Research Article

Jensen's Functionals on Time Scales

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Received 5 July 2012; Accepted 6 September 2012

Academic Editor: Ti-Jun Xiao

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We consider Jensen's functionals on time scales and discuss its properties and applications. Further, we define weighted generalized and power means on time scales. By applying the properties of Jensen's functionals on these means, we obtain several refinements and converses of Hölder's inequality on time scales.

1. Introduction

Time scales theory was initiated by Hilger [1], and now there is a lot of work in this field. For an introduction to the theory of dynamic equations on time scales, we refer to [2–4]. Time scales calculus provides unification, extension, and generalization of classical continuous and discrete results. In this paper, we give results only for Lebesgue Δ -integrals, but all the results obtained are also true if we take instead certain other time scales integrals such as the Cauchy delta, Cauchy nabla, α -diamond, multiple Riemann, or multiple Lebesgue integral.

Now, using the same notations as in [4, Chapter 5], we briefly give an introduction of Lebesgue Δ -integrals. Let $[a, b) \subseteq \mathbb{T}$ be a time scales interval defined by

$$[a, b) = \{t \in \mathbb{T} : a \leq t < b\} \quad \text{with } a, b \in \mathbb{T}, \quad a \leq b. \quad (1.1)$$

Suppose μ_Δ is the Lebesgue Δ -measure on $[a, b)$ and $f : [a, b) \rightarrow \mathbb{R}$ is a μ_Δ -measurable function. Then the Lebesgue Δ -integral of f on $[a, b)$ is denoted by

$$\int_{[a, b)} f d\mu_\Delta, \quad \int_{[a, b)} f(t) d\mu_\Delta(t), \quad \text{or} \quad \int_{[a, b)} f(t) \Delta t. \quad (1.2)$$

All theorems of the general Lebesgue integration theory, including the Lebesgue-dominated convergence theorem, hold also for Lebesgue Δ -integrals on $\mathbb{T}$. The following theorem compares the Lebesgue Δ -integral with the Riemann Δ -integral.

Theorem 1.1 (see [4, Theorem 5.81]). *Let $[a, b]$ be a closed bounded interval in $\mathbb{T}$, and let f be a bounded real-valued function defined on $[a, b]$. If f is Riemann Δ -integrable from a to b , then f is Lebesgue Δ -integrable on $[a, b]$, and*

$$R \int_a^b f(t) \Delta t = L \int_{[a,b]} f(t) \Delta t, \quad (1.3)$$

where R and L indicate the Riemann and Lebesgue integrals, respectively.

The results in this paper are based on the authors' results given in [5]. For related results we refer the reader to [6, 7]. The remaining theorems in this section are taken from [5]. Theorem 1.2 shows that the Lebesgue Δ -integral is a so-called *isotonic linear functional*. Theorem 1.3 recalls *Jensen's inequality* for Lebesgue Δ -integrals, while Theorem 1.4 states *Hölder's inequality* for Lebesgue Δ -integrals. These three results are used in the remainder of this paper.

Theorem 1.2 (see [5, Theorem 3.2]). *If f and g are Δ -integrable functions on $[a, b]$, then*

$$\begin{aligned} \int_{[a,b]} (\alpha f + \beta g) d\mu_\Delta &= \alpha \int_{[a,b]} f d\mu_\Delta + \beta \int_{[a,b]} g d\mu_\Delta \quad \forall \alpha, \beta \in \mathbb{R}, \\ f(t) \geq 0 \quad \forall t \in [a, b] &\text{ implies } \int_{[a,b]} f d\mu_\Delta \geq 0. \end{aligned} \quad (1.4)$$

Theorem 1.3 (see [5, Theorem 4.2]). *Assume $\Phi \in C(I, \mathbb{R})$ is convex, where $I \subseteq \mathbb{R}$ is an interval. Suppose $f : [a, b] \rightarrow I$ is Δ -integrable. Moreover, let $p : [a, b] \rightarrow \mathbb{R}$ be nonnegative and Δ -integrable such that $\int_{[a,b]} p d\mu_\Delta > 0$. Then*

$$\Phi \left(\frac{\int_{[a,b]} p f d\mu_\Delta}{\int_{[a,b]} p d\mu_\Delta} \right) \leq \frac{\int_{[a,b]} p (\Phi \circ f) d\mu_\Delta}{\int_{[a,b]} p d\mu_\Delta}. \quad (1.5)$$

Theorem 1.4 (see [5, Theorem 6.2]). *For $p \neq 1$, define $q = p/(p-1)$. Let w, f, g be nonnegative functions such that wf^p, wg^q, wfg are Δ -integrable on $[a, b]$. If $p > 1$, then*

$$\int_{[a,b]} wfg d\mu_\Delta \leq \left(\int_{[a,b]} wf^p d\mu_\Delta \right)^{1/p} \left(\int_{[a,b]} wg^q d\mu_\Delta \right)^{1/q}. \quad (1.6)$$

If $0 < p < 1$ and $\int_{[a,b]} wg^q d\mu_\Delta > 0$, or if $p < 0$ and $\int_{[a,b]} wf^p d\mu_\Delta > 0$, then (1.6) is reversed.

In Section 2, we define Jensen's functionals and, by using Jensen's inequality on time scales (Theorem 1.3), give some of their properties concerning superadditivity and monotonicity. In Section 3, we apply the properties of Jensen's functionals to generalized

means, defined on time scales, and obtain improvements of several classical inequalities on time scales. Finally, in Section 4, we give applications of Hölder's inequality on time scales (Theorem 1.4) and obtain several refinements and converses of this inequality.

2. Properties of Jensen's Functionals

Definition 2.1 (Jensen's functional). Assume $\Phi \in C(I, \mathbb{R})$, where $I \subseteq \mathbb{R}$ is an interval. Suppose $f : [a, b) \rightarrow I$ is Δ -integrable. Moreover, let $p : [a, b) \rightarrow \mathbb{R}$ be nonnegative and Δ -integrable such that $\int_{[a,b)} p d\mu_\Delta > 0$. Then we define *Jensen's functional* on time scales by

$$\mathcal{J}(\Phi, f, p; \mu_\Delta) := \int_{[a,b)} p(\Phi \circ f) d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta \Phi\left(\frac{\int_{[a,b)} p f d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right). \quad (2.1)$$

Remark 2.2. By Theorem 1.3, the following statements are obvious. If Φ is convex, then

$$\mathcal{J}(\Phi, f, p; \mu_\Delta) \geq 0, \quad (2.2)$$

while if Φ is concave, then

$$\mathcal{J}(\Phi, f, p; \mu_\Delta) \leq 0. \quad (2.3)$$

Theorem 2.3. Assume $\Phi \in C(I, \mathbb{R})$, where $I \subseteq \mathbb{R}$ is an interval. Suppose $f : [a, b) \rightarrow I$ is Δ -integrable. Also, let $p, q : [a, b) \rightarrow \mathbb{R}$ be nonnegative and Δ -integrable such that $\int_{[a,b)} p d\mu_\Delta > 0$ and $\int_{[a,b)} q d\mu_\Delta > 0$. If Φ is convex, then $\mathcal{J}(\Phi, f, \cdot; \mu_\Delta)$ is superadditive, that is,

$$\mathcal{J}(\Phi, f, p + q; \mu_\Delta) \geq \mathcal{J}(\Phi, f, p; \mu_\Delta) + \mathcal{J}(\Phi, f, q; \mu_\Delta), \quad (2.4)$$

and $\mathcal{J}(\Phi, f, \cdot; \mu_\Delta)$ is increasing, that is, $p \geq q$ with $\int_{[a,b)} p d\mu_\Delta > \int_{[a,b)} q d\mu_\Delta$ implies

$$\mathcal{J}(\Phi, f, p; \mu_\Delta) \geq \mathcal{J}(\Phi, f, q; \mu_\Delta). \quad (2.5)$$

Moreover, if Φ is concave, then $\mathcal{J}(\Phi, f, \cdot; \mu_\Delta)$ is subadditive and decreasing, that is, (2.4) and (2.5) hold in reverse order.

Proof. Let Φ be convex. Because the time scales integral is linear (see Theorem 1.2), it follows from Definition 2.1 that

$$\begin{aligned}
& \mathcal{J}(\Phi, f, p+q; \mu_\Delta) \\
&= \int_{[a,b)} (p+q)(\Phi \circ f) d\mu_\Delta - \int_{[a,b)} (p+q) d\mu_\Delta \Phi \left(\frac{\int_{[a,b)} (p+q) f d\mu_\Delta}{\int_{[a,b)} (p+q) d\mu_\Delta} \right) \\
&= \int_{[a,b)} (p+q)(\Phi \circ f) d\mu_\Delta - \int_{[a,b)} (p+q) d\mu_\Delta \times \\
&\quad \times \Phi \left(\frac{\int_{[a,b)} p d\mu_\Delta}{\int_{[a,b)} (p+q) d\mu_\Delta} \frac{\int_{[a,b)} p f d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} + \frac{\int_{[a,b)} q d\mu_\Delta}{\int_{[a,b)} (p+q) d\mu_\Delta} \frac{\int_{[a,b)} q f d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta} \right) \quad (2.6) \\
&\geq \int_{[a,b)} p(\Phi \circ f) d\mu_\Delta + \int_{[a,b)} q(\Phi \circ f) d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta \Phi \left(\frac{\int_{[a,b)} p f d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right) \\
&\quad - \int_{[a,b)} q d\mu_\Delta \Phi \left(\frac{\int_{[a,b)} q f d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta} \right) \\
&= \mathcal{J}(\Phi, f, p; \mu_\Delta) + \mathcal{J}(\Phi, f, q; \mu_\Delta).
\end{aligned}$$

If $p \geq q$, we have $p - q \geq 0$. Now, because Jensen's functional is superadditive (see above) and nonnegative (see Theorem 1.2), we have

$$\begin{aligned}
\mathcal{J}(\Phi, f, p; \mu_\Delta) &= \mathcal{J}(\Phi, f, q + p - q; \mu_\Delta) \\
&\geq \mathcal{J}(\Phi, f, q; \mu_\Delta) + \mathcal{J}(\Phi, f, p - q; \mu_\Delta) \quad (2.7) \\
&\geq \mathcal{J}(\Phi, f, q; \mu_\Delta).
\end{aligned}$$

On the other hand, if Φ is concave, then the reversed inequalities of (2.4) and (2.5) can be obtained in a similar way. $\square$

Corollary 2.4. *Let Φ, f, p, q satisfy the hypotheses of Theorem 2.3. Further, suppose there exist nonnegative constants m and M such that*

$$\begin{aligned}
& Mq(t) \geq p(t) \geq mq(t) \quad \forall t \in [a, b), \\
& M \int_{[a,b)} q d\mu_\Delta > \int_{[a,b)} p d\mu_\Delta > m \int_{[a,b)} q d\mu_\Delta. \quad (2.8)
\end{aligned}$$

If Φ is convex, then

$$M\mathcal{J}(\Phi, f, q; \mu_\Delta) \geq \mathcal{J}(\Phi, f, p; \mu_\Delta) \geq m\mathcal{J}(\Phi, f, q; \mu_\Delta), \quad (2.9)$$

while if Φ is concave, then the inequalities in (2.9) hold in reverse order.

Proof. By using Definition 2.1, we have

$$\begin{aligned}\mathcal{J}(\Phi, f, mq; \mu_\Delta) &= m\mathcal{J}(\Phi, f, q; \mu_\Delta), \\ \mathcal{J}(\Phi, f, Mq; \mu_\Delta) &= M\mathcal{J}(\Phi, f, q; \mu_\Delta).\end{aligned}\tag{2.10}$$

Now the result follows from the second property of Theorem 2.3. $\square$

Corollary 2.5. *Let Φ, f, p satisfy the hypotheses of Theorem 2.3. Further, assume that p attains its minimum value and its maximum value on its domain. If Φ is convex, then*

$$\left[\max_{t \in [a,b]} p(t) \right] \mathfrak{J}(\Phi, f; \mu_\Delta) \geq \mathcal{J}(\Phi, f, p; \mu_\Delta) \geq \left[\min_{t \in [a,b]} p(t) \right] \mathfrak{J}(\Phi, f; \mu_\Delta), \tag{2.11}$$

where

$$\mathfrak{J}(\Phi, f; \mu_\Delta) := \int_{[a,b]} (\Phi \circ f) d\mu_\Delta - (b-a)\Phi\left(\frac{\int_{[a,b]} f d\mu_\Delta}{b-a}\right). \tag{2.12}$$

Moreover, if Φ is concave, then the inequalities in (2.11) hold in reverse order.

Proof. Let p attain its minimum value $\underline{p}$ and its maximum value $\bar{p}$ on its domain $[a, b]$. Then

$$\bar{p} = \max_{t \in [a,b]} p(t) \geq p(x) \geq \min_{t \in [a,b]} p(t) = \underline{p}. \tag{2.13}$$

By Definition 2.1, we have

$$\begin{aligned}\mathcal{J}(\Phi, f, \bar{p}; \mu_\Delta) &= \bar{p}\mathfrak{J}(\Phi, f; \mu_\Delta), \\ \mathcal{J}(\Phi, f, \underline{p}; \mu_\Delta) &= \underline{p}\mathfrak{J}(\Phi, f; \mu_\Delta).\end{aligned}\tag{2.14}$$

Now the result follows from the second property of Theorem 2.3. $\square$

Remark 2.6. The first inequality in (2.11) gives a converse of Jensen's inequality on time scales, and the second inequality in (2.11) gives a refinement of the observed inequality.

Example 2.7 (see [8, Remark 4]). Let us take the discrete form of Jensen's functional (2.1). For this, let $\mathbb{T} = \mathbb{Z}$, $n \in \mathbb{N}$, $a = 1$, $b = n + 1$, and $f(i) = x_i$, $p(i) = p_i$ for $i \in [a, b] = \{1, 2, \dots, n\}$. Then (2.1) becomes

$$\mathcal{J}(\Phi, x, p) = \sum_{i=1}^n p_i \Phi(x_i) - P_n \Phi\left(\frac{\sum_{i=1}^n p_i x_i}{P_n}\right), \tag{2.15}$$

where

$$x := (x_1, x_2, \dots, x_n) \in I^n, \quad p := (p_1, p_2, \dots, p_n) \in \mathbb{R}_+^n, \quad P_n := \sum_{i=1}^n p_i. \quad (2.16)$$

Under these notations, (2.11) takes the form

$$\left[\max_{1 \leq i \leq n} p_i \right] \mathfrak{J}(\Phi, x) \geq \mathcal{J}(\Phi, x, p) \geq \left[\min_{1 \leq i \leq n} p_i \right] \mathfrak{J}(\Phi, x), \quad (2.17)$$

where

$$\mathfrak{J}(\Phi, x) = \sum_{i=1}^n \Phi(x_i) - n\Phi\left(\frac{\sum_{i=1}^n x_i}{n}\right). \quad (2.18)$$

Example 2.8 (see [8, Remark 5]). In addition to the notation introduced in Example 2.7, let $q(i) = q_i > 0$ for $i \in [a, b) = \{1, 2, \dots, n\}$ and put $q := (q_1, q_2, \dots, q_n)$. Using

$$m = \min_{1 \leq i \leq n} \frac{p_i}{q_i}, \quad M = \max_{1 \leq i \leq n} \frac{p_i}{q_i} \quad (2.19)$$

in Corollary 2.4, (2.9) becomes

$$\left[\max_{1 \leq i \leq n} \frac{p_i}{q_i} \right] \mathcal{J}(\Phi, x, q) \geq \mathcal{J}(\Phi, x, p) \geq \left[\min_{1 \leq i \leq n} \frac{p_i}{q_i} \right] \mathcal{J}(\Phi, x, q) \geq 0. \quad (2.20)$$

Example 2.9. Suppose $\mathbb{T} = \mathbb{R}$ and $a, b \in \mathbb{R}$. Then Jensen's functional (2.1) becomes

$$\int_{[a,b)} p(t) \Phi(f(t)) d\mu(t) - \int_{[a,b)} p(t) d\mu(t) \Phi\left(\frac{\int_{[a,b)} p(t) f(t) d\mu(t)}{\int_{[a,b)} p(t) d\mu(t)}\right). \quad (2.21)$$

3. Applications to Weighted Generalized Means

Definition 3.1 (weighted generalized mean). Assume $\chi \in C(I, \mathbb{R})$ is strictly monotone, where $I \subseteq \mathbb{R}$ is an interval. Suppose $f : [a, b) \rightarrow I$ is Δ -integrable. Also, let $p : [a, b) \rightarrow \mathbb{R}$ be nonnegative and Δ -integrable such that $\int_{[a,b)} p d\mu_\Delta > 0$. Then we define the weighted generalized mean on time scales by

$$\mathcal{M}_\chi(f, p; \mu_\Delta) := \chi^{-1}\left(\frac{\int_{[a,b)} p(\chi \circ f) d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right), \quad (3.1)$$

provided (3.1) is well defined.

Theorem 3.2. Assume $\chi, \psi \in C(I, \mathbb{R})$ are strictly monotone, where $I \subseteq \mathbb{R}$ is an interval. Suppose $f : [a, b] \rightarrow I$ is Δ -integrable. Moreover, let $p, q : [a, b] \rightarrow \mathbb{R}$ be nonnegative and Δ -integrable such that the functional

$$\int_{[a,b]} p d\mu_{\Delta} [\chi(\mathcal{M}_{\chi}(f, p; \mu_{\Delta})) - \chi(\mathcal{M}_{\psi}(f, p; \mu_{\Delta}))] \quad (3.2)$$

is well defined. If $\chi \circ \psi^{-1}$ is convex, then (3.2) is superadditive, that is,

$$\begin{aligned} & \int_{[a,b]} (p+q) d\mu_{\Delta} [\chi(\mathcal{M}_{\chi}(f, p+q; \mu_{\Delta})) - \chi(\mathcal{M}_{\psi}(f, p+q; \mu_{\Delta}))] \\ & \geq \int_{[a,b]} p d\mu_{\Delta} [\chi(\mathcal{M}_{\chi}(f, p; \mu_{\Delta})) - \chi(\mathcal{M}_{\psi}(f, p; \mu_{\Delta}))] \\ & \quad + \int_{[a,b]} q d\mu_{\Delta} [\chi(\mathcal{M}_{\chi}(f, q; \mu_{\Delta})) - \chi(\mathcal{M}_{\psi}(f, q; \mu_{\Delta}))], \end{aligned} \quad (3.3)$$

and (3.2) is increasing, that is, $p \geq q$ with $\int_{[a,b]} p d\mu_{\Delta} > \int_{[a,b]} q d\mu_{\Delta}$ implies

$$\begin{aligned} & \int_{[a,b]} p d\mu_{\Delta} [\chi(\mathcal{M}_{\chi}(f, p; \mu_{\Delta})) - \chi(\mathcal{M}_{\psi}(f, p; \mu_{\Delta}))] \\ & \geq \int_{[a,b]} q d\mu_{\Delta} [\chi(\mathcal{M}_{\chi}(f, q; \mu_{\Delta})) - \chi(\mathcal{M}_{\psi}(f, q; \mu_{\Delta}))]. \end{aligned} \quad (3.4)$$

Moreover, if $\chi \circ \psi^{-1}$ is concave, then (3.2) is subadditive and decreasing, that is, (3.3) and (3.4) hold in reverse order.

Proof. The functional defined in (3.2) is obtained by replacing Φ with $\chi \circ \psi^{-1}$ and f with $\psi \circ f$ in Jensen's functional (2.1), that is,

$$\begin{aligned} & \mathcal{J}(\chi \circ \psi^{-1}, \psi \circ f, p; \mu_{\Delta}) \\ & = \int_{[a,b]} p(\chi \circ \psi^{-1} \circ \psi \circ f) d\mu_{\Delta} - \int_{[a,b]} p d\mu_{\Delta} \left(\chi \circ \psi^{-1} \left(\frac{\int_{[a,b]} p(\psi \circ f) d\mu_{\Delta}}{\int_{[a,b]} p d\mu_{\Delta}} \right) \right) \\ & = \int_{[a,b]} p(\chi \circ f) d\mu_{\Delta} - \int_{[a,b]} p d\mu_{\Delta} \chi(\mathcal{M}_{\psi}(f, p; \mu_{\Delta})) \\ & = \int_{[a,b]} p d\mu_{\Delta} \chi(\mathcal{M}_{\chi}(f, p; \mu_{\Delta})) - \int_{[a,b]} p d\mu_{\Delta} \chi(\mathcal{M}_{\psi}(f, p; \mu_{\Delta})) \\ & = \int_{[a,b]} p d\mu_{\Delta} [\chi(\mathcal{M}_{\chi}(f, p; \mu_{\Delta})) - \chi(\mathcal{M}_{\psi}(f, p; \mu_{\Delta}))]. \end{aligned} \quad (3.5)$$

Now, all claims follow immediately from Theorem 2.3. □

Corollary 3.3. Let f, p, χ, ψ satisfy the hypotheses of Theorem 3.2. Further, assume that p attains its minimum value and its maximum value on its domain. If $\chi \circ \psi^{-1}$ is convex, then

$$\begin{aligned} & \left[\max_{t \in [a,b]} p(t) \right] (b-a) [\chi(\mathfrak{M}_\chi(f; \mu_\Delta)) - \chi(\mathfrak{M}_\psi(f; \mu_\Delta))] \\ & \geq \int_{[a,b]} p d\mu_\Delta [\chi(\mathcal{M}_\chi(f, p; \mu_\Delta)) - \chi(\mathcal{M}_\psi(f, p; \mu_\Delta))] \\ & \geq \left[\min_{t \in [a,b]} p(t) \right] (b-a) [\chi(\mathfrak{M}_\chi(f; \mu_\Delta)) - \chi(\mathfrak{M}_\psi(f; \mu_\Delta))], \end{aligned} \quad (3.6)$$

where

$$\mathfrak{M}_\eta(f; \mu_\Delta) := \eta^{-1} \left(\frac{\int_{[a,b]} (\eta \circ f) d\mu_\Delta}{b-a} \right), \quad \eta \in \{\chi, \psi\}. \quad (3.7)$$

Moreover, if $\chi \circ \psi^{-1}$ is concave, then the inequalities in (3.6) hold in reverse order.

Proof. The proof is omitted as it is similar to the proof of Corollary 2.5. $\square$

Definition 3.4 (weighted generalized power mean). Assume $r \in \mathbb{R}$. Suppose $f : [a, b] \rightarrow I$ is positive and Δ -integrable, where $I \subseteq \mathbb{R}$ is an interval. Moreover, let $p : [a, b] \rightarrow \mathbb{R}$ be nonnegative and Δ -integrable such that $\int_{[a,b]} p d\mu_\Delta > 0$. Then we define the weighted generalized power mean on time scales by

$$\mathcal{M}^{[r]}(f, p; \mu_\Delta) := \begin{cases} \left(\frac{\int_{[a,b]} p f^r d\mu_\Delta}{\int_{[a,b]} p d\mu_\Delta} \right)^{1/r} & \text{if } r \neq 0, \\ \exp \left(\frac{\int_{[a,b]} p \ln(f) d\mu_\Delta}{\int_{[a,b]} p d\mu_\Delta} \right) & \text{if } r = 0, \end{cases} \quad (3.8)$$

provided (3.8) is well defined.

Remark 3.5. The weighted generalized power mean defined in (3.8) follows from the weighted generalized mean defined in (3.1) by taking $\chi(x) = x^r$ ($x > 0$) in the weighted generalized mean.

Theorem 3.6. Assume $r, s \in \mathbb{R}$ with $r \neq 0$. Suppose $f : [a, b] \rightarrow I$ is positive and Δ -integrable, where $I \subseteq \mathbb{R}$ is an interval. Moreover, let $p, q : [a, b] \rightarrow \mathbb{R}$ be nonnegative and Δ -integrable such that the functional

$$\int_{[a,b]} p d\mu_\Delta \left\{ \left[\mathcal{M}^{[s]}(f, p; \mu_\Delta) \right]^s - \left[\mathcal{M}^{[r]}(f, p; \mu_\Delta) \right]^s \right\} \quad (3.9)$$

is well defined. If $\min\{0, r\} > s > \max\{0, r\}$, then (3.9) is superadditive (also if $r = 0$), that is,

$$\begin{aligned} & \int_{[a,b]} (p+q) d\mu_{\Delta} \left\{ \left[\mathcal{M}^{[s]}(f, p+q; \mu_{\Delta}) \right]^s - \left[\mathcal{M}^{[r]}(f, p+q; \mu_{\Delta}) \right]^s \right\} \\ & \geq \int_{[a,b]} p d\mu_{\Delta} \left\{ \left[\mathcal{M}^{[s]}(f, p; \mu_{\Delta}) \right]^s - \left[\mathcal{M}^{[r]}(f, p; \mu_{\Delta}) \right]^s \right\} \\ & \quad + \int_{[a,b]} q d\mu_{\Delta} \left\{ \left[\mathcal{M}^{[s]}(f, q; \mu_{\Delta}) \right]^s - \left[\mathcal{M}^{[r]}(f, q; \mu_{\Delta}) \right]^s \right\}, \end{aligned} \quad (3.10)$$

and (3.9) is increasing, that is, $p \geq q$ with $\int_{[a,b]} p d\mu_{\Delta} > \int_{[a,b]} q d\mu_{\Delta}$ implies

$$\begin{aligned} & \int_{[a,b]} p d\mu_{\Delta} \left\{ \left[\mathcal{M}^{[s]}(f, p; \mu_{\Delta}) \right]^s - \left[\mathcal{M}^{[r]}(f, p; \mu_{\Delta}) \right]^s \right\} \\ & \geq \int_{[a,b]} q d\mu_{\Delta} \left\{ \left[\mathcal{M}^{[s]}(f, q; \mu_{\Delta}) \right]^s - \left[\mathcal{M}^{[r]}(f, q; \mu_{\Delta}) \right]^s \right\}. \end{aligned} \quad (3.11)$$

Moreover, if $r > s > 0$ or $0 > s > r$, then (3.9) is subadditive and decreasing, that is, (3.10) and (3.11) hold in reverse order.

Proof. If $r \neq 0$, then let $\chi(x) = x^s$ and $\varphi(x) = x^r$ ($x > 0$) in Theorem 3.2. Then $(\chi \circ \varphi^{-1})(x) = x^{s/r}$ and therefore

$$(\chi \circ \varphi^{-1})''(x) = \frac{s(s-r)}{r^2} x^{s/r-2}. \quad (3.12)$$

Thus $\chi \circ \varphi^{-1}$ is convex if $\min\{0, r\} > s > \max\{0, r\}$ and concave if $r > s > 0$ or $0 > s > r$. If, however, $r = 0$, then let $\chi(x) = x^s$ and $\varphi(x) = \ln(x)$ ($x > 0$) in Theorem 3.2. Then $(\chi \circ \varphi^{-1})(x) = e^{sx}$. Thus $\chi \circ \varphi^{-1}$ is convex for $s \neq 0$. In either case the result follows now immediately from Theorem 3.2. $\square$

Corollary 3.7. Let r, s, f, p satisfy the hypotheses of Theorem 3.6. Further, assume that p attains its minimum value and its maximum value on its domain. If $\min\{0, r\} > s > \max\{0, r\}$, then

$$\begin{aligned} & \left[\max_{t \in [a,b]} p(t) \right] (b-a) \left\{ \left[\mathfrak{M}^{[s]}(f; \mu_{\Delta}) \right]^s - \left[\mathfrak{M}^{[r]}(f; \mu_{\Delta}) \right]^s \right\} \\ & \geq \int_{[a,b]} p d\mu_{\Delta} \left\{ \left[\mathcal{M}^{[s]}(f, p; \mu_{\Delta}) \right]^s - \left[\mathcal{M}^{[r]}(f, p; \mu_{\Delta}) \right]^s \right\} \\ & \geq \left[\min_{t \in [a,b]} p(t) \right] (b-a) \left\{ \left[\mathfrak{M}^{[s]}(f; \mu_{\Delta}) \right]^s - \left[\mathfrak{M}^{[r]}(f; \mu_{\Delta}) \right]^s \right\}, \end{aligned} \quad (3.13)$$

where

$$\mathfrak{M}^{[u]}(f; \mu_\Delta) := \begin{cases} \left(\frac{\int_{[a,b]} f^u d\mu_\Delta}{b-a} \right)^{1/u} & \text{if } u \in \mathbb{R} \setminus \{0\}, \\ \exp\left(\frac{\int_{[a,b]} \ln(f) d\mu_\Delta}{b-a} \right) & \text{if } u = 0. \end{cases} \quad (3.14)$$

Moreover, if $r > s > 0$ or $0 > s > r$, then the inequalities in (3.13) hold in reverse order.

Proof. The proof is omitted as it is similar to the proof of Corollary 2.5 followed by Theorem 3.6. $\square$

Example 3.8 (See [8, Remark 7]). From the discrete form of Corollary 3.7, that is, by using $\mathbb{T} = \mathbb{Z}$, we get a refinement and a converse of the arithmetic-geometric mean inequality. Using the notation as introduced in Example 2.7, let $x_i > 0$ for all $i \in [a, b]$ and $s = 1$, $r = 0$. Then (3.13) becomes

$$\begin{aligned} n \left[\max_{1 \leq i \leq n} p_i \right] [A_n(x) - G_n(x)] &\geq P_n [\mathcal{M}^{[1]}(x, p) - \mathcal{M}^{[0]}(x, p)] \\ &\geq n \left[\min_{1 \leq i \leq n} p_i \right] [A_n(x) - G_n(x)] \geq 0, \end{aligned} \quad (3.15)$$

where

$$\mathcal{M}^{[r]}(x, p) = \begin{cases} \left(\frac{\sum_{i=1}^n p_i x_i^r}{P_n} \right)^{1/r} & \text{if } r \in \mathbb{R} \setminus \{0\}, \\ \left(\prod_{i=1}^n x_i^{p_i} \right)^{1/P_n} & \text{if } r = 0, \end{cases} \quad (3.16)$$

$$A_n(x) = \frac{\sum_{i=1}^n x_i}{n}, \quad G_n(x) = \left(\prod_{i=1}^n x_i \right)^{1/n}.$$

The first inequality in (3.15) gives a converse and the second one gives a refinement of the arithmetic geometric-mean inequality of $\mathcal{M}^{[1]}(x, p)$ and $\mathcal{M}^{[0]}(x, p)$.

Theorem 3.9. Let r, f, p, q satisfy the hypotheses of Theorem 3.6. Suppose that the functional

$$\int_{[a,b]} p d\mu_\Delta \left\{ \frac{\int_{[a,b]} p \ln(f) d\mu_\Delta}{\int_{[a,b]} p d\mu_\Delta} - \ln(\mathcal{M}^{[r]}(f, p; \mu_\Delta)) \right\} \quad (3.17)$$

is well defined. If $r < 0$, then (3.17) is superadditive, that is,

$$\begin{aligned} & \int_{[a,b)} (p+q) d\mu_{\Delta} \left\{ \frac{\int_{[a,b)} (p+q) \ln(f) d\mu_{\Delta}}{\int_{[a,b)} (p+q) d\mu_{\Delta}} - \ln(\mathcal{M}^{[r]}(f, p+q; \mu_{\Delta})) \right\} \\ & \geq \int_{[a,b)} p d\mu_{\Delta} \left\{ \frac{\int_{[a,b)} p \ln(f) d\mu_{\Delta}}{\int_{[a,b)} p d\mu_{\Delta}} - \ln(\mathcal{M}^{[r]}(f, p; \mu_{\Delta})) \right\} \\ & \quad + \int_{[a,b)} q d\mu_{\Delta} \left\{ \frac{\int_{[a,b)} q \ln(f) d\mu_{\Delta}}{\int_{[a,b)} q d\mu_{\Delta}} - \ln(\mathcal{M}^{[r]}(f, q; \mu_{\Delta})) \right\}, \end{aligned} \quad (3.18)$$

and (3.17) is increasing, that is, $p \geq q$ with $\int_{[a,b)} p d\mu_{\Delta} > \int_{[a,b)} q d\mu_{\Delta}$ implies

$$\begin{aligned} & \int_{[a,b)} p d\mu_{\Delta} \left\{ \frac{\int_{[a,b)} p \ln(f) d\mu_{\Delta}}{\int_{[a,b)} p d\mu_{\Delta}} - \ln(\mathcal{M}^{[r]}(f, p; \mu_{\Delta})) \right\} \\ & \geq \int_{[a,b)} q d\mu_{\Delta} \left\{ \frac{\int_{[a,b)} q \ln(f) d\mu_{\Delta}}{\int_{[a,b)} q d\mu_{\Delta}} - \ln(\mathcal{M}^{[r]}(f, q; \mu_{\Delta})) \right\}. \end{aligned} \quad (3.19)$$

Moreover, if $r > 0$, then (3.17) is subadditive and decreasing, that is, (3.18) and (3.19) hold in reverse order.

Proof. Let $\chi(x) = \ln(x)$ and $\varphi(x) = x^r$ in Theorem 3.2. Then $\chi \circ \varphi^{-1}(x) = (1/r) \ln(x)$. Thus $\chi \circ \varphi^{-1}$ is convex if $r < 0$ and concave if $r > 0$. Now the rest of the proof follows immediately from Theorem 3.2. $\square$

Corollary 3.10. Let r, f, p satisfy the hypotheses of Theorem 3.6. Further, assume that p attains its minimum value and its maximum value on its domain. If $r < 0$, then

$$\begin{aligned} & \left[\max_{t \in [a,b)} p(t) \right] (b-a) \left\{ \frac{\int_{[a,b)} \ln(f) d\mu_{\Delta}}{b-a} - \ln(\mathfrak{M}^{[r]}(f; \mu_{\Delta})) \right\} \\ & \geq \int_{[a,b)} p d\mu_{\Delta} \left\{ \frac{\int_{[a,b)} p \ln(f) d\mu_{\Delta}}{\int_{[a,b)} p d\mu_{\Delta}} - \ln(\mathcal{M}^{[r]}(f, p; \mu_{\Delta})) \right\} \\ & \geq \left[\min_{t \in [a,b)} p(t) \right] (b-a) \left\{ \frac{\int_{[a,b)} \ln(f) d\mu_{\Delta}}{b-a} - \ln(\mathfrak{M}^{[r]}(f; \mu_{\Delta})) \right\}, \end{aligned} \quad (3.20)$$

where $\mathfrak{M}^{[r]}(f; \mu_{\Delta})$ is defined in (3.14). Moreover, if $r > 0$, then the inequalities in (3.20) hold in reverse order.

Proof. The proof is omitted as it is similar to the proof of Corollary 2.5 followed by Theorem 3.9. $\square$

Example 3.11 (see [8, Remark 8]). Again we consider $\mathbb{T} = \mathbb{Z}$. Using the notation as introduced in Example 3.8, the term $\int_{[a,b)} p \ln(f) d\mu_\Delta / \int_{[a,b)} p d\mu_\Delta$ takes the form

$$\frac{\sum_{i=1}^n p_i \ln(x_i)}{\sum_{i=1}^n p_i} = \ln \left(\prod_{i=1}^n x_i^{p_i} \right)^{1/P_n} = \ln(\mathcal{M}^{[0]}(x, p)), \quad (3.21)$$

and (3.20) becomes

$$\left[\frac{G_n(x)}{A_n(x)} \right]^{n \max_{1 \leq i \leq n} p_i} \geq \left[\frac{\mathcal{M}^{[0]}(x, p)}{\mathcal{M}^{[1]}(x, p)} \right]^{P_n} \geq \left[\frac{G_n(x)}{A_n(x)} \right]^{n \min_{1 \leq i \leq n} p_i}. \quad (3.22)$$

The inequalities in (3.22) provide a refinement and a converse of the arithmetic-geometric mean inequality in quotient form.

Example 3.12 (See [8, Remark 9]). The relations (3.15) and (3.22) also yield refinements and converses of Young's inequality. To see this, consider again $\mathbb{T} = \mathbb{Z}$. Using the notation as introduced in Example 3.8, define

$$x^p := (x_1^{p_1}, x_2^{p_2}, \dots, x_n^{p_n}), \quad p^{-1} := \left(\frac{1}{p_1}, \frac{1}{p_2}, \dots, \frac{1}{p_n} \right), \quad (3.23)$$

where x and p are positive n -tuples such that $\sum_{i=1}^n (1/p_i) = 1$. Then, (3.15) and (3.22) become

$$\begin{aligned} n \left[\max_{1 \leq i \leq n} \frac{1}{p_i} \right] [A_n(x^p) - G_n(x^p)] &\geq \mathcal{M}^{[1]}(x^p, p^{-1}) - \mathcal{M}^{[0]}(x^p, p^{-1}) \\ &\geq n \left[\min_{1 \leq i \leq n} \frac{1}{p_i} \right] [A_n(x^p) - G_n(x^p)], \end{aligned} \quad (3.24)$$

$$\left[\frac{G_n(x^p)}{A_n(x^p)} \right]^{n \max_{1 \leq i \leq n} (1/p_i)} \geq \frac{\mathcal{M}^{[0]}(x^p, p^{-1})}{\mathcal{M}^{[1]}(x^p, p^{-1})} \geq \left[\frac{G_n(x^p)}{A_n(x^p)} \right]^{n \min_{1 \leq i \leq n} (1/p_i)}. \quad (3.25)$$

The inequalities in (3.24) and (3.25) provide the refinements and converses of Young's inequality in difference and quotient form.

4. Improvements of Hölder's Inequality

Let $n \in \mathbb{N}$ and let $f_i : [a, b) \rightarrow \mathbb{R}$ be Δ -integrable for all $i \in \{1, 2, \dots, n\}$. Assume $p_i > 1$ for all $i \in \{1, 2, \dots, n\}$ are conjugate exponents, that is, $\sum_{i=1}^n (1/p_i) = 1$, and $\prod_{i=1}^n f_i^{1/p_i}$ is Δ -integrable on $[a, b)$. Hölder's inequality on time scales (Theorem 1.4) asserts that

$$\int_{[a,b)} \prod_{i=1}^n f_i^{1/p_i} d\mu_\Delta \leq \prod_{i=1}^n \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/p_i}. \quad (4.1)$$

Theorem 4.1. Let $p_i > 1$, $i \in \{1, 2, \dots, n\}$, be conjugate exponents, and let f_i , $i \in \{1, 2, \dots, n\}$, be nonnegative Δ -integrable functions such that $\prod_{i=1}^n f_i^{1/p_i}$ and $\prod_{i=1}^n f_i^{1/n}$ are nonnegative and Δ -integrable. Then the following inequalities hold:

$$\begin{aligned}
 & n \left[\max_{1 \leq i \leq n} \frac{1}{p_i} \right] \left[\prod_{i=1}^n \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/p_i} - \prod_{i=1}^n \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/p_i - 1/n} \left(\int_{[a,b)} \prod_{i=1}^n f_i^{1/n} d\mu_\Delta \right) \right] \\
 & \geq \prod_{i=1}^n \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/p_i} - \left(\int_{[a,b)} \prod_{i=1}^n f_i^{1/p_i} d\mu_\Delta \right) \\
 & \geq n \left[\min_{1 \leq i \leq n} \frac{1}{p_i} \right] \left[\prod_{i=1}^n \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/p_i} - \prod_{i=1}^n \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/p_i - 1/n} \left(\int_{[a,b)} \prod_{i=1}^n f_i^{\frac{1}{n}} d\mu_\Delta \right) \right].
 \end{aligned} \tag{4.2}$$

Proof. Let $x_i = [f_i / \int_{[a,b)} f_i d\mu_\Delta]^{1/p_i}$, $i \in \{1, 2, \dots, n\}$, in Example 3.12. Then the expressions in (3.24) become

$$\begin{aligned}
 \mathcal{M}_1(x^p, p^{-1}) - \mathcal{M}_0(x^p, p^{-1}) &= \sum_{i=1}^n \frac{f_i}{p_i \int_{[a,b)} f_i d\mu_\Delta} - \prod_{i=1}^n \frac{f_i^{1/p_i}}{\left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/p_i}}, \\
 A_n(x^p) - G_n(x^p) &= \frac{1}{n} \sum_{i=1}^n \frac{f_i}{\int_{[a,b)} f_i d\mu_\Delta} - \prod_{i=1}^n \frac{f_i^{1/n}}{\left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/n}}.
 \end{aligned} \tag{4.3}$$

Now, by applying the Δ -integral to the last two equations, we get

$$\begin{aligned}
 \int_{[a,b)} \left[\mathcal{M}_1(x^p, p^{-1}) - \mathcal{M}_0(x^p, p^{-1}) \right] d\mu_\Delta &= \sum_{i=1}^n \frac{\int_{[a,b)} f_i d\mu_\Delta}{p_i \int_{[a,b)} f_i d\mu_\Delta} - \frac{\int_{[a,b)} \left(\prod_{i=1}^n f_i^{1/p_i} \right) d\mu_\Delta}{\prod_{i=1}^n \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/p_i}} \\
 &= 1 - \frac{\int_{[a,b)} \left(\prod_{i=1}^n f_i^{1/p_i} \right) d\mu_\Delta}{\prod_{i=1}^n \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/p_i}},
 \end{aligned}$$

$$\begin{aligned}
\int_{[a,b)} [A_n(x^p) - G_n(x^p)] d\mu_\Delta &= \frac{1}{n} \sum_{i=1}^n \frac{\int_{[a,b)} f_i d\mu_\Delta}{\int_{[a,b)} f_i d\mu_\Delta} - \frac{\int_{[a,b)} \left(\prod_{i=1}^n f_i^{1/n} \right) d\mu_\Delta}{\prod_{i=1}^n \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/n}} \\
&= 1 - \frac{\int_{[a,b)} \left(\prod_{i=1}^n f_i^{1/n} \right) d\mu_\Delta}{\prod_{i=1}^n \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/n}}.
\end{aligned} \tag{4.4}$$

By applying the Δ -integral to the series of inequalities in (3.24), we obtain the required inequalities. $\square$

Remark 4.2. The first inequality in Theorem 4.1 gives a converse and the second one gives a refinement of Hölder's inequality on time scales.

Theorem 4.3. *Under the same assumption as in Theorem 4.1, the following inequalities hold:*

$$\begin{aligned}
&\left[\frac{n^n}{\prod_{i=1}^n \int_{[a,b)} f_i d\mu_\Delta} \right]^{\min_{1 \leq i \leq n} (1/p_i)} \times \\
&\quad \times \int_{[a,b)} \left[\sum_{i=1}^n \frac{f_i}{p_i \int_{[a,b)} f_i d\mu_\Delta} \right] \left[\frac{\prod_{i=1}^n f_i^{1/n}}{\sum_{i=1}^n \left(f_i / \int_{[a,b)} f_i d\mu_\Delta \right)} \right]^{n \min_{1 \leq i \leq n} (1/p_i)} d\mu_\Delta \\
&\geq \frac{\int_{[a,b)} \prod_{i=1}^n f_i^{1/p_i} d\mu_\Delta}{\prod_{i=1}^n \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/p_i}} \\
&\geq \left[\frac{n^n}{\prod_{i=1}^n \int_{[a,b)} f_i d\mu_\Delta} \right]^{\min_{1 \leq i \leq n} (1/p_i)} \times \\
&\quad \times \int_{[a,b)} \left[\sum_{i=1}^n \frac{f_i}{p_i \int_{[a,b)} f_i d\mu_\Delta} \right] \left[\frac{\prod_{i=1}^n f_i^{1/n}}{\sum_{i=1}^n \left(f_i / \int_{[a,b)} f_i d\mu_\Delta \right)} \right]^{n \max_{1 \leq i \leq n} (1/p_i)} d\mu_\Delta
\end{aligned} \tag{4.5}$$

provided that all expressions are well defined.

Proof. We consider relation (3.25) in the same settings as in Theorem 4.1. By inverting, (3.25) can be rewritten in the form

$$\begin{aligned}
\mathcal{M}_1(x^p, p^{-1}) \left[\frac{G_n(x^p)}{A_n(x^p)} \right]^{n \min_{1 \leq i \leq n} (1/p_i)} &\geq \mathcal{M}_0(x^p, p^{-1}) \\
&\geq \mathcal{M}_1(x^p, p^{-1}) \left[\frac{G_n(x^p)}{A_n(x^p)} \right]^{n \max_{1 \leq i \leq n} (1/p_i)}.
\end{aligned} \tag{4.6}$$

Now, if we consider the n -tuple $x = (x_1, x_2, \dots, x_n)$, where

$$x_i = \left[\frac{f_i}{\int_{[a,b)} f_i d\mu_\Delta} \right]^{1/p_i} \quad \forall i \in \{1, 2, \dots, n\}, \quad (4.7)$$

then the expressions that represent the means in (4.6) become

$$\begin{aligned} \mathcal{M}_1(x^p, p^{-1}) &= \sum_{i=1}^n \frac{f_i}{p_i \int_{[a,b)} f_i d\mu_\Delta}, & \mathcal{M}_0(x^p, p^{-1}) &= \prod_{i=1}^n \frac{f_i^{1/p_i}}{\left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/p_i}}, \\ A_n(x^p) &= \frac{1}{n} \sum_{i=1}^n \frac{f_i}{\int_{[a,b)} f_i d\mu_\Delta}, & G_n(x^p) &= \prod_{i=1}^n \frac{f_i^{1/n}}{\left(\int_{[a,b)} f_i d\mu_\Delta \right)^{1/n}}. \end{aligned} \quad (4.8)$$

Now, by taking the Δ -integral on (4.6) in described setting, we obtain the required inequalities. $\square$

Remark 4.4. The first inequality in Theorem 4.3 gives a refinement and the second one gives a converse of Hölder's inequality on time scales.

Corollary 4.5. *Let $r, s \in \mathbb{R}$ such that $1/r + 1/s = 1$. Further, assume that f, g are positive and Δ -integrable such that f attains its minimum value and its maximum value on its domain. If $r > 1$, then*

$$\begin{aligned} &\left[\max_{t \in [a,b)} f(t) \right] \left[(b-a)^{1/r} \left(\int_{[a,b)} \frac{g}{f} d\mu_\Delta \right)^{1/s} - \int_{[a,b)} \left(\frac{g}{f} \right)^{1/s} d\mu_\Delta \right] \\ &\geq \left(\int_{[a,b)} f d\mu_\Delta \right)^{1/r} \left(\int_{[a,b)} g d\mu_\Delta \right)^{1/s} - \int_{[a,b)} f^{1/r} g^{1/s} d\mu_\Delta \\ &\geq \left[\min_{t \in [a,b)} f(t) \right] \left[(b-a)^{1/r} \left(\int_{[a,b)} \frac{g}{f} d\mu_\Delta \right)^{1/s} - \int_{[a,b)} \left(\frac{g}{f} \right)^{1/s} d\mu_\Delta \right]. \end{aligned} \quad (4.9)$$

Moreover, if $0 < r < 1$, then the inequalities in (4.9) hold in reverse order.

Proof. The result follows from Corollary 2.5 by replacing f with g/f , p with f , and letting $\Phi(x) = -rsx^{1/s}$. Then Φ is convex on $(0, \infty)$, and we have

$$\begin{aligned}
\mathcal{J}\left(\Phi, \frac{g}{f}, f; \mu_\Delta\right) &= \int_{[a,b)} f \Phi\left(\frac{g}{f}\right) d\mu_\Delta - \int_{[a,b)} f d\mu_\Delta \Phi\left(\frac{\int_{[a,b)} g d\mu_\Delta}{\int_{[a,b)} f d\mu_\Delta}\right) \\
&= rs \left[\left(\int_{[a,b)} f d\mu_\Delta\right)^{1-1/s} \left(\int_{[a,b)} g d\mu_\Delta\right)^{1/s} - \int_{[a,b)} f^{1-1/s} g^{1/s} d\mu_\Delta \right] \\
&= rs \left[\left(\int_{[a,b)} f d\mu_\Delta\right)^{1/r} \left(\int_{[a,b)} g d\mu_\Delta\right)^{1/s} - \int_{[a,b)} f^{1/r} g^{1/s} d\mu_\Delta \right], \\
\mathfrak{J}\left(\Phi, \frac{g}{f}; \mu_\Delta\right) &= \int_{[a,b)} \Phi\left(\frac{g}{f}\right) d\mu_\Delta - (b-a) \Phi\left(\frac{\int_{[a,b)} g/f d\mu_\Delta}{b-a}\right) \\
&= rs \left[(b-a)^{1-1/s} \left(\int_{[a,b)} \frac{g}{f} d\mu_\Delta\right)^{1/s} - \int_{[a,b)} \left(\frac{g}{f}\right)^{1/s} d\mu_\Delta \right] \\
&= rs \left[(b-a)^{1/r} \left(\int_{[a,b)} \frac{g}{f} d\mu_\Delta\right)^{1/s} - \int_{[a,b)} \left(\frac{g}{f}\right)^{1/s} d\mu_\Delta \right].
\end{aligned} \tag{4.10}$$

If $r > 1$, then by substituting $\mathcal{J}(\Phi, g/f, f; \mu_\Delta)$ and $\mathfrak{J}(\Phi, g/f; \mu_\Delta)$ in (2.11), we get (4.9). If $0 < r < 1$, then $rs < 0$, and since the expressions $\mathcal{J}(\Phi, g/f, f; \mu_\Delta)$ and $\mathfrak{J}(\Phi, g/f; \mu_\Delta)$ contain the factor rs , we conclude that the inequalities in (4.9) hold in reverse order in that case. $\square$

Remark 4.6. The first inequality in (4.9) gives a converse and the second one gives a refinement of Hölder's inequality on time scales.

Corollary 4.7. Let $r, s \in \mathbb{R}$ such that $r > 0$ and $1/r + 1/s = 1$. Further, assume that f, g are positive and Δ -integrable such that f attains its minimum value and its maximum value on its domain. Then

$$\begin{aligned}
&\left[\max_{t \in [a,b)} f(t) \right] \times \\
&\quad \times \left[\left(\int_{[a,b)} f d\mu_\Delta\right)^{s-1} \int_{[a,b)} \frac{g}{f} d\mu_\Delta - \left(\frac{\int_{[a,b)} f d\mu_\Delta}{b-a}\right)^{s-1} \left(\int_{[a,b)} \left(\frac{g}{f}\right)^{1/s} d\mu_\Delta\right)^s \right] \\
&\geq \left[\left(\int_{[a,b)} f d\mu_\Delta\right)^{1/r} \left(\int_{[a,b)} g d\mu_\Delta\right)^{1/s} \right]^s - \left[\int_{[a,b)} f^{1/r} g^{1/s} d\mu_\Delta \right]^s
\end{aligned}$$

$$\begin{aligned}
&\geq \left[\min_{t \in [a,b)} f(t) \right] \times \\
&\quad \times \left[\left(\int_{[a,b)} f d\mu_{\Delta} \right)^{s-1} \int_{[a,b)} \frac{g}{f} d\mu_{\Delta} - \left(\frac{\int_{[a,b)} f d\mu_{\Delta}}{b-a} \right)^{s-1} \left(\int_{[a,b)} \left(\frac{g}{f} \right)^{1/s} d\mu_{\Delta} \right)^s \right].
\end{aligned} \tag{4.11}$$

Proof. In Corollary 2.5, replace f with $(g/f)^{1/s}$, p with f , and let $\Phi(x) = x^s/(s(s-1))$. Then Φ is convex on $(0, \infty)$. We get

$$\begin{aligned}
&\mathcal{J}\left(\Phi, \left(\frac{g}{f}\right)^{1/s}, f; \mu_{\Delta}\right) \\
&= \int_{[a,b)} f \Phi\left(\left(\frac{g}{f}\right)^{1/s}\right) d\mu_{\Delta} - \int_{[a,b)} f d\mu_{\Delta} \Phi\left(\frac{\int_{[a,b)} f^{1/r} g^{1/s} d\mu_{\Delta}}{\int_{[a,b)} f d\mu_{\Delta}}\right) \\
&= \frac{1}{s(s-1)} \left[\int_{[a,b)} g d\mu_{\Delta} - \left(\int_{[a,b)} f d\mu_{\Delta} \right)^{1-s} \left(\int_{[a,b)} f^{1/r} g^{1/s} d\mu_{\Delta} \right)^s \right], \tag{4.12} \\
&\mathfrak{J}\left(\Phi, \left(\frac{g}{f}\right)^{1/s}; \mu_{\Delta}\right) = \int_{[a,b)} \Phi\left(\left(\frac{g}{f}\right)^{1/s}\right) d\mu_{\Delta} - (b-a) \Phi\left(\frac{\int_{[a,b)} (g/f)^{1/s} d\mu_{\Delta}}{b-a}\right) \\
&= \frac{1}{s(s-1)} \left[\int_{[a,b)} \frac{g}{f} d\mu_{\Delta} - (b-a)^{1-s} \left(\int_{[a,b)} \left(\frac{g}{f}\right)^{1/s} d\mu_{\Delta} \right)^s \right].
\end{aligned}$$

Now, the result follows immediately from (2.11). $\square$

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