $*$ indicates that at least one G_r must be an M . The family

$$\left\{ \Psi_{jm}^G : j \in \mathbb{N}_0, m \in \mathbb{Z}^n, G \in G^j \right\} \quad (4.2)$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$, compare [12].

Definition 4.1. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, and $w \in \mathcal{A}_\infty^{\text{loc}}$. Then $f_{pq}^{s,w}$ is the collection of all sequences

$$\lambda = \{\lambda_{jm} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\} \quad (4.3)$$

such that

$$\|\lambda \mid f_{pq}^{s,w}\| = \left\| \left(\sum_{j,m,G} 2^{jsq} |\lambda_{jm}^G \chi_{jm}^{(p)}|^q \right)^{1/q} \mid L_p^w \right\| < \infty. \quad (4.4)$$

For $w \in \mathcal{A}_\infty^{\text{loc}}$ let us define

$$\begin{aligned} \sigma_p(w) &= n \left(\frac{r_w}{\min(p, r_w)} - 1 \right) + (r_w - 1)n, \\ \sigma_q &= \frac{n}{\min(1, q)} - n, \\ \sigma_{pq}(w) &= \max(\sigma_p(w), \sigma_q). \end{aligned} \quad (4.5)$$

The following theorem was obtained in [9].

Theorem 4.2. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, and $w \in \mathcal{A}_\infty^{\text{loc}}$. For wavelets defined in (4.1), we take*

$$k \geq \max \left(0, [s] + 1, \left\lceil \frac{nr_w}{p} - \frac{n}{p} - s \right\rceil + 1, [\sigma_{pq}(w) - s] \right). \quad (4.6)$$

Let $f \in \mathcal{S}'_e(\mathbb{R}^n)$. Then $f \in F_{pq}^{s,w}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{j,G,m} \lambda_{jm}^G 2^{-jn/2} \Psi_{jm}^G, \quad (4.7)$$

where $\lambda \in f_{pq}^{s,w}$ and the series converges in $\mathcal{S}'_e(\mathbb{R}^n)$. This representation is unique with

$$\begin{aligned} \lambda_{jm}^G &= 2^{jn/2} \langle f, \Psi_{jm}^G \rangle, \\ I : f &\mapsto \left\{ 2^{jn/2} \langle f, \Psi_{jm}^G \rangle \right\} \end{aligned} \quad (4.8)$$

is a linear isomorphism of $F_{pq}^{s,w}(\mathbb{R}^n)$ onto $f_{pq}^{s,w}$.

If $0 < p, q < \infty$, then the system $\{\Psi_{jm}^G\}_{j,m,G}$ is an unconditional basis in $F_{pq}^{s,w}(\mathbb{R}^n)$.

5. L_p Spaces with Local Muckenhoupt Weights

In this section we prove our main result. We use the fact that wavelet projection operators satisfy condition (5.2) below. Furthermore we rely on the wavelet characterizations according to Theorem 4.2. We follow the main idea of Aimar et al. [4]. On the other side, we have the wavelet characterization theorem stated in [9].

Lemma 5.1. *Let φ be a continuous function absolutely bounded by an L_1 radial decreasing function such that $\sum_{k \in \mathbb{Z}^n} \varphi(x - k) \neq 0$ for all $x \in \mathbb{R}^n$. Then $F(x, y) = \sum_{k \in \mathbb{Z}^n} \varphi(x - k) \overline{\varphi(y - k)}$ satisfies*

$$\left\{ (x, y) \in \mathbb{R}^{2n} : |x - y| < \ell \right\} \subset \left\{ (x, y) \in \mathbb{R}^{2n} : F(x, y) > \delta \right\}, \quad (5.1)$$

for some positive real numbers ℓ and δ .

A proof of the above lemma can be found in [4]. Following [4] we can find that lemma applies to $P_0(x, y) = \sum_{k \in \mathbb{Z}^n} \Psi^{G_F}(x - k) \Psi^{G_F}(y - k)$ with $G_F = (F, \dots, F)$, where Ψ^{G_F} is a Daubechies scaling function. Now for a family $\{P_j(x, y)\}_{j \geq 0} = \{2^{jn} P_0(2^j x, 2^j y)\}_{j \geq 0}$, we obtain that it satisfies conditions

$$\left\{ (x, y) \in \mathbb{R}^{2n} : |x - y| < \ell_j \right\} \subset \left\{ (x, y) \in \mathbb{R}^{2n} : P_j(x, y) > C \ell_{j+1}^{-n} \right\} \quad (5.2)$$

for every $j \geq 0$ and a positive constant $C > 0$, where $\{\ell_j\}_{j \geq 0}$ is a decreasing sequence of positive real numbers and $\ell_j \rightarrow 0$ as $j \rightarrow \infty$.

Theorem 5.2. *Let $1 < p < \infty$ and μ be a positive Borel measure on $\mathbb{R}^n$ finite on compact sets. Let $k \geq \max([n - n/p] + 1, n(p - 1))$. There exists an unconditional Daubechies wavelet basis in $L_p(\mathbb{R}^n, d\mu)$ with smoothness k if and only if $d\mu = w(x)dx$ with $w \in \mathcal{A}_p^{\text{loc}}$.*

Proof. Let $w \in \mathcal{A}_p^{\text{loc}}$. From Theorem 4.2, we have an unconditional basis in $F_{pq}^{s,w}(\mathbb{R}^n)$. In [5] Rychkov shows a Littlewood-Paley characterization of spaces with local Muckenhoupt weights, and it means that $F_{p,2}^{0,w}(\mathbb{R}^n) = L_p^w(\mathbb{R}^n)$. Hence we have an unconditional basis in $L_p^w(\mathbb{R}^n)$.

On the other side, Let $\{\Psi_{jm}^G : j \in \mathbb{N}_0, m \in \mathbb{Z}^n, G \in G^j\}$ be a Daubechies wavelet system, which is an unconditional basis in $L_p(\mathbb{R}^n, d\mu)$. So every $f \in L_p(\mathbb{R}^n, d\mu)$ has the representation

$$f(x) = \sum_{j,k,G} \langle f, \Psi_{jk}^G \rangle \Psi_{jk}^G(x). \quad (5.3)$$

Operators

$$\begin{aligned} \tilde{P}_0 f &= \sum_k \langle f, \Psi_{0,k}^{G_F} \rangle \Psi_{0,k}^{G_F}, \\ \tilde{P}_m f &= \sum_{0 \leq j < m, k, G} \langle f, \Psi_{jk}^G \rangle \Psi_{jk}^G, \quad m > 0 \end{aligned} \quad (5.4)$$

are uniformly bounded on $L_p(\mathbb{R}^n, d\mu)$. We can write $\tilde{P}_m f(x) = \int_{\mathbb{R}^n} \tilde{P}_m(x, y) f(y) dy$, where $\tilde{P}_0(x, y) = \sum_k \Psi_{0,k}^{G_F}(x) \Psi_{0,k}^{G_F}(y)$ and $\tilde{P}_m(x, y) = \sum_{0 \leq j < m, k, G} \Psi_{jk}^G(x) \Psi_{jk}^G(y)$ if $m > 0$, because wavelets have compact supports, and we deal with locally finite sums. Hence kernels $\tilde{P}_m(x, y)$ are bounded.

On the other hand, by the properties of the multiresolution analysis, the kernel $\tilde{P}_m(x, y)$ coincides with $P_m(x, y) = 2^{mn} P_0(2^m x, 2^m y)$ if it is regarded as a kernel of projection in L_2 . Again, because wavelets and the scaling function are compactly supported, we deal

with locally finite sums and get $P_m(x, y) = \tilde{P}_m(x, y)$ almost everywhere. So the last equality is valid also for L_p spaces, and the kernels $P_m(x, y)$ are bounded.

We are going to show that μ is absolutely continuous. Let E be a set such that $|E| = 0$. For every $\varepsilon > 0$, there exists an open set F such that $E \subset F$ and $\mu(F \setminus E) < \varepsilon$. Set F can be decomposed into a countable union of disjoint and dyadic cubes Q_i . From Lemma 5.1, we get that $\{P_j(x, y)\}_{j \geq 0}$ satisfies the same conditions as weakly positive family. Let $\{\ell_j\}_{j \geq 0}$ be a sequence connected with that family. Without, being generality, we assume that $\ell_0 > \max(d(Q_i))$, where $d(Q)$ denote a diameter of Q and maximum is taken over all cubes Q_i from decomposition of F . For fixed i let $j_0 \geq 0$ be the integer such that $\ell_{j_0+1} \leq d(Q_i) < \ell_{j_0}$. If $x, y \in Q_i$, we get $|x - y| < \ell_{j_0}$ and $P_{j_0}(x, y) > C\ell_{j_0+1}^{-n}$. So for every $x \in Q_i$, we have

$$|P_{j_0}(\chi_{Q_i \setminus E})(x)| = \left| \int_{Q_i \setminus E} P_{j_0}(x, y) dy \right| > C\ell_{j_0+1}^{-n} |Q_i \setminus E|. \quad (5.5)$$

Therefore, $|P_{j_0}(\chi_{Q_i \setminus E})(x)| > c_n$, for some constant depending only on C and n . From weak type inequality for operators P_j , we get

$$\mu(Q_i) \leq \mu(\{x : |P_{j_0}(\chi_{Q_i \setminus E})(x)| > c_n\}) \leq Cc_n^{-p} \mu(Q_i \setminus E). \quad (5.6)$$

Summing over i , we have

$$\mu(F) = \sum_i \mu(Q_i) \leq Cc_n^{-p} \sum_i \mu(Q_i \setminus E) = Cc_n^{-p} \mu(F \setminus E) < Cc_n^{-p} \varepsilon \quad (5.7)$$

for every $\varepsilon > 0$. Hence $\mu(E) = 0$. From Radon-Nikodym Theorem, we get that there exists locally integrable function w such that $d\mu = w(x)dx$.

Now we can show that $w \in \mathcal{A}_p^{\text{loc}}$. We pick a sequence $\{\ell_j\}_{j \geq 0}$. Let $Q \subset \mathbb{R}^n$ be a cube with $|Q| \leq \ell_0$. We can find $m_0 \geq 0$ with $\ell_{m_0+1} \leq d(Q) < \ell_{m_0}$. Inequalities

$$|P_{m_0}(\sigma_\varepsilon \chi_Q)(x)| = \left| \int_Q P_{m_0}(x, y) \sigma_\varepsilon(y) dy \right| > C\ell_{m_0+1}^{-n} \int_Q \sigma_\varepsilon \geq c_n |Q|^{-1} \int_Q \sigma_\varepsilon \equiv \lambda, \quad (5.8)$$

where $\sigma_\varepsilon = (w + \varepsilon)^{-1/(p-1)}$, $\varepsilon > 0$, holds for every $x \in Q$. Since operators P_m are of weak type (p, p) , we get

$$w(Q) \leq w(\{x : |P_{m_0}(\sigma_\varepsilon \chi_Q)(x)| > \lambda\}) \leq Cc_n^{-p} |Q|^p \left(\int_Q \sigma_\varepsilon \right)^{-p} \int_Q \sigma_\varepsilon^p w. \quad (5.9)$$

Multiplying both sides by $(\int_Q \sigma_\varepsilon)^p (\int_Q \sigma_\varepsilon^p w)^{-1}$ and choosing ε close to zero, we get

$$w(Q) \left(\int_Q w^{-1/(p-1)} \right)^p \left(\int_Q w^{-1/(p-1)} \right)^{-1} \leq C|Q|^p \quad (5.10)$$

for every Q , $|Q| < \ell_0$. From Lemma 1.4 in [5], we know that classes $\mathcal{A}_p^{\text{loc}}$ are independent of the upper bound for the cube size in their definition. So we get a condition for A_p^{loc} . $\square$

Following [5], we can state square-function characterization. Let us define

$$S(f)(x) = \left(\sum_j |\varphi_j * f(x)|^2 \right)^{1/2}, \quad (5.11)$$

where $\varphi_0 \in \mathfrak{D}$ have nonzero integral and $\varphi = \varphi_0 - 2^{-n}\varphi_0(\cdot/2)$ and $\varphi_j(x) = 2^{jn}\varphi(2^jx)$, $j > 0$.

Corollary 5.3. *Let $1 < p < \infty$ and $w \in \mathcal{A}_\infty^{\text{loc}}$. The following equivalence holds:*

$$\|S(f) \| L_p^w\| \sim \|f \| L_p^w\| \quad (5.12)$$

if and only if $w \in \mathcal{A}_p^{\text{loc}}$.

Proof. Let $w \in \mathcal{A}_p^{\text{loc}}$. From [5] we have that $F_{p,2}^{0,w}(\mathbb{R}^n) = L_p^w(\mathbb{R}^n)$ with norm equivalence

$$\int_{\mathbb{R}^n} \left(\sum_{j \in \mathbb{Z}} |\varphi_j * f(x)|^2 \right)^{p/2} w(x) dx = \int_{\mathbb{R}^n} |f(x)|^p w(x) dx. \quad (5.13)$$

Conversely, if we assume that $w \in \mathcal{A}_\infty^{\text{loc}}$, then from Theorem 4.2 we get that in $F_{p,2}^{0,w}(\mathbb{R}^n)$ there exists an unconditional basis. Hence also in $L_p^w(\mathbb{R}^n)$, we have an unconditional basis. So from Theorem 5.2, we obtain that $w \in \mathcal{A}_p^{\text{loc}}$. $\square$

Remark 5.4. It is known that above statements are not true for general Muckenhoupt weights. Taking for example weight,

$$w(x) = \begin{cases} |x|^\alpha & \text{for } |x| \leq 1, \\ |x|^\beta & \text{for } |x| > 1, \end{cases} \quad (5.14)$$

for $\alpha, \beta > -n$, For $\alpha < (p_1 - 1)n$ we have $w \in \mathcal{A}_{p_1}^{\text{loc}}$ and $r_w = \max(0, \alpha)/n + 1$; for $\alpha, \beta < (p_2 - 1)n$ we have $w \in \mathcal{A}_{p_2}$ and $r_w = \max(0, \alpha, \beta)/n + 1$. Taking β big enough, we get that w is in $\mathcal{A}_p^{\text{loc}} \cap \mathcal{A}_\infty$, but not in $\mathcal{A}_p$.

For $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ with $\int \varphi_0(x) dx \neq 0$ and $f \in S'_e$, we introduce the “vertical” maximal function

$$\varphi_0^+ f(x) = \sup_{j \in \mathbb{N}} |(\varphi_0)_j * f(x)|. \quad (5.15)$$

The following corollary follows from Theorem 2.25 in [5] and Corollary 5.3.

Corollary 5.5. *Let $1 < p < \infty$ and $w \in \mathcal{A}_\infty^{\text{loc}}$. The following equivalence holds:*

$$\|\varphi_0^+ f\|_{L_p^w} \sim \|f\|_{L_p^w} \quad (5.16)$$

if and only if $w \in \mathcal{A}_p^{\text{loc}}$.

Please note that it follows from the last corollary that if $1 < p < \infty$ and $w \in \mathcal{A}_\infty^{\text{loc}}$, then the weighted local Hardy space

$$h_p^w = \{f \in S'_e : \|\varphi_0^+ f\| < \infty\} \quad (5.17)$$

coincides with L_p^w if and only if $w \in \mathcal{A}_p^{\text{loc}}$.

References

- [1] P. G. Lemarié-Rieusset, "Ondelettes et poids de Muckenhoupt," *Studia Mathematica*, vol. 108, no. 2, pp. 127–147, 1994.
- [2] K. S. Kazarian, "On bases and unconditional bases in the spaces $L_p(d\mu)$, $1 \leq p < \infty$," *Studia Mathematica*, vol. 71, no. 3, pp. 227–249, 1982.
- [3] J. García-Cuerva and K. S. Kazarian, "Spline wavelet bases of weighted L_p spaces, $1 \leq p < \infty$," *Proceedings of the American Mathematical Society*, vol. 123, no. 2, pp. 433–439, 1995.
- [4] H. A. Aimar, A. L. Bernardis, and F. J. Martín-Reyes, "Multiresolution approximations and wavelet bases of weighted L_p spaces," *The Journal of Fourier Analysis and Applications*, vol. 9, no. 5, pp. 497–510, 2003.
- [5] V. S. Rychkov, "Littlewood-Paley theory and function spaces with A_p^{loc} weights," *Mathematische Nachrichten*, vol. 224, pp. 145–180, 2001.
- [6] D. D. Haroske and H. Triebel, "Wavelet bases and entropy numbers in weighted function spaces," *Mathematische Nachrichten*, vol. 278, no. 1-2, pp. 108–132, 2005.
- [7] D. D. Haroske and L. Skrzypczak, "Entropy and approximation numbers of embeddings of function spaces with Muckenhoupt weights I," *Revista Matemática Complutense*, vol. 21, no. 1, pp. 135–177, 2008.
- [8] M. Izuki and Y. Sawano, "Wavelet bases in the weighted Besov and Triebel-Lizorkin spaces with A_p^{loc} -weights," *Journal of Approximation Theory*, vol. 161, no. 2, pp. 656–673, 2009.
- [9] A. Wojciechowska, "Local means and wavelets in function spaces with local Muckenhoupt weights," *Function spaces IX, Banach Center Publications*, vol. 92, pp. 47–60, 2011.
- [10] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, vol. 43 of *Princeton Mathematical Series*, Princeton University Press, Princeton, NJ, USA, 1993.
- [11] T. Schott, "Function spaces with exponential weights I," *Mathematische Nachrichten*, vol. 189, pp. 221–242, 1998.
- [12] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, vol. 37 of *London Mathematical Society Student Texts*, Cambridge University Press, Cambridge, UK, 1997.