Research Article

Note on Boehmians for Class of Optical Fresnel Wavelet Transforms

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We extend the Fresnel-wavelet transform to the context of generalized functions, namely, Boehmians. At first, we study the Fresnel-wavelet transform in the sense of distributions of compact support. Based on this concept, we introduce two new spaces of Boehmians and proving certain related results. Further, we show that the extended transform establishes a linear and an isomorphic mapping between the Boehmian spaces. Moreover, conditions of continuity of the extended transform and its inverse with respect to δ and Δ convergence are discussed in some details.

1. Introduction

Optical integral transforms have been studied in several works, for example, [1–8]. However, is the Fresnel transform among all the great importance [5, 9] where for which the kernel takes the form of a complex exponential function $\exp[(i/2c)(ax_1^2 + bx_2^2)]$, for some constants a , b and c . The generalization of the Fresnel transform called the linear canonical transform was introduced in [10] and has recently attracted considerable attention in optics, see [4, 11]. One of the very well-known linear transform is the wavelet transform, see [12, 13] we have

$$\Omega_f(\mu, \lambda) = \frac{1}{\sqrt{\mu}} \int_{\mathbb{R}} f(x) \psi^*\left(\frac{x - \lambda}{\mu}\right) dx, \quad (1.1)$$

where $\psi(x)$ is named as the mother wavelet such that

$$\int_{\mathbf{R}} dx \psi(x) = 0, \quad (1.2)$$

$\mu \in \mathbf{R}_+$ and $\lambda \in \mathbf{R}$ are the transform dilate and translate of the wavelet ψ and ψ^* being the complex conjugate of ψ . The optical diffraction transform is described by the Fresnel integration in [5, 9] as follows:

$$f_w(x_2) = \frac{1}{\sqrt{2\pi i \gamma_1}} \int_{\mathbf{R}} \exp \left[\frac{i}{2\gamma_1} (\alpha_1 x_1^2 - 2x_1 x_2 + \alpha_2 x_2^2) \right] f(x_1) dx_1. \quad (1.3)$$

The parameters $(\alpha_1, \gamma_1, \gamma_2, \text{ and } \alpha_2)$ are elements of more ray transfer Matrix M describing optical systems, $\alpha_1 \alpha_2 - \gamma_1 \gamma_2 = 1$. For a details of Fresnel integrals, see [14, 15].

Note that many familiar transforms can be considered as special cases of the diffraction Fresnel transform. For example, if the parameters $\alpha_1, \gamma_1, \gamma_2, \text{ and } \alpha_2$ are written in the following matrix form:

$$\begin{pmatrix} \alpha_1 & \gamma_1 \\ \gamma_2 & \alpha_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (1.4)$$

then the diffraction Fresnel transform, the generalized Fresnel Transform becomes a fractional Fourier transform, see [11, 16, 17].

In the present work, we consider a combined optical transform of Fresnel and wavelet transforms, namely, the optical Fresnel-wavelet transform defined by [9]

$$f_w(x_2) = \frac{1}{\sqrt{2\pi i \gamma_1}} \int_{\mathbf{R}} K_{\lambda, \mu, x_2}(x_1) f(x_1) dx. \quad (1.5)$$

with kernel

$$K_{\lambda, \mu, x_2}(x_1) = \exp \left(\frac{i}{2\gamma_1} \left(\frac{\alpha_1 (x_1 - \lambda)^2}{\mu^2} - \frac{2x_2 (x_1 - \lambda)}{\mu} + \alpha_2 x_2^2 \right) \right). \quad (1.6)$$

The parameters $\alpha_1, \gamma_1, \gamma_2, \text{ and } \alpha_2$ appearing in (1.5) are elements of 2×2 matrix with unit determinant.

As the general single-mode squeezing operator of the generalized Fresnel transform is in wave optics, further its applications are having a faithful representation in the optical Fresnel-wavelet transform, see [9]. Therefore the combined optical Fresnel-wavelet transform can be more conveniently studied by the general single-mode squeezed operation.

However, our discussion is somewhat different and making more interesting. Since the theory of the optical Fresnel-wavelet transform of generalized functions has not been reported in the literature. Thus, we extend the optical Fresnel-wavelet transform to a specific space of generalized functions, namely, known as Boehmian space. In Section 2, we observe that the kernel function of the Fresnel-wavelet transform is a smooth function, and therefore

the optical Fresnel-wavelet transform is defined as an adjoint operator in the space of distributions. In a concrete way, Section 3 builds an appropriate space of Boehmians, whereas Section 4 constructs a new space of all images of Boehmians from Section 3. In Section 5, we define the optical Fresnel-wavelet transform of a Boehmian and study some of its general properties.

2. Optical Fresnel-Wavelet Transforms of Distributions

Let $\mathbb{E}(\mathbf{R})$ be the space of all test functions $\phi(x)$ of arbitrary support and $\mathbb{E}'(\mathbf{R})$ be its dual of distributions of bounded support, see, for example [12, 18–20]. Then, $\mathbb{E}(\mathbf{R})$ is a complete multinormed space with the set of norms as follows:

$$\xi_k(\phi)(x) = \sup_{x \in \mathbf{K}} |D_x^k \phi(x)|, \quad (2.1)$$

where $\mathbf{K}$ run through compact subsets of $\mathbf{R}$ and $\phi \in \mathbb{E}(\mathbf{R})$. It is clear that the kernel function of the optical Fresnel-wavelet transform

$$K_{\lambda, \mu, x_2}(x_1) = \exp \frac{i}{2\gamma_1} \left(\frac{\alpha_1(x_1 - \lambda)^2}{\mu^2} - \frac{2x_2(x_1 - \lambda)}{\mu} + \alpha_2 x_2^2 \right) \quad (2.2)$$

for each $x_2, \lambda \in \mathbf{R}$, $\mu \in \mathbf{R}_+$ is an element of $\mathbb{E}(\mathbf{R})$. This describes the distributional optical Fresnel-wavelet transform of bounded support as an adjoint operator as follows:

$$f_w(x_2) = T_f(x_2) = \left\langle f(x_1), \exp \frac{i}{2\gamma_1} \left(\frac{\alpha_1(x_1 - \lambda)^2}{\mu^2} - \frac{2x_2(x_1 - \lambda)}{\mu} + \alpha_2 x_2^2 \right) \right\rangle. \quad (2.3)$$

For convenience we sometimes write $T_f(x_2)$ instead of $f_w(x_2)$. Moreover, from (2.3), we observe that $T_f(x_2)$ is an analytic function satisfying the expression as follows:

$$D_{x_2}^k T_f(x_2) = \left\langle f(x_1), D_{x_2}^k K_{\lambda, \mu, x_2}(x_1) \right\rangle. \quad (2.4)$$

Further, $D_{x_2}^k T_f(x_2)$ is well defined since $D_{x_2}^k K_{\lambda, \mu, x_2}(x_1) \in \mathbb{E}(\mathbf{R})$, $K_{\lambda, \mu, x_2}(x_1)$ has its usual meaning where denoted by $*$ to be the usual convolution product [18, 20, 21]. Then we have the following lemma.

Lemma 2.1. *Let $f, g \in \mathbb{E}'(\mathbf{R})$ and $T_f = f_w(f(x))(x_2)$, $T_g = f_w(g(\tau))(x_2)$ be their respective optical Fresnel-wavelet transforms and $\lambda^2 = 2x\tau$ for all x and τ , then*

$$T_{f*g}(x_2) = \exp \frac{i}{2\gamma_1} \left(-\frac{2x_2\lambda}{\mu} - \alpha_2 x_2^2 \right) T_f(x_2) T_g(x_2), \quad (2.5)$$

where $T_{f*g}(x_2) = f_w((f * g)(x))(x_2)$.

Proof. Let $f, g \in \mathbb{E}'(\mathbf{R})$, $T_f = f_w(f(x))(x_2)$, and $T_g = f_w(g(\tau))(x_2)$, and then

$$\begin{aligned} T_{f*g}(x_2) &= \left\langle (f * g)(x), \exp \frac{i}{2\gamma_1} \left(\frac{\alpha_1(x-\lambda)^2}{\mu^2} - 2x_2 \frac{(x-\lambda)}{\mu} + \alpha_2 x_2^2 \right) \right\rangle \\ \text{i.e.,} \quad &= \left\langle f(x), \left\langle g(\tau), \exp \frac{i}{2\gamma_1} \left(\frac{\alpha_1(x+\tau-\lambda)^2}{\mu^2} - 2x_2 \frac{(x+\tau-\lambda)}{\mu} + \alpha_2 x_2^2 \right) \right\rangle \right\rangle. \end{aligned} \quad (2.6)$$

Hence, using properties of distributions and simple calculations we get

$$T_{f*g}(x_2) = \exp \frac{i}{2\gamma_1} \left(-\frac{2x_2\lambda}{\mu} - \alpha_2 x_2^2 \right) T_f(x_2) T_g(x_2). \quad (2.7)$$

The above theorem is known as the convolution theorem of the Fresnel-wavelet transform. $\square$

Let δ be the dirac delta function. Then the Fresnel-wavelet transform of δ is described as follows:

$$T_\delta = f_w(\delta(x))(x_2) = \exp \left(\frac{i}{2\gamma_1} \left(\alpha_1 \frac{\lambda^2}{\mu^2} + \frac{2x_2\lambda}{\mu} + \alpha_2 x_2^2 \right) \right), \quad (2.8)$$

Now e can easily deduce a corollary for the Lemma 2.1 as follows.

Corollary 2.2. Let $f, g \in \mathbb{E}'(\mathbf{R})$ and δ be the dirac delta function, and then

$$\begin{aligned} T_{f*\delta}(x_2) &= \exp \frac{i}{2\gamma_1} \left(\alpha_1 \frac{4\lambda^2 + 2x_2\tau}{\mu^2} \right) T_f(x_2), \\ T_{\delta*g}(x_2) &= \exp \frac{i}{2\gamma_1} \left(\alpha_1 \frac{4\lambda^2 + 2x_2\tau}{\mu^2} \right) T_g(x_2). \end{aligned} \quad (2.9)$$

Proof. It is a straightforward result of Lemma 2.1. $\square$

Theorem 2.3. The distributional optical Fresnel-wavelet transform f_w is linear.

Proof. It is obvious. $\square$

Lemma 2.4. Let $f, g \in \mathbb{E}'(\mathbf{R})$, $T_g(x_2) = f_w(g(x))(x_2)$, $T_f(x_2) = f_w(f(\tau))(x_2)$, and then one has

$$\begin{aligned} (1) \quad T_{(f*g)^{(k)}}(x_2) &= \exp \frac{i}{2\gamma_1} \left(-\frac{2x_2\lambda}{\mu} - \alpha_2 x_2^2 \right) T_{f^{(k)}}(x_2) T_g(x_2) \\ (2) \quad T_{(f*g)^{(k)}}(x_2) &= \exp \frac{i}{2\gamma_1} \left(-\frac{2x_2\lambda}{\mu} - \alpha_2 x_2^2 \right) T_f(x_2) T_{g^{(k)}}(x_2). \end{aligned}$$

Proof. It is a straightforward conclusion of the fact [20]. Consider

$$(f * g)^{(k)} = f^{(k)} * g = f * g^{(k)}. \quad (2.10)$$

□

3. The Boehmian Space $\mathbb{B}_1$

In this section, we assume that the reader is acquainted with the general construction of Boehmian spaces [6, 22–27]. Let $\mathbb{D}(\mathbf{R})$ be the Schwartz space of test functions of bounded support see [12, 20, 28]. The operation $\bullet$ between a distribution $f \in \mathbb{E}'(\mathbf{R})$ and a test function $\phi \in \mathbb{D}(\mathbf{R})$ is defined by

$$(f \bullet g)\psi(x) = f(g \odot \tau_x \psi), \quad (3.1)$$

where $\tau_x \psi(y) = \psi(x + y)$ and

$$(g \odot \tau_x \psi)(y) = \int_{\mathbf{R}} g(y) \tau_x \psi(y) dy. \quad (3.2)$$

A sequence (ϕ_n) of functions in $\mathbb{D}(\mathbf{R})$ is said to be a delta sequence if it satisfies Conditions (3.3)–(3.5). Consider

$$\int_{\mathbf{R}} \phi_n(x) dx = 1 \quad (3.3)$$

$$\int_{\mathbf{R}} \phi_n(x) dx \leq M, \quad \exists M > 0, \quad (3.4)$$

$$\text{supp } \phi_n \subset (-\varepsilon_n, \varepsilon_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (3.5)$$

The set of all such sequences is denoted by Δ . To see the extension to certain integral transform, see [29–31].

Lemma 3.1. *Given $\phi \in \mathbb{D}(\mathbf{R})$ and $\varphi \in \mathbb{E}(\mathbf{R})$ then*

$$\frac{d^k}{dx^k} (\phi \odot \tau_x \varphi) = \phi \odot \frac{d^k}{dx^k} \tau_x \varphi, \quad (3.6)$$

for each $k \in \mathbb{N}$.

Proof. We prove the lemma by induction on k . Let $k = 1$ and $\phi \in \mathbb{D}(\mathbf{R})$ with $\text{supp } \phi \subset K$, and then

$$\frac{d}{dx} (\phi \odot \tau_x \varphi) = \lim_{x \rightarrow h} \frac{\phi \odot \tau_x \varphi - \phi \odot \tau_h \varphi}{x - h} = \phi \odot \lim_{x \rightarrow h} \frac{\tau_x \varphi - \tau_h \varphi}{x - h}. \quad (3.7)$$

Hence (3.7) reduces to

$$\frac{d}{dx}(\phi \odot \tau_x \varphi) = \phi \odot \frac{d}{dx} \tau_x \varphi. \quad (3.8)$$

□

Next, assume that the lemma satisfies for k th derivatives, then certainly we get

$$\frac{d^{k+1}}{dx^{k+1}}(\phi \odot \tau_x \varphi) = \frac{d}{dx} \left(\frac{d^k}{dx^k}(\phi \odot \tau_x \varphi) \right) = \phi \odot \frac{d^{k+1}}{dx^{k+1}} \tau_x \varphi, \quad (3.9)$$

by (3.7). Hence the lemma is as follows.

Lemma 3.2. *Let $\phi \in \mathbb{D}(\mathbf{R})$ and $\varphi \in \mathbb{E}(\mathbf{R})$, and then $\phi \odot \tau_x \varphi \in \mathbb{E}(\mathbf{R})$.*

Proof. Let K be a compact subset of $\mathbf{R}$. Then using Lemma 3.1 we get

$$\xi_k(\phi \odot \tau_x \varphi) \leq \sup_{x \in K} \left| \frac{d^{k+1}}{dx^{k+1}} \tau_x \varphi \right|. \quad (3.10)$$

The inequality (3.10) can be explicitly expressed as

$$\xi_k(\phi \odot \tau_x \varphi) \leq \xi_k(\varphi), \quad (3.11)$$

where ξ_k is the norm in the topology equipped with $\mathbb{E}(\mathbf{R})$. Hence, the lemma follows from (3.11). This completes the proof. □

Lemma 3.3. *Let $f \in \mathbb{E}'(\mathbf{R})$ and $\phi \in \mathbb{D}(\mathbf{R})$, and then $f \bullet \phi \in \mathbb{E}'(\mathbf{R})$.*

Proof. In view of Lemma 3.2 we get

$$f \odot \tau_x \phi \in \mathbb{E}(\mathbf{R}). \quad (3.12)$$

Therefore, the righthand side of (3.1) is meaningful. To show that $f \bullet \phi \in \mathbb{E}'(\mathbf{R})$ we are requested to show that $f \bullet \phi$ is continuous and linear. To establish continuity, let $(\varphi_n) \rightarrow 0$ in $\mathbb{E}(\mathbf{R})$, then from (3.11) we get

$$\xi_k(f \odot \tau_x \varphi_n) \leq \xi_k(\varphi_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.13)$$

□

Hence we have

$$(f \bullet \phi) \varphi_n(x) = f(\phi \odot \tau_x \varphi_n) \rightarrow 0 \quad (3.14)$$

as $n \rightarrow \infty$. Linearity condition is obvious. Hence the lemma is completely proved.

Lemma 3.4. Let $\phi_1, \phi_2 \in \mathbb{D}(\mathbf{R})$ and $\psi \in \mathbb{E}(\mathbf{R})$ be given, and then

$$(\phi_1 * \phi_2) \odot \tau_x \psi = \phi_1 \bullet (\phi_2 \odot \tau_x \psi) \quad (3.15)$$

Proof. It is a straightforward consequence of definitions and change of variables. $\square$

Lemma 3.5. Let $\phi_1, \phi_2 \in \mathbb{D}(\mathbf{R})$ and $f \in \mathbb{E}'(\mathbf{R})$, and then

$$f \bullet (\phi_1 * \phi_2) = (f \bullet \phi_1) \bullet \phi_2. \quad (3.16)$$

Proof. Using (3.1) and Lemma 3.4. we get

$$\begin{aligned} (f \bullet (\phi_1 * \phi_2))\psi(x) &= f((\phi_1 * \phi_2) \odot \tau_x \psi) \\ &= f(\phi_1 \bullet (\phi_2 \odot \tau_x \psi)) \\ &= (f \bullet \phi_1) \bullet \phi_2. \end{aligned} \quad (3.17)$$

Hence the lemma is as follows. $\square$

Lemma 3.6. Let $f_1, f_2 \in \mathbb{E}'(\mathbf{R})$ and $\phi \in \mathbb{D}(\mathbf{R})$, and then one has

- (1) $\alpha f \bullet \phi = \alpha(f \bullet \phi)$, $\alpha \in \mathbb{C}$
- (2) $(f_1 + f_2) \bullet \phi = f_1 \bullet \phi + f_2 \bullet \phi$.

Proof. It is a straightforward result of definitions. $\square$

Lemma 3.7. Let $f_n \rightarrow f$ in $\mathbb{E}'(\mathbf{R})$ and $\phi \in \mathbb{D}(\mathbf{R})$ be given then

$$f_n \bullet \phi \longrightarrow f \bullet \phi \quad \text{in } \mathbb{E}'(\mathbf{R}) \quad (3.18)$$

as $n \rightarrow \infty$.

Proof. By virtue of Lemma 3.2, $\phi \odot \tau_x \psi \in \mathbb{E}(\mathbf{R})$. Hence, using (3.1) we get

$$\begin{aligned} (f_n \bullet \phi - f \bullet \phi)\psi(x) &= ((f_n - f) \bullet \phi)\psi(x) \\ &= (f_n - f)(\phi \odot \tau_x \psi). \end{aligned} \quad (3.19)$$

Allowing $n \rightarrow \infty$ completes the proof of the lemma. $\square$

Lemma 3.8. Let $f \in \mathbb{E}'(\mathbf{R})$ and $(\phi_n) \in \Delta$, and then $f \bullet \phi \rightarrow f \in \mathbb{E}'(\mathbf{R})$.

Proof. Considering a compact subset K of $\mathbf{R}$ and a sequence $(\phi_n) \in \Delta$ such that $\text{supp } \phi_n \subset (-\varepsilon_n, \varepsilon_n)$ for each $n \in \mathbb{N}$, we show that $\phi \odot \tau_x \psi \rightarrow \psi(x)$ as $n \rightarrow \infty$ in the sense of $\mathbb{E}(\mathbf{R})$. By Lemma 3.1 we have

$$\left| \frac{d^k}{dx^k} (\phi \odot \tau_x \psi - \psi(x)) \right| \leq \int_{-\varepsilon_n}^{\varepsilon_n} \left| \frac{d^k}{dx^k} (\psi(x+y) - \psi(x)) \right| |\phi_n(y)| dy, \quad (3.20)$$

and by applying (3.4) we get

$$\left| \frac{d^k}{dx^k} (\phi \odot \tau_x \psi - \psi(x)) \right| \leq \int_{-\varepsilon_n}^{\varepsilon_n} M \left| \frac{d^k}{dx^k} (\psi(x+y) - \psi(x)) \right| dx. \quad (3.21)$$

Then the mean value theorem implies that

$$\left| \frac{d^k}{dx^k} (\phi \odot \tau_x \psi - \psi(x)) \right| \leq \int_{-\varepsilon_n}^{\varepsilon_n} M |y| \left| \frac{d^{k+1}}{dx^{k+1}} \psi(x+\xi) \right| dy \quad (3.22)$$

for some $\xi \in (0, y)$. Let $A = \sup_{s \in K} |(d^{k+1}/ds^k)\psi(s)|$ then considering supremum over all $x \in K$ with the fact that $|y| \leq \varepsilon_n$ yields

$$\xi_k(\phi \odot \tau_x \psi - \psi(x)) \leq AM\varepsilon_n. \quad (3.23)$$

Now allowing $n \rightarrow \infty$ in (3.23) yields $\xi_k(\phi \odot \tau_x \psi - \psi(x)) \rightarrow 0$. Hence we have established that

$$\phi \odot \tau_x \psi \rightarrow \psi(x). \quad (3.24)$$

On using (3.24) can be observed as

$$(f \bullet \phi_n)\psi(x) = f(\phi \odot \tau_x \psi) \rightarrow f(\psi)(x) \quad \text{as } n \rightarrow \infty. \quad (3.25)$$

This implies that $f \bullet \phi_n \rightarrow f$ as $n \rightarrow \infty$. The proof is therefore completed. $\square$

Finally, by virtue of the above sequence of results (Lemma 3.1–3.8), our desired Boehmian space $\mathbb{B}_1$ is well defined.

4. The Boehmian Space $\mathbb{B}_{\mathbb{F}_w}$

In this section we construct the space of all Fresnel-wavelet transforms of Boehmians from the space $\mathbb{B}_1$ as follows.

Let $\mathbb{F}_w(\mathbf{R})$ be the space of all analytic functions which are Fresnel-wavelet transforms of distributions in $\mathbb{E}'(\mathbf{R})$. Then, we define convergence as follows. We say $T_n \rightarrow T$ in $\mathbb{F}_w(\mathbf{R})$ if and only if there are $f_n, f \in \mathbb{E}'(\mathbf{R})$ such that $f_n \rightarrow f$ in $\mathbb{E}'(\mathbf{R})$, where $T_n = f_w f_n$ and $T = f_w f$. Let $T \in \mathbb{F}_w(\mathbf{R})$ and $\phi \in \mathbb{D}(\mathbf{R})$ be given, and then define

$$(T \star \phi)(x_2) = \phi \odot \tau_{x_2} T. \quad (4.1)$$

Theorem 4.1. *Let $f \in \mathbb{E}'(\mathbf{R})$ and $\phi \in \mathbb{D}(\mathbf{R})$, and then $f_w(f \bullet \phi) = T \star \phi$, where $T = f_w f$.*

Proof. By the aid of (2.3) we write

$$f_w(f \bullet \phi)(x_2) = \langle f \bullet \phi, K_{\lambda, \mu, x_2}(x_1) \rangle. \quad (4.2)$$

By using (3.1) we get

$$f_w(f \bullet \phi)(x_2) = f(\phi \odot \tau_{x_2} K_{\lambda, \mu, x_2}(x_1)) = \phi(f \odot \tau_{x_2} K_{\lambda, \mu, x_2}(x_1)). \quad (4.3)$$

That is $f_w(f \bullet \phi)(x_2) = \phi \odot \tau_{x_2} T$. Employing (4.1) yields $f_w(f \bullet \phi)(x_2) = (T \star \phi)(x_2)$, where $T = f_w f$. This proves the theorem. $\square$

Lemma 4.2. *Let $T \in \mathbb{F}_w(\mathbf{R})$ and $\phi \in \mathbb{D}(\mathbf{R})$, and then $T \star \phi \in \mathbb{F}_w(\mathbf{R})$.*

Proof. Let $f \in \mathbb{E}'(\mathbf{R})$ be such that $T = f_w f$, and that then Theorem 4.1 implies

$$T \star \phi = f_w(f \bullet \phi). \quad (4.4)$$

Thus $T \star \phi \in \mathbb{F}_w(\mathbf{R})$ by Lemma 3.3. Hence the lemma. $\square$

Lemma 4.3. *Let $T \in \mathbb{F}_w(\mathbf{R})$ and $\phi \in \mathbb{D}(\mathbf{R})$, and then $f_w^{-1}(T \star \phi) = f \bullet \phi$.*

Proof. Theorem 4.1 implies that $f_w^{-1}(T \star \phi) = f_w^{-1}(f_w(f \bullet \phi)) = f \bullet \phi$. This completes the proof of the lemma. $\square$

Lemma 4.4. *Let $T_1, T_2 \in \mathbb{F}_w(\mathbf{R})$ and $\phi_1, \phi_2 \in \mathbb{D}(\mathbf{R})$, then*

- (1) $(\alpha T_1 + \beta T_2) \star \phi = \alpha T_1 \star \phi + \beta T_2 \star \phi$,
- (2) $T \star (\phi_1 \star \phi_2) = (T \star \phi_1) \star \phi_2$.

Proof. It is obvious. $\square$

Lemma 4.5. (i) *Let $T_n \rightarrow T$ as $n \rightarrow \infty$ and $\phi \in \mathbb{D}(\mathbf{R})$, and then $T_n \star \phi \rightarrow T \star \phi$ as $n \rightarrow \infty$ in $\mathbb{F}_w(\mathbf{R})$.*

(ii) *Let $T_n \rightarrow T$ as $n \rightarrow \infty$ and $(\phi_n) \in \Delta$, and then $T_n \star \phi_n \rightarrow T$ as $n \rightarrow \infty$ in $\mathbb{F}_w(\mathbf{R})$.*

Proof. (i) Let $T_n, T \in \mathbb{F}_w(\mathbf{R})$ then $T = f_w f$ and $T_n = f_w f_n$ for some $f, f_n \in \mathbb{E}'(\mathbf{R})$. Hence, using Theorem 4.1, we have

$$T_n \star \phi = f_w f_n \star \phi = f_w(f_n \bullet \phi). \quad (4.5)$$

By Lemma 3.7 we get

$$T_n \star \phi \longrightarrow f_w(f \bullet \phi). \quad (4.6)$$

Once again Theorem 4.1 implies that

$$T_n \star \phi \longrightarrow f_w f \star \phi. \quad (4.7)$$

Thus we have

$$T_n \star \phi \longrightarrow T \star \phi. \quad (4.8)$$

This proves part (i) of the Lemma.

(ii) can be proved similarly by using Lemma 3.8 and Theorem 4.1. The space $\mathbb{B}_{\mathbb{F}_w}$ is therefore established. $\square$

The sum of two Boehmians and multiplication by a scalar in $\mathbb{B}_{\mathbb{F}_w}$ is defined in a natural way as follows:

$$\begin{aligned} \left[\frac{f_n}{\phi_n} \right] + \left[\frac{g_n}{\psi_n} \right] &= \left[\frac{(f_n \star \psi_n) + (g_n \star \phi_n)}{\phi_n \star \psi_n} \right], \\ \alpha \left[\frac{f_n}{\phi_n} \right] &= \left[\alpha \frac{f_n}{\phi_n} \right], \quad \alpha \in \mathbb{C}. \end{aligned} \quad (4.9)$$

The operation $\star$ and the differentiation are defined by

$$\left[\frac{f_n}{\phi_n} \right] \star \left[\frac{g_n}{\psi_n} \right] = \left[\frac{f_n \star g_n}{\phi_n \star \psi_n} \right], \quad D^\alpha \left[\frac{f_n}{\phi_n} \right] = \left[\frac{D^\alpha f_n}{\phi_n} \right]. \quad (4.10)$$

5. Optical Fresnel-Wavelet Transforms of Boehmians

In view of the analysis obtained in Sections 4 and 5 and Theorem 4.1 we are led to state the following definition.

Definition 5.1. Let $[f_n/\phi_n] \in \mathbb{B}_1$, and then

$$\mathfrak{F}_w^{-1} \left[\frac{T_{f_n}}{\phi_n} \right] = \left[\frac{T_{f_n}^{-1}}{\phi_n} \right], \quad \mathfrak{F}_w \left[\frac{f_n}{\phi_n} \right] = \left[\frac{T_{f_n}}{\phi_n} \right], \quad (5.1)$$

for each $(\phi_n) \in \Delta$, where $T_{f_n} = f_w f_n$.

Lemma 5.2. The optical Fresnel-wavelet transform $\mathfrak{F}_w : \mathbb{B}_1 \rightarrow \mathbb{B}_{\mathbb{F}_w}$ is well defined.

Proof. It is a straightforward. $\square$

Lemma 5.3. The optical Fresnel-wavelet transform $\mathfrak{F}_w : \mathbb{B}_1 \rightarrow \mathbb{B}_{\mathbb{F}_w}$ is linear.

Proof. It is straightforward by using Definition 5.1. $\square$

Lemma 5.4. The optical Fresnel-wavelet transform $\mathfrak{F}_w : \mathbb{B}_1 \rightarrow \mathbb{B}_{\mathbb{F}_w}$ is an isomorphism.

Proof. Assume that $\mathfrak{F}_w[f_n/\phi_n] = \mathfrak{F}_w[g_n/\psi_n]$, and then using (5.1) and the concept of quotients we get $T_{f_n} \star \psi_m = T_{g_m} \star \phi_n$, where $T_{f_n} = f_w f_n$ and $T_{g_m} = f_w g_m$. Therefore, Theorem 4.1 implies that $f_w(f_n \bullet \psi_m) = f_w(g_m \bullet \phi_n)$. Properties of f_w imply that $f_n \bullet \psi_m = g_m \bullet \phi_n$. Therefore,

$[f_n/\phi_n] = [g_n/\psi_n]$. To establish f_w is surjective, and let $[T_{f_n}/\phi_n] \in \mathbb{B}_{F_w}$. Then $T_{f_n} \star \phi_m = T_{f_m} \star \phi_n$ for every $m, n \in \mathbb{N}$. Hence $f_n, f_m \in \mathbb{E}'(\mathbf{R})$ are such that $T_{f_n} = f_w f_n$ and $T_{f_m} = f_w f_m$. Theorem 4.1 implies that $f_w(f_n \bullet \phi_m) = f_w(f_m \bullet \phi_n)$. Hence $[f_n/\phi_n] \in \mathbb{B}_1$ is such that $\mathfrak{F}_w[f_n/\phi_n] = [T_{f_n}/\phi_n]$. This completes the proof of the lemma. $\square$

Now, Let $[f_n/\phi_n] \in \mathbb{B}_1$, and then we define the inverse optical Fresnel-wavelet transform of $[f_n/\phi_n]$ as

$$\mathfrak{F}_w^{-1} \left[\frac{f_n}{\phi_n} \right] = \left[\frac{T_{f_n}^{-1}}{\phi_n} \right], \quad (5.2)$$

where $T_{f_n}^{-1} = f_w^{-1} f_n$.

Lemma 5.5. Let $[T_{f_n}/\phi_n] \in \mathbb{B}_{F_w}$, $T_{f_n} = f_w f_n$, and $\phi \in \mathbb{D}(\mathbf{R})$ then

$$\mathfrak{F}_w^{-1} \left(\left[\frac{T_{f_n}}{\phi_n} \right] \star \phi \right) = \left[\frac{T_{f_n}^{-1}}{\phi_n} \right] \bullet \phi, \quad \mathfrak{F}_w \left(\left[\frac{f_n}{\phi_n} \right] \bullet \phi \right) = \left[\frac{T_{f_n}}{\phi_n} \right] \star \phi. \quad (5.3)$$

Proof. Applying Definition 5.1 yields

$$\mathfrak{F}_w^{-1} \left(\left[\frac{T_{f_n}}{\phi_n} \right] \star \phi \right) = \mathfrak{F}_w^{-1} \left(\left[\frac{T_{f_n} \star \phi}{\phi_n} \right] \right) = \left[\frac{f_w^{-1}(T_{f_n} \star \phi)}{\phi_n} \right]. \quad (5.4)$$

Using Lemma 4.3 we obtain

$$\mathfrak{F}_w^{-1} \left(\left[\frac{T_{f_n}}{\phi_n} \right] \star \phi \right) = \left[\frac{f_w^{-1} T_{f_n}}{\phi_n} \right] \bullet \phi = \left[\frac{T_{f_n}^{-1}}{\phi_n} \right] \bullet \phi. \quad (5.5)$$

This completes the proof of the Lemma. $\square$

Theorem 5.6. $\mathfrak{F}_w : \mathbb{B}_1 \rightarrow \mathbb{B}_{F_w}$ and $\mathfrak{F}_w^{-1} : \mathbb{B}_{F_w} \rightarrow \mathbb{B}_1$ are continuous with respect to δ and Δ convergences.

Proof. First of all, we show that $\mathfrak{F}_w : \mathbb{B}_1 \rightarrow \mathbb{B}_{F_w}$ and $\mathfrak{F}_w^{-1} : \mathbb{B}_{F_w} \rightarrow \mathbb{B}_1$ are continuous with respect to δ convergence.

Let $\beta_n \xrightarrow{\delta} \beta$ in $\mathbb{B}_1$ as $n \rightarrow \infty$, and then we show that $\mathfrak{F}_w \beta_n \rightarrow \mathfrak{F}_w \beta$ as $n \rightarrow \infty$. By virtue of [25] we can find $f_{n,k}$ and f_k in $\mathbb{E}'(\mathbf{R})$ such that

$$\beta_n = \left[\frac{f_{n,k}}{\phi_k} \right], \quad \beta = \left[\frac{f_k}{\phi_k} \right] \quad (5.6)$$

such that $f_{n,k} \rightarrow f_k$ as $n \rightarrow \infty$ for every $k \in \mathbb{N}$. Employing the continuity condition of the optical Fresnel-wavelet transform implies that $T_{f_{n,k}} \rightarrow T_{f_k}$ as $n \rightarrow \infty$ in the space $\mathbb{F}_w(\mathbf{R})$. Thus, $[T_{f_{n,k}}/\phi_k] \rightarrow [T_{f_k}/\phi_k]$ as $n \rightarrow \infty$ in $\mathbb{B}_{F_w}$.

To prove the second part, let $g_n \xrightarrow{\delta} g$ in $\mathbb{B}_{\mathbb{F}_w}$ as $n \rightarrow \infty$. Then, once again, by [4], $g_n = [T_{f_{n,k}}/\phi_k]$ and $g = [T_{f_k}/\phi_k]$ for some $T_{f_{n,k}}, T_{f_k} \in \mathbb{F}_w(\mathbf{R})$ and $T_{f_{n,k}} \rightarrow T_{f_k}$ as $n \rightarrow \infty$. Hence $f_w^{-1}T_{f_{n,k}} \rightarrow f_w^{-1}T_{f_k}$ in $\mathbb{B}_1$ as $n \rightarrow \infty$. Or, $[T_{f_{n,k}}^{-1}/\phi_k] \rightarrow [T_{f_k}^{-1}/\phi_k]$ as $n \rightarrow \infty$. Using Definition 5.1 we get $\mathfrak{F}_w^{-1}[T_{f_{n,k}}/\phi_k] \rightarrow \mathfrak{F}_w^{-1}[T_{f_k}/\phi_k]$ as $n \rightarrow \infty$.

Now, we establish continuity of $\mathfrak{F}_w$ and $\mathfrak{F}_w^{-1}$ with respect to Δ convergence. Let $\beta_n \xrightarrow{\Delta} \beta$ in $\mathbb{B}_1$ as $n \rightarrow \infty$. Then, there exist $f_n \in \mathbb{E}'(\mathbf{R})$ and $\phi_n \in \Delta$ such that $(\beta_n - \beta) \bullet \phi_n = [(f_n \bullet \phi_k)/\phi_k]$ and $f_n \rightarrow 0$ as $n \rightarrow \infty$. Employing Definition 5.1 we get

$$\mathfrak{F}_w((\beta_n - \beta) \bullet \phi_n) = \left[\frac{f_w(f_n \bullet \phi_k)}{\phi_k} \right]. \quad (5.7)$$

Hence, from Lemma 4.2 we have $\mathfrak{F}_w((\beta_n - \beta) \bullet \phi_n) = [(T_{f_n} \star \phi_k)/\phi_k] = T_{f_n} \rightarrow 0$ as $n \rightarrow \infty$ in $\mathbb{F}_w$. Therefore consider

$$\mathfrak{F}_w((\beta_n - \beta) \bullet \phi_n) = (\mathfrak{F}_w\beta_n - \mathfrak{F}_w\beta) \star \phi_n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (5.8)$$

Hence, $\mathfrak{F}_w\beta_n \xrightarrow{\Delta} \mathfrak{F}_w\beta$ as $n \rightarrow \infty$. Finally, let $g_n \xrightarrow{\Delta} g$ in $\mathbb{B}_{\mathbb{F}_w}$ as $n \rightarrow \infty$, and then we find $T_{f_k} \in \mathbb{F}_w(\mathbf{R})$ such that $(g_n - g) \star \phi_n = [(T_{f_k} \star \phi_k)/\phi_k]$ and $T_{f_k} \rightarrow 0$ as $n \rightarrow \infty$ for some $(\phi_n) \in \Delta$ and $T_{f_k} = f_w f_n$. Now, using Definition 5.1, we obtain that

$$\mathfrak{F}_w^{-1}((g_n - g) \star \phi_n) = \left[\frac{f_w^{-1}(T_{f_k} \star \phi_k)}{\phi_k} \right]. \quad (5.9)$$

Lemma 5.5 implies that

$$\mathfrak{F}_w^{-1}((g_n - g) \star \phi_n) = \left[\frac{f_n \bullet \phi_k}{\phi_k} \right] = f_n \rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ in } \mathbb{E}'(\mathbf{R}). \quad (5.10)$$

Thus we have

$$\mathfrak{F}_w^{-1}((g_n - g) \star \phi_n) = (\mathfrak{F}_w^{-1}g_n - \mathfrak{F}_w^{-1}g) \bullet \phi_n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (5.11)$$

From this we find that $\mathfrak{F}_w^{-1}g_n \xrightarrow{\Delta} \mathfrak{F}_w^{-1}g$ as $n \rightarrow \infty$ in $\mathbb{B}_1$. This completes the proof of the theorem. $\square$

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