Research Article

Duality of Variable Exponent Triebel-Lizorkin and Besov Spaces

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We will prove the duality and reflexivity of variable exponent Triebel-Lizorkin and Besov spaces. It was shown by many authors that variable exponent Triebel-Lizorkin spaces coincide with variable exponent Bessel potential spaces, Sobolev spaces, and Lebesgue spaces when appropriate indices are chosen. In consequence of the results, these variable exponent function spaces are shown to be reflexive.

1. Introduction

Recently, variable exponent function spaces have been studied by many authors [1–13], and in particular, the papers about the variable exponent Triebel-Lizorkin and Besov spaces have been published in [14–18]. Diening et al. [15] and Almeida and Hästö [14] studied the spaces $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ and $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ and Xu studied [16–18] the spaces $F_{p(\cdot),q}^s(\mathbb{R}^n)$ and $B_{p(\cdot),q}^s(\mathbb{R}^n)$.

In this paper, we will show the duality of variable exponent Triebel-Lizorkin spaces $F_{p(\cdot),q}^s(\mathbb{R}^n)$ and Besov spaces $B_{p(\cdot),q}^s(\mathbb{R}^n)$ under suitable conditions by using the same arguments in Triebel [19]. The duality follows from the fact that $F_{p(\cdot),q}^s(\mathbb{R}^n)$ and $B_{p(\cdot),q}^s(\mathbb{R}^n)$ are reflexive under same conditions.

Xu [16] showed that variable exponent Bessel potential space $L^{s,p(\cdot)}(\mathbb{R}^n)$ coincides with $F_{p(\cdot),2}^s(\mathbb{R}^n)$ if $s \geq 0$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Diening et al. [15] showed that the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ coincides with $F_{p(\cdot),2}^0(\mathbb{R}^n)$ under suitable assumptions on $p(\cdot)$. Gurka et al. [7] showed that $L^{k,p(\cdot)}(\mathbb{R}^n)$ coincides with variable exponent Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$ if $k \in \mathbb{N}$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. In consequences of these results, we have the duality and reflexivity

of $L^{s,p(\cdot)}(\mathbb{R}^n)$ ($s \geq 0$), although the duality and reflexivity of $L^{p(\cdot)}(\mathbb{R}^n)$ have been obtained in Kováčik and Rákosník [9] under the assumptions on $p(\cdot)$ which are weaker than ours.

2. Definitions of Variable Exponent Function Spaces

We first introduce variable exponent Lebesgue spaces. Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ with range in $(1, \infty)$. Let $L^{p(\cdot)}(\mathbb{R}^n)$ denote the set of all complex-valued functions f on $\mathbb{R}^n$ such that, for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty. \quad (2.1)$$

The set becomes a Banach function space when it is equipped with the Luxemburg-Nakano norm:

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}. \quad (2.2)$$

If $p(x) \equiv p$ is a constant function, then the above norm coincides with the usual L^p -norm and so the notation is not confusional. In Kováčik and Rákosník [9], variable exponent Lebesgue spaces are defined on arbitrary measurable subset of $\mathbb{R}^n$. Denote by $\mathcal{P}(\mathbb{R}^n)$ the set of measurable functions $p(\cdot)$ on $\mathbb{R}^n$ with range in $(1, \infty)$ such that

$$1 < p_- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x) = p_+ < \infty. \quad (2.3)$$

For a complex-valued locally Lebesgue-integrable function $f(\cdot)$ on $\mathbb{R}^n$, let

$$(\mathcal{M}f)(x) = \sup \frac{1}{|B|} \int_B |f(y)| dy \quad (2.4)$$

denote the Hardy-Littlewood maximal function, where the supremum is taken over all balls B centered at x . There exists $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator $\mathcal{M}$ is not bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ [11], although the operator $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$ for $p > 1$. Some sufficient conditions on $p(\cdot)$ for maximal operator $\mathcal{M}$ to be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ are known. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^n$. Let $\mathcal{S}'(\mathbb{R}^n)$ be the set of all the tempered distribution on $\mathbb{R}^n$. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, let $\mathcal{F}\varphi$ denote the Fourier transform of φ and $\mathcal{F}^{-1}\varphi$ the inverse Fourier transform of φ . We write $\mathcal{F}^{-1}m\mathcal{F}f = \mathcal{F}^{-1}[m \cdot \mathcal{F}f]$ for the sake of simplicity.

Let $s \geq 0$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. The variable exponent Bessel potential space $L^{s,p(\cdot)}(\mathbb{R}^n)$ is the collection of $f \in L^{p(\cdot)}(\mathbb{R}^n)$ such that the norm:

$$\|f\|_{L^{s,p(\cdot)}} = \left\| \mathcal{F}^{-1} \left(1 + |\cdot|^2 \right)^{s/2} \mathcal{F}f(\cdot) \right\|_{L^{p(\cdot)}} < \infty. \quad (2.5)$$

Let $k \in \mathbb{N}$ and $p \in \mathcal{P}(\mathbb{R}^n)$. The variable exponent Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^n)$ is the collection of $f \in L^{p(\cdot)}(\mathbb{R}^n)$ such that the derivatives (in the sense of distribution) up to the order k belong to $L^{p(\cdot)}(\mathbb{R}^n)$ and the norm:

$$\|f\|_{W^{k,p(\cdot)}} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^{p(\cdot)}} < \infty, \quad (2.6)$$

where α is a multi-index and $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be a function such that $\text{supp } \mathcal{F}\Phi \subset B(0,1)$ and $\mathcal{F}\Phi = 1$ on $B(0,1/2)$, where $B(0,\lambda)$ is an open ball centered at 0 with radius λ . Set $\Phi_j(x) = 2^{nj}\Phi(2^j x)$ for $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$. We also set

$$\theta_j(x) = \Phi_j(x) - \Phi_{j-1}(x). \quad (2.7)$$

It follows that

$$\sum_{j=0}^{\infty} \mathcal{F}\theta_j(\xi) \equiv 1, \quad (2.8)$$

where $\theta_0 = \Phi_0$.

Definition 2.1. Let $s \in \mathbb{R}$, $0 < q < \infty$, and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Let θ_j , $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as above. The variable exponent Triebel-Lizorkin space $F_{p(\cdot),q}^s(\mathbb{R}^n)$ is the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F_{p(\cdot),q}^s} = \left\| \left\{ 2^{sj} \theta_j * f \right\}_0^\infty \right\|_{L^{p(\cdot)}(\ell^q)} < \infty. \quad (2.9)$$

Let $s \in \mathbb{R}$, $0 < q \leq \infty$, and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The variable exponent Besov space $B_{p(\cdot),q}^s(\mathbb{R}^n)$ is the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p(\cdot),q}^s} = \left\| \left\{ 2^{sj} \theta_j * f \right\}_0^\infty \right\|_{\ell^q(L^{p(\cdot)})} < \infty. \quad (2.10)$$

Here, $L^{p(\cdot)}(\ell^q)$ and $\ell^q(L^{p(\cdot)})$ are the spaces of all sequences $\{g_j\}$ of measurable functions on $\mathbb{R}^n$ such that quasi-norms:

$$\begin{aligned} \left\| \{g_j\}_0^\infty \right\|_{L^{p(\cdot)}(\ell^q)} &= \left\| \left\{ \|g_j\|_{\ell^q} \right\}_0^\infty \right\|_{L^{p(\cdot)}} = \left\| \left(\sum_{j=0}^\infty |g_j(\cdot)|^q \right)^{1/q} \right\|_{L^{p(\cdot)}}, \\ \left\| \{g_j\}_0^\infty \right\|_{\ell^q(L^{p(\cdot)})} &= \left\| \left\{ \|g_j\|_{L^{p(\cdot)}} \right\}_0^\infty \right\|_{\ell^q} = \left(\sum_{j=0}^\infty \|g_j(\cdot)\|_{L^{p(\cdot)}}^q \right)^{1/q} \end{aligned} \quad (2.11)$$

are finite, respectively.

Definition 2.2. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $0 < q \leq \infty$. For a sequence of compact subsets $\Omega = \{\Omega_k\}_{k=0}^\infty$ of $\mathbb{R}^n$, $L_{p(\cdot)}^\Omega(\ell^q)$ denotes the space of all sequences $\{f_k\}_{k=0}^\infty$ of $\mathcal{S}'(\mathbb{R}^n)$ such that

$$\text{supp } \mathcal{F}f_k \subset \Omega_k \quad \text{for } k = 0, 1, 2, \dots, \quad (2.12)$$

and $\|f_k\|_{L_{p(\cdot)}(\ell^q)} < \infty$. For a compact subset Γ of $\mathbb{R}^n$, $L_{p(\cdot)}^\Gamma(\mathbb{R}^n)$ denotes the space of all elements $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\text{supp } \mathcal{F}f \subset \Gamma, \quad (2.13)$$

and $\|f\|_{L_{p(\cdot)}} < \infty$.

3. Preliminaries

We need the following fundamental properties of $L_{p(\cdot)}(\mathbb{R}^n)$.

Theorem 3.1 (see [5, Theorem 8.1]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then, the following conditions are equivalent:*

- (a) $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$,
- (b) $p(\cdot)/t \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < t < p_-$,
- (c) $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, where

$$p'(\cdot) = \frac{p(\cdot)}{p(\cdot) - 1}. \quad (3.1)$$

In [1, 5], some other conditions equivalent to the above are given. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $0 < q < \infty$. Then we write

$$r_{p-,q} = \sup \left\{ r \in \mathbb{R} : 0 < r < \min(p_-, q), \frac{p(\cdot)}{r} \in \mathcal{B}(\mathbb{R}^n) \right\}. \quad (3.2)$$

If $q = p_-$, then, we write $r_{p-,p_-} = r_{p_-}$.

Remark 3.2. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and t as in Theorem 3.1. Then, for any $w \in (1, t]$, we have $p(\cdot)/w \in \mathcal{B}(\mathbb{R}^n)$. Furthermore, for any $w \in (0, 1]$, we have also $p(\cdot)/w \in \mathcal{B}(\mathbb{R}^n)$ by Jensen inequality. The proofs are found in [5]. Consequently, if $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then, $p(\cdot)/r \in \mathcal{B}(\mathbb{R}^n)$ for any $r \in (0, r_{p-,q})$. For any $r \in (0, r_{p-,q})$, we have also $p(\cdot)/r \in \mathcal{B}(\mathbb{R}^n)$.

The next theorem gives a generalized Hölder inequality, which is shown in [9, 13].

Theorem 3.3 (see [9, 13, generalized Hölder inequality]). *Let $p \in \mathcal{P}(\mathbb{R}^n)$. Then,*

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|f\|_{L_{p(\cdot)}} \|g\|_{L_{p'(\cdot)}}, \quad (3.3)$$

for every $f \in L_{p(\cdot)}(\mathbb{R}^n)$ and $g \in L_{p'(\cdot)}(\mathbb{R}^n)$.

The next theorem is shown in [6].

Theorem 3.4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Then,*

$$\begin{aligned} \min \left\{ \left(\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \right)^{1/p_-}, \left(\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \right)^{1/p_+} \right\} \\ \leq \|f\|_{L^{p(\cdot)}} \leq \max \left\{ \left(\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \right)^{1/p_-}, \left(\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \right)^{1/p_+} \right\}. \end{aligned} \quad (3.4)$$

The next theorem is shown in [9].

Theorem 3.5. (i) *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then, $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space.*

(ii) *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$.*

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $p(\cdot)$ is not a constant function. Then, according to [9, Example 2.9, Theorem 2.10], for every $L^{p(\cdot)}$, there exists a function $f(\cdot) \in L^{p(\cdot)}(\mathbb{R}^n)$ such that its translation $f(\beta + \cdot) \notin L^{p(\cdot)}(\mathbb{R}^n)$ if $\beta \in \mathbb{R}^n \setminus \{(0, 0, \dots, 0)\}$. However, we can prove that all elements of $\mathcal{S}(\mathbb{R}^n)$ belong to $L^{p(\cdot)}(\mathbb{R}^n)$ for every $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Let $f(\cdot) \in \mathcal{S}(\mathbb{R}^n)$. Then, we have

$$\sup_{\beta \in \mathbb{R}^n} \|f(\beta + \cdot)\|_{L^{p(\cdot)}} \leq \begin{cases} |\alpha|^{-n/p_+} \pi^{n/p_-} \sup_{x \in \mathbb{R}^n} \left\{ (1 + |x|^2)^n |f(x)| \right\} & \text{if } |\alpha| \geq 1, \\ |\alpha|^{-n/p_-} \pi^{n/p_+} \sup_{x \in \mathbb{R}^n} \left\{ (1 + |x|^2)^n |f(x)| \right\} & \text{if } |\alpha| < 1, \end{cases} \quad (3.5)$$

where $\alpha \in \mathbb{R} \setminus \{0\}$.

Hence, by the same arguments of Triebel [20], we have the following two theorems.

Theorem 3.6. *Let Ω be a compact subset of $\mathbb{R}^n$ and α an arbitrary multi-index. Let $p(\cdot), q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ such that $1 < p(x) \leq q(x) < \infty$. Then, there exists a positive constant c such that*

$$\|D^\alpha f\|_{L^{q(\cdot)}} \leq c \|f\|_{L^{p(\cdot)}}, \quad (3.6)$$

$$\|D^\alpha f\|_{L^\infty} \leq c \|f\|_{L^{p(\cdot)}}, \quad (3.7)$$

for all $f \in L_{p(\cdot)}^\Omega(\mathbb{R}^n)$.

Let $f(\cdot) \in L_{p(\cdot)}^\Omega(\mathbb{R}^n)$ and $\mathcal{F}^{-1}M \in L^1(\mathbb{R}^n)$. Then,

$$(\mathcal{F}^{-1}M\mathcal{F}f)(x) = c \int_{\mathbb{R}^n} (\mathcal{F}^{-1}M)(x - y) f(y) dy \quad (3.8)$$

makes sense for any $x \in \mathbb{R}^n$ by the classical Hölder inequality and (3.7).

Let s be a real number and $H_2^s(\mathbb{R}^n)$ the spaces of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H_2^s} = \left\| \left(1 + |\cdot|^2 \right)^{s/2} (\mathcal{F}f)(\cdot) \right\|_{L^2} < \infty. \quad (3.9)$$

Theorem 3.7. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $0 < q < \infty$. Let $\Omega = \{\Omega_k\}_{k=0}^\infty$ be a sequence of compact subsets of $\mathbb{R}^n$. Let $d_k > 0$ be the diameter of Ω_k . If $v > n/2 + n/r_{p,q}$, then there exists a number c such that

$$\left\| \mathcal{F}^{-1} M_k \mathcal{F} f_k \right\|_{L^{p(\cdot)}(\ell^q)} \leq c \sup_l \|M_l(d_l \cdot)\|_{H_2^v} \|f_k\|_{L^{p(\cdot)}(\ell^q)} \quad (3.10)$$

for $\{f_k(x)\}_{k=0}^\infty \in L_{p(\cdot)}^\Omega(\ell^q)$ and $\{M_k(x)\}_{k=0}^\infty \in H_2^v(\mathbb{R}^n)$.

4. Basic Properties of Variable Exponent Triebel-Lizorkin and Besov Spaces

We will first consider the quasi-norms on $F_{p(\cdot),q}^s(\mathbb{R}^n)$ and $B_{p(\cdot),q}^s(\mathbb{R}^n)$.

Definition 4.1. (i) The set $\Psi(\mathbb{R}^n)$ is the collection of all systems $\varphi = \{\varphi_j\}_{j=0}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ with $\mathcal{F}\varphi_k(\xi) = \mathcal{F}\varphi(2^{-k}\xi)$ for $k = 1, 2, \dots$, such that

$$\begin{aligned} \text{supp } \mathcal{F}\varphi_0 &\subset \{x : |x| \leq 2\}, \\ \text{supp } \mathcal{F}\varphi_j &\subset \left\{x : 2^{j-1} \leq |x| \leq 2^{j+1}\right\} \quad \text{for } j = 1, 2, \dots, \end{aligned} \quad (4.1)$$

for every multi-index α , there exists a positive number c_α such that

$$2^{j|\alpha|} |D^\alpha \mathcal{F}\varphi_j(x)| \leq c_\alpha, \quad (4.2)$$

for $j = 0, 1, \dots$, and $x \in \mathbb{R}^n$ and there exists a positive number c such that

$$c \leq \sum_{j=0}^\infty \mathcal{F}\varphi_j(x), \quad (4.3)$$

for $x \in \mathbb{R}^n$.

(ii) The set $\Phi(\mathbb{R}^n)$ is the collection of all systems $\varphi = \{\varphi_j\}_{j=0}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$\begin{aligned} \text{supp } \varphi_0 &\subset \{x : |x| \leq 2\}, \\ \text{supp } \varphi_j &\subset \left\{x : 2^{j-1} \leq |x| \leq 2^{j+1}\right\} \quad \text{for } j = 1, 2, \dots, \end{aligned} \quad (4.4)$$

for every multi-index α , there exists a positive number c_α such that

$$2^{j|\alpha|} |D^\alpha \varphi_j(x)| \leq c_\alpha, \quad (4.5)$$

for $j = 0, 1, \dots$, and $x \in \mathbb{R}^n$, and

$$\sum_{j=0}^\infty \varphi_j(x) = 1, \quad (4.6)$$

for $x \in \mathbb{R}^n$.

Theorem 4.2. (i) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < q \leq \infty$, and $s \in \mathbb{R}$. Let θ_j be as in Definition 2.1 and $\varphi_j = \mathcal{F}\theta_j$. Then, the norm $\|f\|_{F_{p(\cdot),q}^s}$ coincides with $\|2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f\|_{L_{p(\cdot)}(\ell^q)}$ and the norm $\|f\|_{B_{p(\cdot),q}^s}$ with $\|2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f\|_{\ell^q(L_{p(\cdot)})}$.

(ii) Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q \leq \infty$, and $s \in \mathbb{R}$. Let $\varphi, \psi \in \Phi(\mathbb{R}^n)$. Then, $\|2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f\|_{L_{p(\cdot)}(\ell^q)}$ and $\|2^{js}\mathcal{F}^{-1}\psi_j\mathcal{F}f\|_{L_{p(\cdot)}(\ell^q)}$ are equivalent quasi-norms on $F_{p(\cdot),q}^s(\mathbb{R}^n)$. Similarly, $\|2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f\|_{\ell^q(L_{p(\cdot)})}$ and $\|2^{js}\mathcal{F}^{-1}\psi_j\mathcal{F}f\|_{\ell^q(L_{p(\cdot)})}$ are equivalent quasi-norms on $B_{p(\cdot),q}^s(\mathbb{R}^n)$.

(iii) Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q \leq \infty$, and $s \in \mathbb{R}$. Let $\varphi \in \Phi(\mathbb{R}^n)$ and $\psi \in \Psi(\mathbb{R}^n)$. Then, $\|2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f\|_{L_{p(\cdot)}(\ell^q)}$ and $\|2^{js}\psi_j * f\|_{L_{p(\cdot)}(\ell^q)}$ are equivalent quasi-norms on $F_{p(\cdot),q}^s(\mathbb{R}^n)$. Similarly, $\|2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f\|_{\ell^q(L_{p(\cdot)})}$ and $\|2^{js}\psi_j * f\|_{\ell^q(L_{p(\cdot)})}$ are equivalent quasi norms on $B_{p(\cdot),q}^s(\mathbb{R}^n)$.

Proof. Let θ_j be as in Definition 2.1. It is obvious that $\theta_j * f = d_{\mathcal{F}}\mathcal{F}^{-1}(\mathcal{F}\theta_j \cdot \mathcal{F}f)$ for some positive number $d_{\mathcal{F}}$, which depends only on the definition of Fourier transform $\mathcal{F}$. Let $\varphi_j = \mathcal{F}\theta_j$. Then $\{\varphi_j\}_{j=0}^\infty \in \Phi(\mathbb{R}^n)$. Hence, the norm $\|f\|_{F_{p(\cdot),q}^s}$ coincides with $\|2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f\|_{L_{p(\cdot)}(\ell^q)}$. Similarly, the variable exponent Besov norm $\|f\|_{B_{p(\cdot),q}^s}$ coincides with $\|2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f\|_{\ell^q(L_{p(\cdot)})}$. This proves (i). The proof of (ii) is done as in the proof of [20, page 46, Proposition 1] with using Theorem 3.7, the scalar case of Theorem 3.7, and Definition 4.1.

We will prove (iii). Let $f \in F_{p(\cdot),q}^s(\mathbb{R}^n)$, $g_j = \mathcal{F}^{-1}(\varphi_j\mathcal{F}f)$, and $\psi \in \Psi(\mathbb{R}^n)$. Let $\psi_k \equiv g_k \equiv 0$ for $k < 0$. Then,

$$f * \psi_k(x) \underset{\mathcal{S}'}{=} \sum_{j=0}^{\infty} g_j * \psi_k(x) = \sum_{j=k-2}^{k+2} (g_j * \psi_k)(x), \quad (4.7)$$

because

$$g_j * \psi_k = \mathcal{F}^{-1}\mathcal{F}(g_j * \psi_k) = d_{\mathcal{F}}\mathcal{F}^{-1}(\mathcal{F}\psi_k \cdot \mathcal{F}g_j) = 0, \quad (4.8)$$

for $j < k-2$ and $j > k+2$. Here, $f \underset{\mathcal{S}'}{=} \sum_0^\infty a_j$ means $\sum_0^N a_j$ converges to f in $\mathcal{S}'(\mathbb{R}^n)$. It follows that

$$\begin{aligned} \left\| \left\{ 2^{sk} f * \psi_k \right\}_0^\infty \right\|_{L_{p(\cdot)}(\ell^q)} &\leq \sum_{r=-2}^2 \left\| \left\{ 2^{sk} g_{k+r} * \psi_k \right\}_0^\infty \right\|_{L_{p(\cdot)}(\ell^q)} \\ &\leq \sum_{r=-2}^2 \left\| \mathcal{F}\psi(2^{r+1}\cdot) \right\|_{H_2^s} \left\| \left\{ 2^{sk} g_{k+r} \right\}_0^\infty \right\|_{L_{p(\cdot)}(\ell^q)}, \end{aligned} \quad (4.9)$$

where we use Theorem 3.7 with $f_k = g_{k+r}$, $M_k = \mathcal{F}\psi_k$, and $d_k = 2^{k+1+r}$. Hence, we have

$$\left\| \left\{ 2^{sk} f * \psi_k \right\}_0^\infty \right\|_{L_{p(\cdot)}(\ell^q)} \leq \sum_{r=-2}^2 \left\| \mathcal{F}\psi(2^{r+1}\cdot) \right\|_{H_2^s} \|f\|_{F_{p(\cdot),q}^s}. \quad (4.10)$$

We will prove the opposite inequality. Let $\rho(x)$ be a real function with $\rho(x) = 1$ for $2^{-1} \leq |x| \leq 2$ and

$$\rho(\cdot) \in C_0^\infty \left(\left\{ \xi : 2^{-2} \leq |\xi| \leq 2^2 \right\} \right). \quad (4.11)$$

We set $\rho_k(x) = \rho(2^{-k}x)$, for $k = 1, 2, \dots$. Furthermore, let $\rho_0(x)$ be a real function such that $\rho_0(x) = 1$ for $|x| \leq 2$ and

$$\rho_0(\cdot) \in C_0^\infty\left(\left\{\xi : |\xi| \leq 2^2\right\}\right). \quad (4.12)$$

We define $h_k(x)$ by

$$\mathcal{F}h_k(\xi) = \left(\sum_{l=0}^{\infty} \mathcal{F}\psi_l(\xi)\right)^{-1} \rho_k(\xi), \quad (4.13)$$

for $k = 0, 1, 2, \dots$ ($h_k = 0$ for $k < 0$). Then, $\mathcal{F}h_k(\cdot) \in C_0^\infty(\mathbb{R}^n)$ and $h_k(\cdot) \in \mathcal{S}(\mathbb{R}^n)$. Let $\gamma_k = \psi_k * h_k$. It is obvious that $\{\mathcal{F}\gamma_k/d\mathcal{F}\} \in \Phi(\mathbb{R}^n)$ and $\{2^{ks}\psi_k * f\}_0^\infty \in L_{p(\cdot)}^\Omega(\ell^q)$ by Theorem 4.8 and (4.10), where Ω is some sequence of compact subsets of $\mathbb{R}^n$. Then,

$$\begin{aligned} \left\|\{2^{ks}f * \gamma_k\}_0^\infty\right\|_{L^{p(\cdot)}(\ell^q)} &= \left\|\{2^{ks}f * \psi_k * h_k\}_0^\infty\right\|_{L^{p(\cdot)}(\ell^q)} \\ &\leq \sup_k \left\|\mathcal{F}h_k(2^{k+1}\cdot)\right\|_{H_2^\infty} \left\|\{2^{ks}f * \psi_k\}_0^\infty\right\|_{L^{p(\cdot)}(\ell^q)}, \end{aligned} \quad (4.14)$$

$$\mathcal{F}h_k(2^{k+1}x) = \frac{\rho_k(2^{k+1}x)}{\sum_{l=0}^\infty \mathcal{F}\psi_l(2^{k+1}x)} = \frac{\rho_k(2^{k+1}x)}{\sum_{l=k-3}^{k+3} \mathcal{F}\psi_l(2^{k+1}x)} = \frac{\rho(2x)}{\sum_{r=-4}^4 \mathcal{F}\psi(2^r x)},$$

for any $k = 1, 2, \dots$. This means that $\mathcal{F}h_k(2^{k+1}x)$ is independent of $k \in \mathbb{N}$ and in $C_0^\infty(\mathbb{R}^n)$. Hence, we have $\sup_k \|\mathcal{F}h_k(2^{k+1}\cdot)\|_{H_2^\infty} < \infty$. This implies that

$$\|f\|_{F_{p(\cdot),q}^s} \leq \sup_k \left\|\mathcal{F}h_k(2^{k+1}\cdot)\right\|_{H_2^\infty} \left\|\{2^{ks}f * \psi_k\}_0^\infty\right\|_{L^{p(\cdot)}(\ell^q)}. \quad (4.15)$$

This proves the $F_{p(\cdot),q}^s$ case of (iii). The proof of the $B_{p(\cdot),q}^s$ case of (iii) is essentially the same as one of the $F_{p(\cdot),q}^s$ case. $\square$

Corollary 4.3. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $s \in \mathbb{R}$, and $1 < q < \infty$.

(i) $f \in F_{p(\cdot),q}^s(\mathbb{R}^n)$ if and only if there exist continuous functions $\{a_j\}_{j \in \mathbb{N}_0}$ such that $f = \sum_{j=0}^\infty a_j$, $\|\{2^{js}a_j\}_0^\infty\|_{L^{p(\cdot)}(\ell^q)} < \infty$,

$$\text{supp } \mathcal{F}a_0 \subset \{\xi : |\xi| \leq 2\}, \quad \text{supp } \mathcal{F}a_j \subset \{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}, \quad \text{for } j \in \mathbb{N}_0. \quad (4.16)$$

In particular, $\|f\|_{F_{p(\cdot),q}^s}$ coincides with $\inf_{f=\sum a_j} \|\{2^{js}a_j\}_0^\infty\|_{L^{p(\cdot)}(\ell^q)}$, where the infimum is taken over all representation $f = \sum a_j$.

(ii) $f \in B_{p(\cdot),q}^s(\mathbb{R}^n)$ if and only if, there exist continuous functions $\{a_j\}_{j \in \mathbb{N}_0}$ such that $f = \sum_{j=0}^\infty a_j$, $\|\{2^{js}a_j\}_0^\infty\|_{\ell^q(L^{p(\cdot)})} < \infty$, $\text{supp } \mathcal{F}a_0 \subset \{\xi : |\xi| \leq 2\}$, and $\text{supp } \mathcal{F}a_j \subset \{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ for $j \in \mathbb{N}_0$. In particular, the norm $\|f\|_{B_{p(\cdot),q}^s}$ coincides with $\inf_{f=\sum a_j} \|\{2^{js}a_j\}_0^\infty\|_{\ell^q(L^{p(\cdot)})}$.

Proposition 4.4. (i) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $s \in \mathbb{R}$, and $0 < q_1 \leq q_2 \leq \infty$. Then,

$$\begin{aligned} B_{p(\cdot),q_1}^s(\mathbb{R}^n) &\subset B_{p(\cdot),q_2}^s(\mathbb{R}^n), \\ F_{p(\cdot),q_1}^s(\mathbb{R}^n) &\subset F_{p(\cdot),q_2}^s(\mathbb{R}^n). \end{aligned} \quad (4.17)$$

(ii) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$, and $\epsilon > 0$. Then

$$\begin{aligned} B_{p(\cdot),q_1}^{s+\epsilon}(\mathbb{R}^n) &\subset B_{p(\cdot),q_2}^s(\mathbb{R}^n), \\ F_{p(\cdot),q_1}^{s+\epsilon}(\mathbb{R}^n) &\subset F_{p(\cdot),q_2}^s(\mathbb{R}^n). \end{aligned} \quad (4.18)$$

(iii) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $s \in \mathbb{R}$. If $0 < q \leq \infty$, then

$$B_{p(\cdot),\min\{p_-,q\}}^s(\mathbb{R}^n) \subset F_{p(\cdot),q}^s(\mathbb{R}^n) \subset B_{p(\cdot),\max\{p_+,q\}}^s(\mathbb{R}^n). \quad (4.19)$$

The proof is almost same as in [20]. Furthermore, Almeida and Hästö [14] proved the above inclusion for $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ and $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Theorem 4.5. Let $0 < q < \infty$, $s \in \mathbb{R}$, and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Let $A_{p(\cdot),q}^s$ be either $B_{p(\cdot),q}^s(\mathbb{R}^n)$ or $F_{p(\cdot),q}^s(\mathbb{R}^n)$. Then, $\mathcal{S}(\mathbb{R}^n) \subset A_{p(\cdot),q}^s \subset \mathcal{S}'(\mathbb{R}^n)$. Furthermore, $\mathcal{S}(\mathbb{R}^n)$ is dense in $A_{p(\cdot),q}^s(\mathbb{R}^n)$ and $A_{p(\cdot),q}^s(\mathbb{R}^n)$ is a quasi Banach space (a Banach space if $1 \leq q \leq \infty$).

Proof. To prove the inclusion, we may restrict ourselves to $B_{p(\cdot),q}^s(\mathbb{R}^n)$ with $q = \infty$ by Proposition 4.4. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi = \{\varphi_k(x)\}_{k=0}^\infty \in \Phi(\mathbb{R}^n)$. We recall that the topology of the complete locally convex space $\mathcal{S}(\mathbb{R}^n)$ is generated by seminorms:

$$p_N(f) = \sup_{x \in \mathbb{R}^n} \sum_{|\alpha|+k \leq N} (1 + |x|^2)^k |D^\alpha f(x)|, \quad N \in \mathbb{N}, \quad (4.20)$$

$\mathcal{F}$ yields a one-to-one mapping from $\mathcal{S}(\mathbb{R}^n)$ onto itself, and, in particular, $p_N(\mathcal{F}\varphi)$ with $N = 1, 2, \dots$ generates the topology of $\mathcal{S}(\mathbb{R}^n)$. If N is a sufficiently large natural number, then

$$\|f\|_{B_{p(\cdot),q}^s} = \sup_k 2^{ks} \left\| \mathcal{F}^{-1} \varphi_k \mathcal{F} f \right\|_{L^{p(\cdot)}} \leq c p_N(\mathcal{F} f), \quad (4.21)$$

where c is a positive number. This proves the left-hand side of the inclusion with $q = \infty$. We prove the right-hand side of the inclusion with $q = \infty$. Let $\varphi \in \Phi(\mathbb{R}^n)$ be the above system. It is well known that

$$f(\varphi) = \sum_{k=0}^{\infty} \mathcal{F}^{-1} \varphi_k \mathcal{F} f(\varphi) \quad (4.22)$$

in $\mathcal{S}'(\mathbb{R}^n)$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. By the Paley-Wiener-Schwartz Theorem, the definition of φ_k implies that $\mathcal{F}^{-1} \varphi_k \mathcal{F} f$ are analytic functions, and that these functions are

regular distributions. We put $\chi_k(x) = \varphi_{k-1}(x) + \varphi_k(x) + \varphi_{k+1}(x)$ if $k = 0, 1, \dots$, with $\varphi_{-1} = 0$. If $f \in B_{p(\cdot), \infty}^s(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, then $f(\psi)$ denotes the value of the functional f of $\mathcal{S}'(\mathbb{R}^n)$ of the test function ψ . We obtain

$$|f(\psi)| \leq \left| \sum_{k=0}^{\infty} \mathcal{F}^{-1} \varphi_k \mathcal{F} f \left(\mathcal{F} \chi_k \mathcal{F}^{-1} \psi \right) \right| \leq \sum_{k=0}^{\infty} \left\| \mathcal{F}^{-1} \varphi_k \mathcal{F} f \right\|_{L^\infty} \left\| \mathcal{F} \chi_k \mathcal{F}^{-1} \psi \right\|_{L^1} \quad (4.23)$$

by the classical Hölder inequality. Because both $\mathcal{F}^{-1} \varphi_k \mathcal{F} f$ and $\mathcal{F} \chi_k \mathcal{F}^{-1} \psi$ are analytic functions, the last estimate makes sense. By applying (3.7) with $\alpha = (0, 0, \dots, 0)$ to $\mathcal{F}^{-1} \varphi_k \mathcal{F} f$, we have

$$|f(\psi)| \leq c \|f\|_{B_{p(\cdot), \infty}^s} \sum_{k=0}^{\infty} 2^{-sk} \left\| \mathcal{F} \chi_k \mathcal{F}^{-1} \psi \right\|_{L^1} \leq c \|f\|_{B_{p(\cdot), \infty}^s} \|\psi\|_{B_{1,1}^{-s}}. \quad (4.24)$$

If N is a sufficiently large natural number, then, for any $\psi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$|f(\psi)| \leq c \|f\|_{B_{p(\cdot), \infty}^s} p_N(\psi) \quad (4.25)$$

by the left-hand side of the inclusion for $B_{1,1}^{-s}(\mathbb{R}^n)$. Hence, we have the inclusion for $A_{p(\cdot), q}^s$.

The proof of the completeness and the density of $\mathcal{S}(\mathbb{R}^n)$ is given by the same arguments of step 4 and step 5 of the proof of [20, page 48, Theorem] with p replaced by $p(\cdot)$. $\square$

As we mentioned in Section 1, the next theorem is found in [15, 16].

Theorem 4.6. *Let $s \geq 0$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then, $F_{p(\cdot), 2}^s(\mathbb{R}^n)$ coincides with $L^{s, p(\cdot)}(\mathbb{R}^n)$. Let $k \in \mathbb{N}$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then, $F_{p(\cdot), 2}^k(\mathbb{R}^n)$ coincides with $W^{k, p(\cdot)}(\mathbb{R}^n)$.*

Definition 4.7. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $(\mathcal{F}\varphi)(\xi) = 1$ for $1/\sqrt{2} \leq |\xi| \leq \sqrt{2}$ and

$$0 \leq \mathcal{F}\varphi \in C_0^\infty \left(\left\{ \xi \in \mathbb{R}^n : \frac{1}{\sqrt{2}} - \epsilon < |\xi| < \sqrt{2} + \epsilon \right\} \right), \quad (4.26)$$

and let $\rho \in \mathcal{S}(\mathbb{R}^n)$ such that

$$0 \leq \mathcal{F}\rho \in C_0^\infty \left(\left\{ \xi \in \mathbb{R}^n : \frac{1}{\sqrt{2}} + \delta < |\xi| < \sqrt{2} - \delta \right\} \right), \quad (4.27)$$

where ϵ and δ are positive numbers. We construct $\{\varphi_k\}_{k=0}^\infty$ and $\{\rho_k\}_{k=0}^\infty$ by

$$(\mathcal{F}\varphi_k)(\xi) = (\mathcal{F}\varphi)(2^{-k}\xi), \quad (\mathcal{F}\rho_k)(\xi) = (\mathcal{F}\rho)(2^{-k}\xi). \quad (4.28)$$

We choose ϵ and δ sufficiently small so that

$$\mathcal{F}\varphi_k \cdot \mathcal{F}\rho_j \equiv 0 \quad (4.29)$$

for $k \neq j$.

Diening [3] shows that Young inequality $\|f * g\|_{L^{p(\cdot)}} \leq \|f\|_{L^{p(\cdot)}} \|g\|_{L^1}$ holds if and only if $p(\cdot)$ is a constant function. However, the next theorem, which is a part of Corollary 3.6 of [3], holds.

Theorem 4.8. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Let ϕ be an integrable function on $\mathbb{R}^n$ with range in $\mathbb{R}$ and set $\phi_\epsilon(x) = \epsilon^{-n} \phi(x/\epsilon)$ for all $\epsilon > 0$. Assume that the least decreasing radial majorant of ϕ is integrable, that is, $A = \int_{\mathbb{R}^n} \sup_{|y| \geq |x|} |\phi(y)| dx < \infty$. Then, for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$, one has*

$$\sup_{\epsilon > 0} |(f * \phi_\epsilon)(x)| \leq 2A \mathcal{M}f(x). \quad (4.30)$$

Hence, there exist a positive number $C(A, p)$ such that

$$\|f * \phi_\epsilon\|_{L^{p(\cdot)}} \leq C(A, p) \|f\|_{L^{p(\cdot)}}, \quad (4.31)$$

where $C(A, p)$ only depends on A and $p(\cdot)$.

The duality pair of $g \in (F_{p(\cdot), q}^s(\mathbb{R}^n))'$ and $f \in F_{p(\cdot), q}^s(\mathbb{R}^n)$ is denoted by $\langle g, f \rangle$, although, for $\varphi \in \mathcal{S}(\mathbb{R}^n) \subset F_{p(\cdot), q}^s(\mathbb{R}^n)$, we also write $g(\varphi)$ as $\langle g, \varphi \rangle$. In order to prove the duality of $F_{p(\cdot), q}^s(\mathbb{R}^n)$ and $B_{p(\cdot), q}^s(\mathbb{R}^n)$ (Theorem 5.1), we will first prove the following two lemmas which correspond to [19, Lemmas 7.1.3, and 7.1.5].

Lemma 4.9. *Let $g \in (F_{p(\cdot), q}^s)^'$, $\{\rho_k\}$ as in Definition 4.7 and*

$$\tilde{g} = \sum_{k=0}^{\infty} \mathcal{F}^{-1}(\mathcal{F}\rho_k \cdot \mathcal{F}g) \quad (4.32)$$

in $\mathcal{S}'(\mathbb{R}^n)$. Then, $\tilde{g} \in (F_{p(\cdot), q}^s)^'$ and

$$\|\tilde{g}\|_{(F_{p(\cdot), q}^s)^'} \leq c \|g\|_{(F_{p(\cdot), q}^s)^'}, \quad (4.33)$$

where c does not depend on g .

Proof. We use the same argument of the proof of [19, Lemma 7.1.3]. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then, it holds that

$$\tilde{g}(f) = (\mathcal{F}\tilde{g})(\mathcal{F}^{-1}f) = \sum_{k=0}^{\infty} \mathcal{F}g(\mathcal{F}\rho_k \mathcal{F}^{-1}f) = c \sum_{k=0}^{\infty} g(\rho_k^\vee * f) = c g\left(\sum_{k=0}^{\infty} \rho_k^\vee * f\right), \quad (4.34)$$

where $\rho_k^\vee(x) = \rho_k(-x)$. Using $g \in (F_{p(\cdot), q}^s)^'$ and Theorem 3.7 with $M_k = \mathcal{F}(\rho_k^\vee)$ and $d_k = 2^{k+1}$, we find that

$$|\tilde{g}(f)| \leq \|g\|_{(F_{p(\cdot), q}^s)^'} \left\| \left\{ 2^{ks} \rho_k^\vee * \varphi_k * f \right\}_0^\infty \right\|_{L^{p(\cdot)}(\ell^q)} \leq c \|g\|_{(F_{p(\cdot), q}^s)^'} \left\| \left\{ 2^{ks} \varphi_k * f \right\}_0^\infty \right\|_{L^{p(\cdot)}(\ell^q)}, \quad (4.35)$$

where $\{\varphi_k\}_{k=0}^\infty$ is a system of Definition 4.7. Thus, we get

$$|\tilde{g}(f)| \leq c \|g\|_{(F_{p(\cdot),q}^s)'} \|f\|_{F_{p(\cdot),q}^s}, \quad (4.36)$$

from Definition 2.1 and Theorem 4.2. $\square$

Lemma 4.10. *Let $\{\rho_k\}$ and $\{\varphi_k\}$ be ones as in Definition 4.7, $g \in (F_{p(\cdot),q}^s)'$, and $\tilde{g}$ the functional in Lemma 4.9. Let $\{b_k\}_{k=0}^N$ be a system of functions such that $b_k \in L^{p(\cdot)}(\mathbb{R}^n)$. Set*

$$a_k = \mathcal{F}^{-1}(\mathcal{F}\rho_k \cdot \mathcal{F}g), \quad (4.37)$$

$$c_k = b_k * \varphi_k, \quad (4.38)$$

for $k = 0, 1, 2, \dots, N$, and

$$f = \sum_{k=0}^N c_k. \quad (4.39)$$

Then,

$$\langle \tilde{g}, f \rangle = c \int_{\mathbb{R}^n} \sum_{k=0}^N a_k b_k dx, \quad (4.40)$$

where c depends only on the definition of Fourier transform.

Proof. We use the same argument of the proof of [19, Lemma 7.1.5]. We first prove that $a_k \in L^{p'(\cdot)}$, where $p'(\cdot)$ is defined by (3.1). The consideration of the proof of Lemma 4.9 implies that for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$|a_k(f)| \leq c \|g\|_{(F_{p(\cdot),q}^s)'} \left\| \left\{ 2^{sj} \delta_{j,k} \varphi_k * f \right\}_j \right\|_{L^{p(\cdot)}(\ell^q)} \leq c' \|f\|_{L^{p(\cdot)}}, \quad (4.41)$$

where $\delta_{k,k} = 1$ and $\delta_{j,k} = 0$ if $j \neq k$. This implies $a_k \in L^{p'(\cdot)}$ and the right-hand side of (4.40) has a sense. We assume $b_k \in \mathcal{S}(\mathbb{R}^n)$. Then, $c_k \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. By (4.32), we have

$$\langle \tilde{g}, f \rangle = \tilde{g}(f) = \sum_{k=0}^\infty a_k(f) \quad (4.42)$$

in $\mathcal{S}'(\mathbb{R}^n)$ and

$$a_k(f) = \sum_{l=0}^N a_k(c_l) = \sum_{l=0}^N \mathcal{F}a_k(\mathcal{F}^{-1}c_l) = c \sum_{l=0}^N (\mathcal{F}a_k \cdot \mathcal{F}^{-1}\varphi_l)(\mathcal{F}^{-1}b_l). \quad (4.43)$$

Since $(\mathcal{F}^{-1}h)(\xi) = (\mathcal{F}h)(-\xi)$, (4.29), (4.37), and (4.38) imply that

$$a_k(f) = c\mathcal{F}a_k(\mathcal{F}^{-1}b_k) = c \int_{\mathbb{R}^n} a_k b_k dx. \quad (4.44)$$

From this, (4.40) follows for $b_k \in \mathcal{S}(\mathbb{R}^n)$. Let $b_k \in L^{p(\cdot)}(\mathbb{R}^n)$, $b_{k,j} \in \mathcal{S}(\mathbb{R}^n)$ such that

$$b_{k,j} \longrightarrow b_k \quad (4.45)$$

in $L^{p(\cdot)}(\mathbb{R}^n)$ as $j \rightarrow \infty$ and

$$c_{k,j} = b_{k,j} * \varphi_k, \quad f_j = \sum_{k=0}^N c_{k,j}. \quad (4.46)$$

Thus, (4.40) holds for f_j and $b_{k,j}$. For any $h \in \mathcal{S}(\mathbb{R}^n)$, the least decreasing radial majorant of h is integrable. By the definition of $\varphi_k(x)$,

$$\varphi_k(x) = 2^{kn} \varphi(2^k x). \quad (4.47)$$

Hence, $c_{k,j}$ converges to c_k in $L^{p(\cdot)}(\mathbb{R}^n)$ as $j \rightarrow \infty$ by Theorem 4.8 and f_j converges to f in $F_{p(\cdot),q}^s(\mathbb{R}^n)$ as $j \rightarrow \infty$. Equation (4.40) follows from the last relation. $\square$

5. Duality and Reflexivity of Variable Exponent Function Spaces

Theorem 5.1. *Let $1 < q < \infty$, $s \in \mathbb{R}$, and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Let $A_{p(\cdot),q}^s$ be either $B_{p(\cdot),q}^s(\mathbb{R}^n)$ or $F_{p(\cdot),q}^s(\mathbb{R}^n)$. Then,*

$$\left(A_{p(\cdot),q}^s(\mathbb{R}^n) \right)' = A_{p'(\cdot),q'}^{-s}(\mathbb{R}^n), \quad (5.1)$$

where

$$p'(\cdot) = \frac{p(\cdot)}{p(\cdot) - 1}, \quad (5.2)$$

and $1/q + 1/q' = 1$. The spaces $F_{p(\cdot),q}^s(\mathbb{R}^n)$ and $B_{p(\cdot),q}^s(\mathbb{R}^n)$ are reflexive.

Proof. We will only prove (5.1) because the reflexivity of $A_{p(\cdot),q(\cdot)}^{s(\cdot)}$ follows from Theorem 3.1.

Step 1. We will first prove the $F_{p(\cdot),q}^s$ case. We use the arguments of the proof of [19, Theorems 7.1.7, and 7.2.2]. Let $f \in \mathcal{S}(\mathbb{R}^n)$, $g \in F_{p'(\cdot),q'}^{-s}(\mathbb{R}^n)$, and $\{\psi_k\}_{k=0}^\infty$ replaced by $\{\theta_k\}_{k=0}^\infty$

as in Definition 2.1. Then, $f = \sum_{k=0}^{\infty} f * \psi_k$ in $\mathcal{S}(\mathbb{R}^n)$, $f * \psi_k \in \mathcal{S}(\mathbb{R}^n)$, and $g = \sum_{k=0}^{\infty} g * \psi_k$ in $\mathcal{S}'(\mathbb{R}^n)$. It follows from Definition 2.1 that

$$g(f) = \sum_{k=0}^{\infty} (g * \psi_k)(f) = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{\infty} (g * \psi_k)(f * \psi_l) \right) = \sum_{k=0}^{\infty} \left(\sum_{r=-2N}^{2N} (g * \psi_k)(f * \psi_{k+r}) \right) \quad (5.3)$$

in $\mathcal{S}'(\mathbb{R}^n)$. Here, $\psi_j = 0$ for $j < 0$. Using Hölder inequality, we get

$$\begin{aligned} |g(f)| &\leq c \int_{\mathbb{R}^n} \left\| \{2^{-ks} g * \psi_k\} \right\|_{\ell^{q'}} \left\| \{2^{ks} f * \psi_k\} \right\|_{\ell^q} dx \\ &\leq c \left\| \{2^{-ks} g * \psi_k\} \right\|_{L^{p'(\cdot)}(\ell^{q'})} \left\| \{2^{ks} f * \psi_k\} \right\|_{L^{p(\cdot)}(\ell^q)} \\ &\leq c' \|g\|_{F_{p'(\cdot),q'}^{-s}} \|f\|_{F_{p(\cdot),q}^s}, \end{aligned} \quad (5.4)$$

where c' does not depend on g , f and ψ . Hence, we see $g \in (F_{p(\cdot),q}^s)'$ and $\|g\|_{(F_{p(\cdot),q}^s)'} \leq c \|g\|_{F_{p'(\cdot),q'}^{-s}}$ since $\mathcal{S}(\mathbb{R}^n)$ is dense in $F_{p(\cdot),q}^s(\mathbb{R}^n)$.

Step 2. Let $g \in (F_{p(\cdot),q}^s)'$. We assume that $\tilde{g}$ and the functions φ_k and ρ_k have the same sense as in Definition 4.7, Lemmas 4.9, and 4.10. We show that $\tilde{g} \in F_{p'(\cdot),q'}^{-s}(\mathbb{R}^n)$. For this purpose, we construct functions a_k as (4.37) and set

$$b_k(x) = \text{sgn } a_k \cdot |a_k(x)|^{q'-1} 2^{-sk'q'} \left\| \{2^{-js} a_j\}_{j=0}^N \right\|_{\ell^{q'}}^{p'(x)-q'}, \quad (5.5)$$

for $k = 0, 1, \dots, N$. If $a_k = 0$, we set $b_k = 0$. We have

$$\begin{aligned} \left\| \{2^{ks} b_k(x)\}_{k=0}^N \right\|_{\ell^q} &= \left\{ \sum_{k=0}^N \left(|a_k|^{q'-1} 2^{ks(1-q')} \left\| \{2^{-js} a_j\}_{j=0}^N \right\|_{\ell^{q'}}^{p'(x)-q'} \right)^q \right\}^{1/q} \\ &= \left\| \{2^{-js} a_j\}_{j=0}^N \right\|_{\ell^{q'}}^{p'(x)-q'} \left\{ \sum_{k=0}^N |a_k|^{q'} 2^{-ksq'} \right\}^{1/q} \\ &= \left\| \{2^{-js} a_j\}_{j=0}^N \right\|_{\ell^{q'}}^{q'/q + p'(x)-q'}, \end{aligned} \quad (5.6)$$

where we used $(q' - 1)q = q'$. We set

$$\alpha = \begin{cases} \text{ess inf}_{x \in \mathbb{R}^n} \frac{p'(x)}{p(x)}, & \text{if } \left\| \{2^{-ks} a_k\}_{k=0}^N \right\|_{L^{p'(\cdot)}(\ell^{q'})} \leq 1, \\ \text{ess sup}_{x \in \mathbb{R}^n} \frac{p'(x)}{p(x)}, & \text{if } \left\| \{2^{-ks} a_k\}_{k=0}^N \right\|_{L^{p'(\cdot)}(\ell^{q'})} > 1, \end{cases} \quad (5.7)$$

and $\lambda = \|\{2^{-ks}a_k\}_{k=0}^N\|_{L^{p(\cdot)}(\ell^{q'})}$. Then,

$$\begin{aligned}
 \int_{\mathbb{R}^n} \left(\frac{\|\{2^{ks}b_k\}_{k=0}^N\|_{\ell^q}}{\lambda^\alpha} \right)^{p(x)} dx &= \int_{\mathbb{R}^n} \left(\frac{\|\{2^{-js}a_j\}_{j=0}^N\|_{\ell^{q'}}^{q'/q+p'(x)-q'}}{\lambda^\alpha} \right)^{p(x)} dx \\
 &= \int_{\mathbb{R}^n} \frac{\|\{2^{-js}a_j\}_{j=0}^N\|_{\ell^{q'}}^{(q'/q+p'(x)-q')p(x)}}{\lambda^{\alpha p(x)}} dx \\
 &= \int_{\mathbb{R}^n} \left(\frac{\|\{2^{-js}a_j\}_{j=0}^N\|_{\ell^{q'}}}{\lambda^{\alpha(p(x)/p'(x))}} \right)^{p'(x)} dx \\
 &\leq \int_{\mathbb{R}^n} \left(\frac{\|\{2^{-js}a_j\}_{j=0}^N\|_{\ell^{q'}}}{\lambda} \right)^{p'(x)} dx \leq 1,
 \end{aligned} \tag{5.8}$$

where we used

$$\begin{aligned}
 \frac{q'}{q}p(x) + p(x)(p'(x) - q') &= p'(x), \\
 \operatorname{ess\,sup}_{x \in \mathbb{R}^n} \frac{p'(x)}{p(x)} &= \left(\operatorname{ess\,inf}_{x \in \mathbb{R}^n} \frac{p(x)}{p'(x)} \right)^{-1}, \\
 \operatorname{ess\,inf}_{x \in \mathbb{R}^n} \frac{p'(x)}{p(x)} &= \left(\operatorname{ess\,sup}_{x \in \mathbb{R}^n} \frac{p(x)}{p'(x)} \right)^{-1}.
 \end{aligned} \tag{5.9}$$

Hence, we have

$$\left\| \{2^{ks}b_k\}_{k=0}^N \right\|_{L^{p(\cdot)}(\ell^q)} \leq \left\| \{2^{-ks}a_k\}_{k=0}^N \right\|_{L^{p(\cdot)}(\ell^{q'})}^\alpha. \tag{5.10}$$

In particular, $b_k \in L^{p(\cdot)}(\mathbb{R}^n)$. Now, we construct the same function f by (4.39). Using Lemma 4.10, (4.37) and Theorem 4.8, we have

$$\begin{aligned}
 \int_{\mathbb{R}^n} \left\| \{2^{-ks}a_k\}_0^N \right\|_{\ell^{q'}}^{p'(x)} dx &= \int_{\mathbb{R}^n} \sum_{k=0}^N 2^{-skq'} |a_k|^{q'} \left\| \{2^{-js}a_j\}_{j=0}^N \right\|_{\ell^{q'}}^{p'(x)-q'} dx \\
 &= \int_{\mathbb{R}^n} \sum_{k=0}^N a_k b_k dx = \langle \tilde{g}, f \rangle \\
 &\leq c \|g\|_{(F_{p(\cdot),q}^s)'} \left\| \{2^{ks}b_k * \varphi_k\}_0^N \right\|_{L^{p(\cdot)}(\ell^q)} \\
 &\leq c \|g\|_{(F_{p(\cdot),q}^s)'} \left\| \{2^{ks}b_k\}_{k=0}^N \right\|_{L^{p(\cdot)}(\ell^q)}.
 \end{aligned} \tag{5.11}$$

We set

$$\beta = \begin{cases} \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p'(x), & \text{if } \left\| \{2^{-ks} a_k\}_{k=0}^N \right\|_{L^{p'(\cdot)}(\ell^{q'})} \leq 1, \\ \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p'(x), & \text{if } \left\| \{2^{-ks} a_k\}_{k=0}^N \right\|_{L^{p'(\cdot)}(\ell^{q'})} > 1. \end{cases} \quad (5.12)$$

Then, it follows from Theorem 3.4, (5.11), and (5.10) that

$$\left\| \{2^{-ks} a_k\}_{k=0}^N \right\|_{L^{p'(\cdot)}(\ell^{q'})}^{\beta-\alpha} \leq c \|g\|_{(F_{p(\cdot),q}^s)'}. \quad (5.13)$$

If $p'(x)$ is not a constant function, then $\beta - \alpha \neq 1$. So, we normalize the a_k . We put $a'_k = a_k/\lambda$ and

$$b'_k = \operatorname{sgn} a'_k \cdot |a'_k|^{q'-1} 2^{-sk'q'} \left\| \{2^{-js} a'_j\}_{j=0}^N \right\|_{\ell_{q'}}^{p'(\cdot)-q'}. \quad (5.14)$$

Then, we have

$$\begin{aligned} \left\| \{2^{ks} b'_k\}_{k=0}^N \right\|_{L^{p(\cdot)}(\ell^q)} &\leq 1, \\ \left\| \{2^{-ks} a'_k\}_{k=0}^N \right\|_{L^{p'(\cdot)}(\ell^{q'})}^{\beta-\alpha} &\leq c \left\| \frac{g}{\lambda} \right\|_{(F_{p(\cdot),q}^s)'}. \end{aligned} \quad (5.15)$$

Hence we have

$$\left\| \{2^{-ks} a_k\}_{k=0}^N \right\|_{L^{p'(\cdot)}(\ell^{q'})} \leq c \|g\|_{(F_{p(\cdot),q}^s)'}. \quad (5.16)$$

The last estimate, (4.32), (4.37), and Corollary 4.3 lead us to

$$\tilde{g} \in F_{p'(\cdot),q'}^{-s}, \quad \|\tilde{g}\|_{F_{p'(\cdot),q'}^{-s}} \leq c \|g\|_{(F_{p(\cdot),q}^s)'}. \quad (5.17)$$

We specialize the function $\rho(x)$ in Definition 4.7 by setting $(\mathcal{F}\rho)(\xi) = 1$ for

$$\frac{1}{\sqrt{2}} + 2\delta \leq |\xi| \leq \sqrt{2} - 2\delta, \quad (5.18)$$

where δ is a sufficiently small positive number. Then, we find a function $\tilde{\rho}(x) \in \mathcal{S}(\mathbb{R}^n)$ such that

$$0 \leq \mathcal{F}\tilde{\rho} \in C_0^\infty \left(\left\{ \xi \in \mathbb{R}^n : \sqrt{2} - 2\delta < |\xi| < \sqrt{2} + 5\delta \right\} \right), \quad (5.19)$$

$(\mathcal{F}\rho + \mathcal{F}\tilde{\rho} + \mathcal{F}\rho_1) = 1$ for $1/\sqrt{2} + 2\delta < |\xi| \leq 2(\sqrt{2} - 2\delta)$. As in (4.28), we construct $\tilde{\rho}_k(x)$, and, as in (4.32), we put

$$\tilde{\tilde{g}} = \sum_{k=0}^{\infty} \mathcal{F}^{-1}(\mathcal{F}\tilde{\rho}_k \mathcal{F}g) \quad (5.20)$$

in $\mathcal{S}'(\mathbb{R}^n)$. The function $\tilde{\tilde{g}}$ has properties similar to $\tilde{g}$, notably, (5.17) holds. Changing the function $\tilde{\rho}_0 = \tilde{\rho}$, we obtain $g = \tilde{g} + \tilde{\tilde{g}}$. Then, (5.17) implies $g \in F_{p'(\cdot),q'}^{-s}(\mathbb{R}^n)$:

$$\|g\|_{F_{p'(\cdot),q'}^{-s}} \leq c \|g\|_{(F_{p(\cdot),q}^s)'} \quad (5.21)$$

This completes Theorem 5.1 for the $F_{p(\cdot),q}^s(\mathbb{R}^n)$.

Step 3. Finally, we will prove the $B_{p(\cdot),q}^s$ case. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in B_{p'(\cdot),q'}^{-s}(\mathbb{R}^n)$. Then, we have $g \in (B_{p(\cdot),q}^s(\mathbb{R}^n))'$ and see that there exists a positive number c such that

$$\|g\|_{(B_{p(\cdot),q}^s)'} \leq c \|g\|_{B_{p'(\cdot),q'}^{-s}} \quad (5.22)$$

by the same manner as in the Step 1.

Step 4. Let $g \in (B_{p(\cdot),q}^s)'$ and $\tilde{g}$ as in (4.32). Then, we have $\tilde{g} \in (B_{p(\cdot),q}^s)'$ and

$$\|\tilde{g}\|_{(B_{p(\cdot),q}^s)'} \leq c \|g\|_{(B_{p(\cdot),q}^s)'} \quad (5.23)$$

by the same argument in the proof of Lemma 4.9. Furthermore, Lemma 4.10 also holds for $g \in (B_{p(\cdot),q}^s)'$ by the same argument in the proof of Lemma 4.10. Let a_k as in (4.37). Then, $a_k \in L^{p'(\cdot)}(\mathbb{R}^n)$. We set

$$b_k(x) = \text{sgn } a_k \cdot |a_k(x)|^{p'(x)-1} 2^{-skq'} \|a_k\|_{L^{p'(\cdot)}}^{q'-p'(x)}, \quad (5.24)$$

for $k = 0, 1, 2, \dots$. Then, we have $\|b_k\|_{L^{p(\cdot)}} \leq 2^{-skq'} \|a_k\|_{L^{p'(\cdot)}}^{q'/q}$ by the following calculating:

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\frac{|b_k(x)|}{2^{-ksq'} \|a_k\|_{L^{p'(\cdot)}}^{q'/q}} \right)^{p(x)} dx &= \int_{\mathbb{R}^n} \left(\frac{|a_k(x)|^{p'(x)-1} \|a_k\|_{L^{p'(\cdot)}}^{q'-p'(x)}}{\|a_k\|_{L^{p'(\cdot)}}^{q'/q}} \right)^{p(x)} dx \\ &= \int_{\mathbb{R}^n} \frac{|a_k(x)|^{p'(x)} \|a_k\|_{L^{p'(\cdot)}}^{(q'/q)p(x)-p'(x)}}{\|a_k\|_{L^{p'(\cdot)}}^{(q'/q)p(x)}} dx \\ &= \int_{\mathbb{R}^n} \left(\frac{|a_k(x)|}{\|a_k\|_{L^{p'(\cdot)}}} \right)^{p'(x)} dx = 1. \end{aligned} \quad (5.25)$$

Hence, we have

$$\begin{aligned} \left\| \left\{ 2^{ks} b_k \right\}_0^N \right\|_{\ell^q(L^{p(\cdot)})} &\leq \left(\sum_{k=0}^N 2^{skq(1-q')} \|a_k\|_{L^{p'(\cdot)}}^{q'} \right)^{1/q} = \left(\sum_{k=0}^N 2^{-skq'} \|a_k\|_{L^{p'(\cdot)}}^{q'} \right)^{q'/(q \cdot q)} \\ &= \left\| \left\{ 2^{-ks} a_k \right\}_0^N \right\|_{\ell^{q'}(L^{p'(\cdot)})}^{q'/q}. \end{aligned} \quad (5.26)$$

We have

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{k=0}^N a_k b_k dx &= \sum_{k=0}^N 2^{-skq'} \int_{\mathbb{R}^n} |a_k(x)|^{p'(x)} \|a_k\|_{L^{p'(\cdot)}}^{q'-p'(x)} dx \\ &= \sum_{k=0}^N 2^{-skq'} \|a_k\|_{L^{p'(\cdot)}}^{q'} = \left\| \left\{ 2^{-ks} a_k \right\}_0^N \right\|_{\ell^{q'}(L^{p'(\cdot)})}^{q'}. \end{aligned} \quad (5.27)$$

Hence, we have

$$\left\| \left\{ 2^{-ks} a_k \right\}_0^N \right\|_{\ell^{q'}(L^{p'(\cdot)})}^{q'} \leq \|g\|_{(B_{p(\cdot),q}^s)^{p'(\cdot)'}} \left\| \left\{ 2^{-ks} a_k \right\}_0^N \right\|_{\ell^{q'}(L^{p'(\cdot)})}^{q'/q} \quad (5.28)$$

by the same manner as in Step 2. It follows from this, as in Step 2, that

$$g \in B_{p'(\cdot),q'}^{-s}(\mathbb{R}^n), \quad \|g\|_{B_{p'(\cdot),q'}^{-s}} \leq c \|g\|_{(B_{p(\cdot),q}^s)^{p'(\cdot)'}}. \quad (5.29)$$

This complete the proof. □

The next corollary is an immediate consequence of Theorems 4.6 and 5.1.

Corollary 5.2. *Let $s \geq 0$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then,*

$$\left(L^{s,p(\cdot)}(\mathbb{R}^n) \right)' = F_{p'(\cdot),2}^{-s}(\mathbb{R}^n). \quad (5.30)$$

In addition, for $k \in \mathbb{N}_0$, one has

$$\left(W^{k,p(\cdot)}(\mathbb{R}^n) \right)' = F_{p'(\cdot),2}^{-k}(\mathbb{R}^n). \quad (5.31)$$

Hence, $L^{s,p(\cdot)}(\mathbb{R}^n)$ and $W^{k,p(\cdot)}(\mathbb{R}^n)$ are reflexive.

Remark 5.3. Let Ω be $\mathbb{R}^n$ or any measurable subset of $\mathbb{R}^n$. As we mentioned in Section 1, Kováčik and Rákosník [9] showed that the dual of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$ if $p(\cdot)$ is a measurable function on Ω with a range in $[1, \infty]$ and $p(\cdot) \in L^\infty(\Omega)$. Furthermore, $L^{p(\cdot)}(\Omega)$ is reflexive if $1 < p_- \leq p_+ < \infty$. There is no assumption of boundedness of Hardy-Littlewood maximal operator $\mathcal{M}$ on $L^{p(\cdot)}(\Omega)$.

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