Research Article

C^* -Algebras Generated by a System of Unilateral Weighted Shifts and Their Application

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We study the structure C^* -algebras generated by a system of unilateral weighted shifts. Finally the obtained results are applied to a class of integral equations.

1. Introduction

The structure of C^* -algebras generated by isometry is determined in [1, 2]. The structure is the same with the structure of C^* -algebras generated by unilateral weighted shift operators, that is, the structure of C^* -algebras generated by multiplication operators with the independent variable in the Hardy space on the unit disc. The analogue of the unit disc on $\mathbb{C}^N$ is the polydisc

$$\Delta^N = \left\{ z = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N : |z_i| < 1, 1 \leq i \leq N \right\} \quad (1.1)$$

or the unit ball

$$B^N = \left\{ z = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N : |z_1|^2 + |z_2|^2 + \dots + |z_N|^2 < 1 \right\}. \quad (1.2)$$

The structures of C^* -algebras generated by multiplication operators with the independent variable in the Hardy space on the unit ball and polydisc are different. To understand this difference we study the structure of C^* -algebras generated by system of unilateral weighted shifts.

Let I be a multiindex $(i_1, \dots, i_N)$ of integers and $I \pm \varepsilon_j$ denotes $(i_1, \dots, i_j \pm 1, \dots, i_N)$ for the multi-index I . Here ε_j is another multi-index $(\delta_{1j}, \dots, \delta_{Nj})$, where δ_{ij} is the Kronocker symbol.

Let $\{e_I\}_{I \geq 0}$ be an orthonormal basis of a separable complex Hilbert space H and let $\{\omega_{I,j} : I \geq 0, 1 \leq j \leq N\}$ be a bounded net of complex numbers. Denote by A_j the bounded linear operators whose effect on the elements of basis $\{e_I\}_{I \geq 0}$ of H is given as $A_j e_I = \omega_{I,j} e_{I+\varepsilon_j}$, $1 \leq j \leq N$. A family of N operators, denoted by $A = (A_1, A_2, \dots, A_N)$, is called a system of unilateral weighted shifts, and the numbers of $\{\omega_{I,j} : I \geq 0, 1 \leq j \leq N\}$ are called the weights of the system. It is known from [3, page 209, Corollary 2] that for shifts with nonzero weights $\{\omega_{I,j}\}$, without loss of generality we may always assume the weights are a set of positive real numbers, that is, the system A positive. It is possible to show that if there exists a solution for the multivariable moment problem for the net $\{\beta_I^2\}_{I \geq 0}$ where $\beta_{I+\varepsilon_j} = \omega_{I,j} \beta_I$, $\beta_0 = 1$, that is, if there exists the probability measure ν_A on $[0, 1]^N = [0, 1] \times \dots \times [0, 1]$ (N times) such that

$$\beta_I^2 = \int_{[0,1]^N} r_1^{2i_1} r_2^{2i_2} \dots r_N^{2i_N} d\nu(r_1, r_2, \dots, r_N), \quad (1.3)$$

then the system A is unitarily equivalent to the system of multiplication operators by the independent variables z_j , $1 \leq j \leq N$, on the space

$$H^2(\Delta^N, \mu_A) = \left\{ f = \sum_{I \geq 0} f_I z^I : z \in \Delta^N, \sum_{I \geq 0} |f_I|^2 \beta_I^2 < \infty \right\}, \quad (1.4)$$

where $\mu_A = \nu_A d\theta_1 d\theta_2 \dots d\theta_N$, $0 \leq \theta_i \leq 2\pi$.

The reader can find for more details of such operators in the article by Jewell and Lubin [3] and Ergezen and Sadik [4]. Furthermore the papers of Curto and Yoon [5] and Curto and Yan [6] are closely related to our study.

2. C^* -Algebras Generated by a System of Unilateral Weighted Shifts

Let Ω denote the family of the systems A which satisfy the functional model defined above. Moreover, let Ω_1 be a subset of Ω defined by

$$\Omega_1 = \left\{ A \in \Omega : \nu_A(U(1, 1, \dots, 1)) > 0 \text{ for arbitrary neighborhood } U(1, 1, \dots, 1) \right. \\ \left. \text{of the point } (1, 1, \dots, 1) \in [0, 1]^N \right\}. \quad (2.1)$$

Theorem 2.1 (see [4, page 25, Theorem 2]). *Let $A \in \Omega$. A necessary and sufficient condition for the operator algebra generated by the system A to be isometrically isomorphic to the polydisc algebra is that A belongs to Ω_1 .*

This theorem will be helpful in studying the structure of C^* -algebra $C^*(A)$ generated by $A \in \Omega_1$.

Let P denote the orthogonal projection of $L_2(\Delta^N, \mu_A)$ onto $H^2(\Delta^N, \mu_A)$ and let ψ lie in $C(\text{supp } \mu_A)$. Then the Toeplitz operator $T_\psi f = P(\psi f)$ for f in $H^2(\Delta^N, \mu_A)$.

Without loss of generality we may take $N = 2$. The following theorems for $N = 1$ were given by Sadikov [7].

Theorem 2.2. *Let $A \in \Omega$. If the algebra generated by the system A is polydisc algebra then the commutator ideal $\mathcal{J}$ of $C^*(A)$ contains properly the ideal of compact operators $\mathcal{K}$ and the quotient space $\mathcal{J}/\mathcal{K}$ is isometrically isomorphic to $C(T \times \{0, 1\}) \oplus \mathcal{K}$ and $C^*(A)/\mathcal{J} = C(T \times T)$, where T is unit circle and $\{0, 1\}$ is the two point space.*

Corollary 2.3. *Let $T_\psi \in C^*(A)$. Then necessary and sufficient condition for T_ψ to be Fredholm is that $\psi(z)$ is nonvanishing for $z \in T^2$ and $\psi|_{T^2}$ is homotopic to constant.*

Theorem 2.2 and Corollary 2.3 are proved by using the methods of Douglas and Howe in their study [8] and Curto and Muhly in [9].

Let

$$\begin{aligned} S_1 &= \{(r_1, r_2) \in [0, 1] \times [0, 1] : r_1^2 + r_2^2 \leq 1\}, \\ \widetilde{S}_1 &= \{(r_1, r_2) \in [0, 1] \times [0, 1] : r_1^2 + r_2^2 = 1\}, \end{aligned} \quad (2.2)$$

and let Ω_2 be subset of Ω defined by

$$\begin{aligned} \Omega_2 &= \left\{ A \in \Omega : \nu_A(U(a)) > 0 \text{ for arbitrary neighborhood } U(a) \right. \\ &\quad \left. \text{of the arbitrary point } a \in \widetilde{S}_1 \right\}. \end{aligned} \quad (2.3)$$

Theorem 2.4 (see [10, p. 1932, Theorem 2]). *Let $A \in \Omega$. A necessary and sufficient condition for the operator algebra generated by the system A to be isometrically isometric to the ball algebra is that A belongs to Ω_2 .*

Theorem 2.5. *Let $A \in \Omega$. If the algebra generated by the system A is ball algebra then $C^*(A)$ contains $\mathcal{K}$ ideal of compact operators and $C^*(A) = \{T_\psi + K : \psi \in C(\text{supp } \mu_A), K \in \mathcal{K}\}$. The quotient $C^*(A)/\mathcal{K}$ is naturally identified with $C(S_3)$ by a map $\sigma(T_\psi + K) = \psi|_{\widetilde{S}_3}$.*

It is enough to show compactness of the operators $A_i^* A_i - A_i A_i^*$, $i = 1, 2$ for proving that commutant of the algebra $C^*(A)$ is $\mathcal{K}$. For this, we just study the case $i = 2$. Then the case $i = 1$ is similarly showed, as well. With the basic computations, we obtain $(A_2^* A_2 - A_2 A_2^*)e_{(m,n)} = (\omega_{(m,n),2}^2 - \omega_{(m,n-1),2}^2)e_{(m,n)}$. If we show $\omega_{(m,n),2}^2 - \omega_{(m,n-1),2}^2$ converges the zero when $|I| \rightarrow \infty$ then the proof is completed. For this, we need the following lemma.

Lemma 2.6. *Let ν_A be a measure determined by the system $A \in \Omega_2$ and $I = (m, n)$. then the expression*

$$\frac{\int_{S_1} r_1^{2m} r_2^{2n+2} d\nu(r_1, r_2)}{\int_{S_1} r_1^{2m} r_2^{2n} d\nu(r_1, r_2)} - \frac{\int_{S_1} r_1^{2m} r_2^{2n} d\nu(r_1, r_2)}{\int_{S_1} r_1^{2m} r_2^{2n-2} d\nu(r_1, r_2)} \quad (*)$$

converges to zero when $|I| \rightarrow \infty$.

In view of the following process, the lemma is proved. Without loss of generality, we can take $n = 1$. Hence the second fractional of the expression $(*)$ becomes $I_1/I_2 := \int_{S_1} r_1^{2m} r_2^2 d\nu(r_1, r_2) / \int_{S_1} r_1^{2m} d\nu(r_1, r_2)$. It is enough to show I_1/I_2 goes to zero when $m \rightarrow \infty$. Take $r_{20} = \sqrt{\varepsilon/2}$ and $r_{10} = \sqrt{1 - (\varepsilon/2)} = \sqrt{1 - r_{20}^2}$ for given $\varepsilon > 0$. Consider $S_{10} := \{(r_1, r_2) : r_{10} < r_1 \leq 1\} \cap S_1$ and $S_{11} := S_1/S_{10}$. We have $I_1 = I_{10} + I_{11}$, where $I_{10} = \int_{S_{10}} r_1^{2m} r_2^2 d\nu(r_1, r_2)$. It easily shows that $I_{10} \leq (\varepsilon/2)I_2$ and $I_{11} \leq r_{10}^{2m} \nu_A(S_1)$ for all m . Moreover, take $r_{11} = (1 + r_{10})/2$; then we have $I_2 \geq r_{11}^{2m} \nu_A(S_{101})$ for all m , where $S_{101} = \{(r_1, r_2) : r_{11} < r_1 < 1\} \cap S_1$. Hence, there exists $M > 0$ such that for all $m > M$ it is obtained $I_1/I_2 < \varepsilon$.

It follows from Lemma 2.6 and a well-known result in C^* -algebras [2, page 212, Proposition 1] that $\mathcal{K} \subset C^*(A)$.

3. An Application

Throughout this section, we follow the notations and definitions in the preceding section.

Let H be separable complex Hilbert space, let $B(H)$ denote the algebra of linear bounded operators on H and let I be identity operator. Consider an operator

$$T = B_1 + B_2 S + K, \quad (3.1)$$

where B_1 and B_2 are the elements of a subalgebra Λ of $B(H)$ such that the image $\tau(\Lambda)$ is a commutative subalgebra of the algebra $B(H)/\mathcal{K}$ under the natural quotient map τ from $B(H)$ to $B(H)/\mathcal{K}$ and S denotes an automorphism in the algebra $\tau(\Lambda)$, that is, $\tau(S)\tau(B)\tau(S^{-1}) = \tau(B')$, where B and B' belong to Λ . It is obvious that if $B \in \Lambda$ then $SBS^{-1} = B' + K$, where $B' \in \Lambda$ and $K \in \mathcal{K}$.

Theorem 3.1 (see [11]). *If the operator $\tau(B_1)\tau(B'_1) - \tau(B_2)\tau(B'_2)$ has an inverse in $\tau(\Lambda)$ then $T = B_1 + B_2 S + K$ is Fredholm operator.*

Using the Theorem 3.1 we take $A \in \Omega_2$ and consider the operator $T_{\varphi_1} + T_{\varphi_2} S + K$, where T_{φ_1} and T_{φ_2} are Toeplitz operators in $C^*(A)$ and the operators S and T satisfy the conditions given above.

Moreover if we take into account the orthonormal projection P has the form

$$(Pf)(z_1, z_2) = \int_{\Delta^2} K(z, l) f(l_1, l_2) d\mu_A(l_1, l_2), \quad (3.2)$$

where K is the reproducing Bergman kernel of the functional space $H^2(\Delta^2, \mu_A)$, then the equation $Tf = \varphi$ is written in the form

$$\begin{aligned} & \int_{\Delta^2} K(z, l) \varphi_1(l_1, l_2) f(l_1, l_2) d\mu_A(l_1, l_2) + \int_{\Delta^2} K(z, l) \varphi_2(l_1, l_2) f(l_1, l_2) d\mu_A(l_1, l_2) \\ & + (Kf)(z_1, z_2) = \varphi(z_1, z_2). \end{aligned} \quad (3.3)$$

Hence we have the following theorem.

Theorem 3.2. *If the function $\psi_1(z_1, z_2)\psi_1(z_2, z_1) - \psi_2(z_1, z_2)\psi_2(z_2, z_1)$ does not vanish in S_3 , then all of Noether's theorems is true for (3.3). In particular, if we take $\omega_{I,1} = \sqrt{(m+1)/(2+m+n)}$, $\omega_{I,2} = \sqrt{(n+1)/(2+m+n)}$ and $Sf(l_1, l_2) = f(l_2, l_1)$ then (3.3) has the form*

$$\int_{S_3} \frac{\psi_1(l_1, l_2)f(l_1, l_2)}{(1 - z_1\bar{e}_1 - z_2\bar{e}_2)^2} ds + \int_{S_3} \frac{\psi_2(l_1, l_2)f(l_2, l_1)}{(1 - z_1\bar{e}_1 - z_2\bar{e}_2)^2} ds + (Kf)(z_1, z_2) = \varphi(z_1, z_2), \quad (3.4)$$

where ds is the surface measure in S_3 .

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