Research Article

On Necessary Condition for the Variable Exponent Hardy Inequality

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We derive a necessary condition for exponent functions $p, \beta$ such that the variable exponent Hardy inequality

$$\|x^\beta |x|^{-1} \int_0^x f(t) dt\|_{L^p([0, l])} \leq C \|x^\beta f\|_{L^p([0, l])}$$

holds for all $f \geq 0$ have been known (see [1–3]). According to mentioned works, if $\beta(0) < 1 - (1/p(0))$, $p^- := \inf\{p(x) : x \in (0, l)\} > 1$, then a sufficient condition is $p, \beta \in \Lambda$, where $\Lambda$ means a class of measurable functions $g : (0, l) \to (-\infty, \infty)$ for which there exist $\exists \ g(0), C > 0$

$$|g(x) - g(0)| \ln \frac{1}{|x|} \leq C, \quad 0 < x < \frac{1}{2}.$$
Theorems 2.1 and 2.2 for a precise statement condition and/or suitable continuous, and the problem of determining which exponent conditions are necessary and/or sufficient is an open one.

If the powers are not monotone, it follows from the results of the paper [2] that condition (1.2) is close to be sharp. Also in [2], the necessity of conditions $p^- > 1$ and $\beta(0) < 1 - 1/p(0)$ was proved. Recently, there have been quite a number of papers discussing the Hardy inequality in norms of the variable exponent Lebesgue spaces [3–11]. For problems of boundedness of classical integral operators in variable exponent Lebesgue spaces and regularity results for nonlinear equations with nonstandard growth condition, see monograph [12] and references therein.

2. Main Results

As to the basic properties of spaces $L^{p(\cdot)}$, we refer to [13]. Throughout this paper it is assumed that $p(x)$ is a measurable function on $(0, l)$ taking its values from the interval $[1, \infty)$ with $p^- := \sup \{ p(x) : x \in (0, l) \} < \infty$. The space of functions $L^{p(\cdot)}(0, l)$ is introduced as the class of measurable functions $f(x)$ in $(0, l)$, which have a finite $I_{p} f := \int_{0}^{l} |f(x)|^{p(x)} \, dx$-modular. A norm in $L^{p(\cdot)}(0, l)$ is given in the form

$$
\| f \|_{L^{p(\cdot)}(0, l)} = \inf \left\{ \lambda > 0 : I_{p} \left( \frac{f}{\lambda} \right) \leq 1 \right\}. 
$$

(2.1)

There exists a relation between modular and norm, which is expressed by the following inequalities:

$$
\| f \|_{p^-}^{p(\cdot)}(0, l) \leq I_{p} (f) \leq \| f \|_{p^+}^{p(\cdot)}(0, l), \quad 1 \geq \| f \|_{p(\cdot)^{+}}.
$$

(2.2)

$$
\| f \|_{L^{p(\cdot)}(0, l)} \leq I_{p} (f) \leq \| f \|_{L^{p(\cdot)'}(0, l)}, \quad 1 \leq \| f \|_{p(\cdot)'}.
$$

(2.3)

Such estimates allow us to perform our estimates in terms of a modular. In the following two theorems, we show that if functions $p, \beta$ are monotone, then condition (1.3) for them is necessary for inequality (1.1) to hold.

**Theorem 2.1.** Let $\beta \in \mathbb{R}$ a function $p : (0, l) \to [1, \infty)$ be increasing on $(0, \varepsilon)$ and such that $p(0) = \lim_{x \to 0} p(x)$ exists, $\beta(0) < 1 - 1/p(0)$, $p^- > 1$. Then for inequality (1.1) to hold, it is necessary that for the function $p(\cdot)$ condition (1.3) is satisfied.
Theorem 2.2. Let \( p \in \mathbb{R} \), let \( \beta : (0, l) \to [-\infty, \infty) \) be a function decreasing on \((0, \epsilon)\) such that \( \beta(0) = \lim_{x \to 0} \beta(x) \) exists, and let the conditions \( \beta(0) < 1 - 1/p(0) \), \( p^- > 1 \) be satisfied. Then for inequality (1.1) to hold, it is necessary that for the function \( \beta(\cdot) \) condition (1.3) is satisfied.

The following two theorems show that the logarithmic regularity conditions (1.2) for the functions \( p, \beta \) are essential for inequality (1.1) to hold.

Theorem 2.3. Let \( \beta \in \mathbb{R} \), and \( \delta_n = 4^{-n}, n \in \mathbb{N} \). There exist a sequence of functions \( \{ f_n \} \) and a function \( p : (0, l) \to [1, \infty) \) satisfying the conditions \( \beta < 1 - 1/p(0) \), \( p^- > 1 \) such that

\[
\lim_{n \to \infty} |p(\delta_n) - p(0)| \ln \frac{1}{\delta_n} = \infty,
\]

and inequality (1.1) is violated.

Theorem 2.4. Let \( p > 1 \), \( \delta_n = 4^{-n}, n \in \mathbb{N} \). Then there exist a sequence of functions \( \{ f_n \} \) and a function \( \beta : (0, l) \to (-\infty, \infty) \) satisfying the conditions \( \beta(0) < 1 - 1/p \),

\[
\lim_{n \to \infty} |\beta(\delta_n) - \beta(0)| \ln \frac{1}{\delta_n} = \infty,
\]

such that inequality (1.1) is violated.

3. Proofs of Main Results

Proof of Theorem 2.1. Denote \( I_{p(\cdot)}(f) = \int_0^l |f(t)|^{p(t)} \, dt \). By (2.2) note that the condition \( I_{p(\cdot)}(f) \leq 1 \) is equivalent to \( \|f\|_{L^{p(\cdot)}(0, l)} \leq 1 \).

Put \( \delta_k = \epsilon 4^{-k}, k \in \mathbb{N}, \) and \( f_k(x) = x^{-1/p(x)}-\beta \chi_{(\delta_k, 2\delta_k)}(x), x \in (0, l) \). Then for sufficiently large \( k \),

\[
I_{p(\cdot)}(x^{\beta(x)} f_k(x)) = \int_{\delta_k}^{2\delta_k} \left( t^{\beta(1/p(\cdot)-\beta)} \right)^{p(t)} \, dt\]
\[
= \int_{\delta_k}^{2\delta_k} t^{-1} \, dt = \ln 2.
\]

Also

\[
I_{p(\cdot)}(x^{\beta(x)-1} H(f_k(x))) \geq \int_{3\delta_k}^{4\delta_k} \left( x^{(\beta-1)} \int_{\delta_k}^{2\delta_k} t^{-(1/p(\cdot)-\beta)} \, dt \right)^{p(x)} \, dx
\]
\[
\geq C \int_{3\delta_k}^{4\delta_k} \delta_k^{(1-1/p(2\delta_k))} x^{(\beta-1)p(2\delta_k)} \, dx
\]
\[
\geq C \delta_k^{1-p(3\delta_k)/p(2\delta_k)} = C \delta_k^{(1/p^*) [p(3\delta_k)-p(2\delta_k)] ln(1/2\delta_k)}.
\]
Applying inequality (1.1), we have

$$|p(3\delta_k) - p(2\delta_k)| \ln \frac{1}{2\delta_k} \leq C, \quad k \in \mathbb{N},$$

(3.3)

which by using of monotony of $p$ and its boundedness implies (1.3).

This completes the proof of Theorem 2.1.

Proof of Theorem 2.2. Put $\delta_k = \varepsilon 4^{-k}, k \in \mathbb{N}$ and $f_k(x) = x^{-1/p} \beta(x)\chi_{(\delta_k, 2\delta_k)}(x), x \in (0, l)$. Then

\[
I_p(x^\beta f_k(x)) = \int_{\delta_k}^{2\delta_k} \left( t^{-1/p} t^{-\beta(t)} \right)^{p(t)} dt \\
= \int_{\delta_k}^{2\delta_k} t^{-1} dt = \ln 2.
\]

(3.4)

Also

\[
I_p(x^{\beta(x)-1} H(f_k(x))) \geq \int_{3\delta_k}^{4\delta_k} \left( x^{\beta(x)-1} \int_{\delta_k}^{2\delta_k} t^{-1/p} t^{-\beta(t)} dt \right)^{p(x)} dx \\
\geq C\delta_k^{[\beta(3\delta_k)-\beta(2\delta_k)]p} \geq C\delta_k^{[\beta(3\delta_k)-\beta(2\delta_k)] \ln(1/\delta_k)}.
\]

(3.5)

Applying inequality (1.1), we have

$$|\beta(2\delta_k) - \beta(3\delta_k)| \ln \frac{1}{\delta_k} \leq C, \quad k \in \mathbb{N}$$

(3.6)

which by using monotony of $\beta$ implies (1.3).

This completes the proof of Theorem 2.2.

Proof of Theorem 2.3. Let us assume that $f_k(x) = x^{-1/p} \beta(x)\chi_{(\delta_k, 2\delta_k)}(x), x \in (0, l)$. Fix $k \in \mathbb{N}$. We define the step function

$$p(x) = \begin{cases} 
  p_0 + \alpha_n & \text{if } x \in (2\delta_n, 4\delta_n), \\
  p_0 & \text{if } x \in (\delta_n, 2\delta_n), \quad n \in \mathbb{N}.
\end{cases}$$

(3.7)

Here $\{\alpha_n\}$ is a sequence of positive numbers that satisfies the condition

$$n\alpha_n \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty.$$
Then $\alpha_n \ln(1/\delta_n) \to \infty$ as $n \to \infty$; that is, condition (1.2) is not satisfied for the function $p(x)$. Also note that this function $p(\cdot)$ is not monotone. We have

$$I_{p(\cdot)}\left(x^{\beta} f_k(x)\right) = \int_{\delta_n}^{2\delta_n} \left(t^{\beta} t^{-1/p(t)-\beta}\right)^{p(t)} dt = \int_{\delta_n}^{2\delta_n} t^{-1} dt = \ln 2,$$

(3.9)

$$I_{p(\cdot)}\left(x^{\beta-1} H(f_k(x))\right) \geq \int_{\delta_n}^{2\delta_n} \left(\int_{\delta_n}^{2\delta_n} t^{-1/p(t)-\beta} dt\right)^{(p_0+a_1)} x^{(\beta-1)(p_0+a_3)} dx \geq C \int_{\delta_n}^{2\delta_n} x^{(\beta-1)(p_0+a_3)} dx \geq C \delta_n^{-a_1/p_0} = C e^{a_2/p_0 \ln(1/\delta_n)} \to \infty \text{ as } k \to \infty.$$

(3.10)

The last relation shows violating of inequality (1.1) for sufficiently large $k$.

Proof of Theorem 2.4. Let us assume that $f_k(x) = x^{-1/p(\beta(x))} \chi_{(\delta_n,2\delta_n)}(x), x \in (0,l), n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. We define the step function $\beta$ as

$$\beta(x) = \begin{cases} 
\beta_0 + \alpha_n & \text{if } x \in (\delta_n, 2\delta_n), \\
\beta_0 & \text{if } x \in (2\delta_n, 4\delta_n), \ n \in \mathbb{N},
\end{cases}$$

(3.11)

where $\alpha_n \ln(1/\delta_n) \to \infty$; that is, condition (1.2) is not satisfied for the function $\beta$. Note that this function $\beta$ is not monotone.

We have

$$I_{p(\cdot)}\left(x^{\beta(x)-1} f_n(x)\right) \geq C \delta_n^{-a_1} \to \infty \text{ as } n \to \infty,$$

(3.12)

$$I_{p(\cdot)}\left(x^{\beta(x)} f_n(x)\right) \leq C_0 \ln 2.$$

(3.13)

The last relation contradicts to inequality (1.1) for sufficiently large $k$.

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References


