Research Article

Simple Harmonic Oscillator Equation and Its Hyers-Ulam Stability

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Received 6 March 2008; Accepted 21 April 2008

Academic Editor: George Isac

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We solve the inhomogeneous simple harmonic oscillator equation and apply this result to obtain a partial solution to the Hyers-Ulam stability problem for the simple harmonic oscillator equation.

1. Introduction

Let X be a normed space over a scalar field $\mathbb{K}$ and let I be an open interval, where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Assume that $a_0, a_1, \dots, a_n : I \rightarrow \mathbb{K}$ are given continuous functions, $g : I \rightarrow X$ is a given continuous function, and $y : I \rightarrow X$ is an n times continuously differentiable function satisfying the inequality:

$$\left\| a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) + g(t) \right\| \leq \varepsilon, \quad (1.1)$$

for all $t \in I$ and for a given $\varepsilon > 0$. If there exists an n times continuously differentiable function $y_0 : I \rightarrow X$ satisfying

$$a_n(t)y_0^{(n)}(t) + a_{n-1}(t)y_0^{(n-1)}(t) + \dots + a_1(t)y_0'(t) + a_0(t)y_0(t) + g(t) = 0, \quad (1.2)$$

and $\|y(t) - y_0(t)\| \leq K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression of ε with $\lim_{\varepsilon \rightarrow 0} K(\varepsilon) = 0$, then we say that the above differential equation has the Hyers-Ulam stability. For more detailed definitions of the Hyers-Ulam stability, we refer the reader to [1–7].

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [8, 9]). Here, we will introduce a result of Alsina and Ger

(see [10]). If a differentiable function $f : I \rightarrow \mathbb{R}$ satisfies the inequality $|y'(t) - y(t)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \rightarrow \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \leq 3\varepsilon$ for any $t \in I$. This result has been generalized by Takahasi et al. Indeed, it was proved in [11] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $y'(t) = \lambda y(t)$ (see also [12, 13]).

Moreover, Miura et al. [14] investigated the Hyers-Ulam stability of n th order linear differential equation with complex coefficients. They [15] also proved the Hyers-Ulam stability of linear differential equations of first order, $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function.

Jung also proved the Hyers-Ulam stability of various linear differential equations of first order [16–19]. Moreover, he could successfully apply the power series method to the study of the Hyers-Ulam stability of Legendre differential equation (see [20]). Subsequently, the authors [21] investigated the Hyers-Ulam stability problem for Bessel differential equation by applying the same method.

In Section 2 of this paper, by using the ideas from [20, 21], we investigate the general solution of the inhomogeneous simple harmonic oscillator equation of the form:

$$y''(x) + \omega^2 y(x) = \sum_{m=0}^{\infty} a_m x^m, \quad (1.3)$$

where ω is a given positive number. Section 3 will be devoted to a partial solution of the Hyers-Ulam stability problem for the simple harmonic oscillator equation (2.1) in a subclass of analytic functions.

2. Inhomogeneous Simple Harmonic Oscillator Equation

A function is called a simple harmonic oscillator function if it satisfies the simple harmonic oscillator equation:

$$y''(x) + \omega^2 y(x) = 0. \quad (2.1)$$

The simple harmonic oscillator equation plays a great role in physics and engineering. In particular, it describes quantum particles confined in potential wells in quantum mechanics and the Hyers-Ulam stability of solutions of this equation is very important.

In this section, we define $c_0 = c_1 = 0$ and for $m \geq 1$,

$$\begin{aligned} c_{2m} &= \sum_{i=0}^{m-1} (-1)^{m-i-1} a_{2i} \frac{(2i)!}{(2m)!} \omega^{2m-2i-2}, \\ c_{2m+1} &= \sum_{i=0}^{m-1} (-1)^{m-i-1} a_{2i+1} \frac{(2i+1)!}{(2m+1)!} \omega^{2m-2i-2}, \end{aligned} \quad (2.2)$$

where we refer to (1.3) for the a_m . We can easily check that these c_m satisfy the following

$$a_m = (m+2)(m+1)c_{m+2} + \omega^2 c_m, \quad (2.3)$$

for any $m \in \{0, 1, 2, \dots\}$.

Lemma 2.1. (a) If the power series $\sum_{m=0}^{\infty} a_m x^m$ converges for all $x \in (-\rho, \rho)$ with $\rho > 1$, then the power series $\sum_{m=2}^{\infty} c_m x^m$ with c_m given in (2.2) satisfies the inequality $|\sum_{m=2}^{\infty} c_m x^m| \leq C_1/(1 - |x|)$ for some positive constant C_1 and for any $x \in (-1, 1)$.

(b) If the power series $\sum_{m=0}^{\infty} a_m x^m$ converges for all $x \in (-\rho, \rho)$ with $\rho \leq 1$, then for any positive $\rho_0 < \rho$, the power series $\sum_{m=2}^{\infty} c_m x^m$ with c_m given in (2.2) satisfies the inequality $|\sum_{m=2}^{\infty} c_m x^m| \leq C_2$ for any $x \in [-\rho_0, \rho_0]$ and for some positive constant C_2 which depends on ρ_0 . Since ρ_0 is arbitrarily close to ρ , this means that $\sum_{m=2}^{\infty} c_m x^m$ is convergent for all $x \in (-\rho, \rho)$.

Proof. (a) Since the power series $\sum_{m=0}^{\infty} a_m x^m$ is absolutely convergent on its interval of convergence, with $x = 1$, $\sum_{m=0}^{\infty} a_m$ converges absolutely, that is, $\sum_{m=0}^{\infty} |a_m| < M_1$ by some number M_1 .

We know that

$$\begin{aligned} |a_{2i}| \frac{(2i)!}{(2m)!} \omega^{2m-2i-2} &= \frac{|a_{2i}|}{2m(2m-1)} \frac{\omega}{(2m-2)} \cdots \frac{\omega}{(2i+1)} \\ &\leq \begin{cases} \frac{|a_{2i}|}{2m(2m-1)} & (\text{for } 0 < \omega \leq 1) \\ \frac{|a_{2i}|}{2m(2m-1)} \omega^{[\omega]} & (\text{for } \omega > 1) \end{cases} \\ &\leq \frac{|a_{2i}|}{2m(2m-1)} \max\{1, \omega^{\omega}\}, \end{aligned} \quad (2.4)$$

since for $\omega > 1$, each factor of the form ω/ℓ in the summand is either less than 1 if $\ell > [\omega]$, or is bigger than or equal to 1 if $\ell \leq [\omega]$, where $[\omega]$ denotes the largest integer less than or equal to ω . Thus, we obtain

$$\begin{aligned} |c_{2m}| &\leq \sum_{i=0}^{m-1} |a_{2i}| \frac{(2i)!}{(2m)!} \omega^{2m-2i-2} \\ &\leq \sum_{i=0}^{m-1} \frac{|a_{2i}|}{2m(2m-1)} \max\{1, \omega^{\omega}\} \\ &\leq \max\{1, \omega^{\omega}\} \sum_{i=0}^{\infty} |a_i| \\ &\leq \max\{M_1, M_1 \omega^{\omega}\} \equiv C_1. \end{aligned} \quad (2.5)$$

Similarly, we have

$$|a_{2i+1}| \frac{(2i+1)!}{(2m+1)!} \omega^{2m-2i-2} \leq \frac{|a_{2i+1}|}{2m(2m-1)} \max\{1, \omega^{\omega}\}, \quad (2.6)$$

and $|c_{2m+1}| \leq C_1$ for all $m \geq 1$.

Therefore, we get

$$\left| \sum_{m=2}^{\infty} c_m x^m \right| \leq \sum_{m=2}^{\infty} |c_m| |x|^m \leq C_1 \sum_{m=2}^{\infty} |x|^m \leq \frac{C_1}{1-|x|}, \quad (2.7)$$

for every $x \in (-1, 1)$.

(b) The power series $\sum_{m=0}^{\infty} a_m x^m$ is absolutely convergent on its interval of convergence, and, therefore, for any given $\rho_0 < \rho$, the series $\sum_{m=0}^{\infty} |a_m x^m|$ is convergent on $[-\rho_0, \rho_0]$ and

$$\sum_{m=0}^{\infty} |a_m| |x|^m \leq \sum_{m=0}^{\infty} |a_m| \rho_0^m \equiv M_2 \quad (2.8)$$

for any $x \in [-\rho_0, \rho_0]$. Now, it follows from (2.2), (2.4), (2.6), and (2.8) that

$$\begin{aligned} \left| \sum_{m=2}^{\infty} c_m x^m \right| &\leq \sum_{m=1}^{\infty} |c_{2m}| \rho_0^{2m} + \sum_{m=1}^{\infty} |c_{2m+1}| \rho_0^{2m+1} \\ &\leq \sum_{m=1}^{\infty} \rho_0^{2m} \sum_{i=0}^{m-1} |a_{2i}| \frac{(2i)!}{(2m)!} \omega^{2m-2i-2} \\ &\quad + \sum_{m=1}^{\infty} \rho_0^{2m+1} \sum_{i=0}^{m-1} |a_{2i+1}| \frac{(2i+1)!}{(2m+1)!} \omega^{2m-2i-2} \\ &\leq \sum_{m=1}^{\infty} \sum_{i=0}^{m-1} |a_{2i}| \rho_0^{2i} \frac{\max\{1, \omega^\omega\}}{2m(2m-1)} \\ &\quad + \sum_{m=1}^{\infty} \sum_{i=0}^{m-1} |a_{2i+1}| \rho_0^{2i+1} \frac{\max\{1, \omega^\omega\}}{2m(2m-1)} \\ &\leq \sum_{m=1}^{\infty} \frac{M_2 \max\{1, \omega^\omega\}}{2m(2m-1)} + \sum_{m=1}^{\infty} \frac{M_2 \max\{1, \omega^\omega\}}{2m(2m-1)} \\ &\leq \max\{M_2, M_2 \omega^\omega\} \sum_{m=1}^{\infty} \frac{1}{m(2m-1)} \\ &\leq \frac{3}{2} \max\{M_2, M_2 \omega^\omega\} \equiv C_2, \end{aligned} \quad (2.9)$$

for any $x \in [-\rho_0, \rho_0]$. □

Lemma 2.2. Suppose that the power series $\sum_{m=0}^{\infty} a_m x^m$ converges for all $x \in (-\rho, \rho)$ with some positive ρ . Let $\rho_1 = \min\{1, \rho\}$. Then, the power series $\sum_{m=2}^{\infty} c_m x^m$ with c_m given in (2.2) is convergent for all $x \in (-\rho_1, \rho_1)$. Further, for any positive $\rho_0 < \rho_1$, $|\sum_{m=2}^{\infty} c_m x^m| \leq C$ for any $x \in [-\rho_0, \rho_0]$ and for some positive constant C which depends on ρ_0 .

Proof. The first statement follows from the latter statement. Therefore, let us prove the latter statement. If $\rho \leq 1$, then $\rho_1 = \rho$. By Lemma 2.1(b), for any positive $\rho_0 < \rho = \rho_1$, $|\sum_{m=2}^{\infty} c_m x^m| \leq C_2$ for each $x \in [-\rho_0, \rho_0]$ and for some positive constant C_2 which depends on ρ_0 .

If $\rho > 1$, then by Lemma 2.1(a), for any positive $\rho_0 < 1 = \rho_1$, we get

$$\left| \sum_{m=2}^{\infty} c_m x^m \right| \leq \frac{C_1}{1-|x|} \leq \frac{C_1}{1-\rho_0} \leq \max \left\{ \frac{C_1}{1-\rho_0}, C_2 \right\} \equiv C, \quad (2.10)$$

for all $x \in [-\rho_0, \rho_0]$ and for some positive constant C which depends on ρ_0 . $\square$

Using these definitions and the lemmas above, we will now show that $\sum_{m=2}^{\infty} c_m x^m$ is a particular solution of the inhomogeneous simple harmonic oscillator equation (1.3).

Theorem 2.3. *Assume that ω is a given positive number and the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is $\rho > 0$. Let $\rho_1 = \min\{1, \rho\}$. Then, every solution $y : (-\rho_1, \rho_1) \rightarrow \mathbb{C}$ of the simple harmonic oscillator equation (1.3) can be expressed by*

$$y(x) = y_h(x) + \sum_{m=2}^{\infty} c_m x^m, \quad (2.11)$$

where $y_h(x)$ is a simple harmonic oscillator function and c_m are given by (2.2).

Proof. We show that $\sum_{m=2}^{\infty} c_m x^m$ satisfies (1.3). By Lemma 2.2, the power series $\sum_{m=2}^{\infty} c_m x^m$ is convergent for each $x \in (-\rho_1, \rho_1)$.

Substituting $\sum_{m=2}^{\infty} c_m x^m$ for $y(x)$ in (1.3) and collecting like powers together, it follows from (2.2) and (2.3) that (with $c_0 = c_1 = 0$)

$$y''(x) + \omega^2 y(x) = \sum_{m=0}^{\infty} \left[(m+2)(m+1)c_{m+2} + \omega^2 c_m \right] x^m = \sum_{m=0}^{\infty} a_m x^m, \quad (2.12)$$

for all $x \in (-\rho_1, \rho_1)$.

Therefore, every solution $y : (-\rho_1, \rho_1) \rightarrow \mathbb{C}$ of the inhomogeneous simple harmonic oscillator equation (1.3) can be expressed by

$$y(x) = y_h(x) + \sum_{m=2}^{\infty} c_m x^m, \quad (2.13)$$

where $y_h(x)$ is a simple harmonic oscillator function. $\square$

3. Partial Solution to Hyers-Ulam Stability Problem

In this section, we will investigate a property of the simple harmonic oscillator equation (2.1) concerning the Hyers-Ulam stability problem. That is, we will try to answer the question whether there exists a simple harmonic oscillator function near any approximate simple harmonic oscillator function.

Theorem 3.1. Let $y : (-\rho, \rho) \rightarrow \mathbb{C}$ be a given analytic function which can be represented by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least $\rho > 0$. Suppose there exists a constant $\varepsilon > 0$ such that

$$|y''(x) + \omega^2 y(x)| \leq \varepsilon, \quad (3.1)$$

for all $x \in (-\rho, \rho)$ and for some positive number ω . Let $\rho_1 = \min\{1, \rho\}$. Define $a_m = (m+2)(m+1)b_{m+2} + \omega^2 b_m$ for all $m \in \{0, 1, 2, \dots\}$ and suppose further that

$$\sum_{m=0}^{\infty} |a_m x^m| \leq K \left| \sum_{m=0}^{\infty} a_m x^m \right|, \quad (3.2)$$

for all $x \in (-\rho, \rho)$ and for some constant K . Then, there exists a simple harmonic oscillator function $y_h : (-\rho_1, \rho_1) \rightarrow \mathbb{C}$ such that

$$|y(x) - y_h(x)| \leq C\varepsilon, \quad (3.3)$$

for all $x \in [-\rho_0, \rho_0]$, where $\rho_0 < \rho_1$ is any positive number and C is some constant which depends on ρ_0 .

Proof. We assumed that $y(x)$ can be represented by a power series and

$$y''(x) + \omega^2 y(x) = \sum_{m=0}^{\infty} a_m x^m \quad (3.4)$$

also satisfies

$$\sum_{m=0}^{\infty} |a_m x^m| \leq K \left| \sum_{m=0}^{\infty} a_m x^m \right| \leq K\varepsilon, \quad (3.5)$$

for all $x \in (-\rho, \rho)$ from (3.1).

According to Theorem 2.3, $y(x)$ can be written as $y_h(x) + \sum_{m=2}^{\infty} c_m x^m$ for all $x \in (-\rho_1, \rho_1)$, where y_h is some simple harmonic oscillator function and c_m are given by (2.2). Then by Lemmas 2.1 and 2.2 and their proofs (replace M_1 and M_2 with $K\varepsilon$ in Lemma 2.1),

$$|y(x) - y_h(x)| = \left| \sum_{m=2}^{\infty} c_m x^m \right| \leq C\varepsilon \quad (3.6)$$

for all $x \in [-\rho_0, \rho_0]$, where $\rho_0 < \rho_1$ is any positive number and C is some constant which depends on ρ_0 .

Actually from the proof of Lemma 2.1, with both M_1 and M_2 replaced by $K\varepsilon$, we find $C_1 = \max\{K\varepsilon\omega^\omega, K\varepsilon\}$ and $C_2 = 3/2C_1$. Further from the proof of Lemma 2.2, we have

$$C\varepsilon = \max\left\{\frac{C_1}{1-\rho_0}, C_2\right\} = \max\left\{\frac{K\varepsilon}{1-\rho_0}\omega^\omega, \frac{K\varepsilon}{1-\rho_0}, \frac{3}{2}K\varepsilon\omega^\omega, \frac{3}{2}K\varepsilon\right\}, \quad (3.7)$$

we find

$$C = \max \left\{ \frac{K}{1-\rho_0} \omega^\omega, \frac{K}{1-\rho_0}, \frac{3}{2} K \omega^\omega, \frac{3}{2} K \right\}, \quad (3.8)$$

which completes the proof of our theorem. $\square$

4. Example

In this section, we show that there certainly exist functions $y(x)$ which satisfy all the conditions given in Theorem 3.1. We introduce an example related to the simple harmonic oscillator equation (1.3) for $\omega = 1/4$.

Let $y_h(x)$ be a simple harmonic oscillator function for $\omega = 1/4$ and let $y : (-1, 1) \rightarrow \mathbb{R}$ be an analytic function given by

$$y(x) = y_h(x) + \varepsilon \sum_{m=0}^{\infty} \frac{x^{2m}}{4^{m+1}}, \quad (4.1)$$

where ε is a positive constant. (We can easily show that the radius of convergence of the power series $\sum_{m=0}^{\infty} x^{2m}/4^{m+1}$ is 2). Then, we have

$$y''(x) + \frac{1}{16}y(x) = \sum_{m=0}^{\infty} a_m x^m, \quad (4.2)$$

where

$$a_m = \begin{cases} \frac{4m^2 + 12m + 9}{2^{m+6}} & (\text{for } m \in \{0, 2, 4, \dots\}) \\ 0 & (\text{for } m \in \{1, 3, 5, \dots\}), \end{cases} \quad (4.3)$$

$$\sum_{m=0}^{\infty} |a_m x^m| = \left| \sum_{m=0}^{\infty} a_m x^m \right|$$

for any $x \in (-1, 1)$. So we can here choose $K = 1$.

Furthermore, we get

$$\begin{aligned} \left| y''(x) + \frac{1}{16}y(x) \right| &\leq \sum_{m=0}^{\infty} \frac{8\varepsilon(m+1)(2m+1)}{4^{m+3}} |x|^{2m} + \sum_{m=0}^{\infty} \frac{\varepsilon}{4^{m+3}} |x|^{2m} \\ &\leq \sum_{m=0}^{\infty} \frac{15\varepsilon}{32} \frac{|x|^{2m}}{2^m} + \sum_{m=0}^{\infty} \frac{\varepsilon}{4^3} \frac{|x|^{2m}}{4^m} \\ &\leq \sum_{m=0}^{\infty} \frac{15\varepsilon}{32} \frac{1}{2^m} + \sum_{m=0}^{\infty} \frac{\varepsilon}{4^3} \frac{1}{4^m} \\ &< \varepsilon, \end{aligned} \quad (4.4)$$

and it follows from (4.1) that

$$|y(x) - y_h(x)| = \left| \varepsilon \sum_{m=0}^{\infty} \frac{x^{2m}}{4^{m+1}} \right| \leq \frac{\varepsilon}{4} \sum_{m=0}^{\infty} \frac{1}{4^m} = C\varepsilon, \quad (4.5)$$

for all $x \in (-1, 1)$, where we set $C = 1/3$.

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