Research Article

A Class of Integral Operators and Bessel Plancherel Transform on L^2_α

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For the relation between Bessel Plancherel transform and a wide class of integral operators we establish some results generalizing the corresponding results for the cosine transform, given by Goldberg (1972) and Titchmarsh (1937). Building on these results we obtain a new properties of certain well-known integral transforms associated with the eigenfunction of the Bessel differential operator defined on $(0, \infty)$ by $l_\alpha u = u'' + ((2\alpha + 1)/x)u'$, $\alpha > -1/2$. We also construct a class of integral operators which commute with Bessel Plancherel transform.

1. Introduction

Titchmarsh [1] studied the relation between the cosine transform and the modified Hardy operator $H(f)(x) = (1/x) \int_0^x f(y)dy$, $f \in L^2([0, \infty[)$. Then, Goldberg [2] considered a wide class of integral transform and established some results generalizing the corresponding theorems obtained by Goldberg in [2]. More precisely Goldberg proved that if g is the cosine transform of $f \in L^2([0, \infty[)$ then

$$G(x) = \frac{1}{x} \int_0^\infty \varphi\left(\frac{y}{x}\right) g(y) dy \quad (1.1)$$

is the cosine transform of

$$F(y) = \int_0^\infty \frac{1}{x} \varphi\left(\frac{y}{x}\right) f(x) dx. \quad (1.2)$$

The same result applies to sine transforms.

The Riemann-Liouville transform and Weyl transform associated with the eigenfunctions of the differential operator

$$l_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha+1}{x} \frac{d}{dx}, \quad \alpha > -\frac{1}{2} \quad (1.3)$$

are, respectively, defined for all measurable functions by

$$\mathcal{R}_\alpha(f)(x) = \begin{cases} C_\alpha x^{-2\alpha} \int_0^x (x^2 - y^2)^{\alpha-1/2} f(y) dy, & \text{if } x > 0 \\ f(0) & \text{if } x = 0, \end{cases} \quad (1.4)$$

respectively,

$$\mathcal{W}_\alpha(f)(x) = C_\alpha \int_x^\infty (y^2 - x^2)^{\alpha-1/2} f(y) y dy, \quad (1.5)$$

where

$$C_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}. \quad (1.6)$$

These operators have been studied on regular spaces of functions. In particular in [3] Trimèche has proved that the Riemann-Liouville transform is an isomorphism from $\xi_*(\mathbb{R})$ (the space of even infinitely differentiable functions on $\mathbb{R}$) onto itself and that the Weyl transform is an isomorphism from $\mathfrak{D}_*(\mathbb{R})$ (the space of even infinitely differentiable functions on $\mathbb{R}$, with compact support) onto itself. The Weyl transform has also been studied on Schwartz spaces $\mathcal{S}_*(\mathbb{R})$. As for the Sonine transform associated with l_α defined by

$$\mathcal{S}_{\beta,\alpha}(f)(x) = \begin{cases} C_{\beta,\alpha} x^{-2\beta} \int_0^x (x^2 - y^2)^{\beta-\alpha-1} f(y) y^{2\alpha+1} dy, & \text{if } x > 0 \\ f(0) & \text{if } x = 0, \end{cases} \quad (1.7)$$

where

$$\alpha, \beta > -\frac{1}{2}, \quad \beta - \alpha > 0, \quad C_{\beta,\alpha} = \frac{2\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)} \quad (1.8)$$

is linked with the Riemann-Liouville transform and Weyl transform.

Such integral transforms and many types integral operators have been studied by many authors [4–9]. They have many applications to science and engineering [5, 10].

In this work we consider a class of integral operators generated by a measurable nonnegative function φ which we denoted T_φ . We study the boundedness of these operators on L_α^2 the space of all real-valued measurable functions f defined on $[0, \infty[$ with norm $\|f\|_{2,\alpha} = (\int_0^\infty |f(x)|^2 x^{2\alpha+1} dx)^{1/2}$ is finite. Building on the continuity of the operator T_φ on L_α^2 , we

establish a relation between Bessel-Plancherel transform Φ_α , $\alpha > -1/2$ (which we will define in Section 2) and this class of operators. We also construct a class of self-adjoint operator which commutes with Φ_α . Then, we derive new results concerning the relation between the Bessel-Plancherel transform, and the Riemann-Liouville transform, Weyl transform, and Sonine transform. Finally we give a self adjoint operator which commutes with the transform Φ_α . Since the eigenfunction of the Dunkl operator defined on real axis $\mathbb{R}$ by

$$D_\alpha u(x) = \frac{du}{dx}(x) + \frac{2\alpha+1}{x} \frac{u(x) - u(-x)}{2}, \quad \alpha \geq -\frac{1}{2} \quad (1.9)$$

is in connection with the special functions associated with the second-order differential operator l_α defined above, the present paper paves the way for the coming paper which deals with the relation between a class of integral operators and the Dunkl transform $\mathcal{F}_\alpha$ defined on the Schwartz space $S(\mathbb{R})$ by

$$\forall \lambda \in \mathbb{R}, \quad \mathcal{F}_\alpha(f)(\lambda) = \int_{\mathbb{R}} f(x) \psi_\lambda^\alpha(x) d\mu_\alpha(x), \quad (1.10)$$

where the kernel ψ_λ^α is given by

$$\psi_\lambda^\alpha(x) = j_\alpha(\lambda x) + \frac{i\lambda x}{2(\alpha+1)} j_{\alpha+1}(\lambda x), \quad (1.11)$$

with j_α the normalized Bessel function of index α defined by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}. \quad (1.12)$$

For more details see [11]. The content of this paper is as follows. In Section 2 we recall some properties of the Bessel transform (also called Hankel transform) associated with the singular differential operator l_α defined above. In Section 3 we study the boundedness of a class of operators generated by a measurable function $\varphi \geq 0$ on the space L_α^2 . Relation of the Bessel Plancherel transform, and this class of bounded operators on L_α^2 is presented in Section 4. Section 5 deals with the connection between Plancherel transform, Riemann-Liouville transform, Weyl transform and Sonine transform. In this section we give an operator which commutes with Plancherel transform associated with l_α .

2. Bessel Transform and Bessel Plancherel Transform

In the following we give some definitions and some results concerning Bessel transform and Bessel Plancherel transform. For more details see [3, 12–16].

For fixed, $\alpha > -1/2$ we define a measure μ_α on $[0, \infty[$ depending on α by

$$d\mu_\alpha(x) = \frac{x^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} dx, \quad x \geq 0. \quad (2.1)$$

We denote by L_{α}^p , $1 \leq p < \infty$ the space of all real-valued measurable functions f defined on $[0, \infty[$ and the norm

$$\|f\|_{p,\alpha} = \left(\int_0^{\infty} |f(x)|^p d\mu_{\alpha}(x) \right)^{1/p} \quad (2.2)$$

is finite. Whereas $L_{\alpha}^{\infty} = L^{\infty}$ which does not depend on α denotes the space of those measurable functions defined on $[0, \infty[$ for which

$$\|f\|_{\infty} = \text{esssup}_{x>0} |f(x)| \quad (2.3)$$

is finite.

If T is a linear transformation on L_{α}^2 into itself then $\|T\|$ is defined as

$$\|T\| = \sup_{g \in L_{\alpha}^2} \frac{\|T(g)\|_{2,\alpha}}{\|g\|_{2,\alpha}}. \quad (2.4)$$

We will make use of the *Schwarz type inequality* if $f, g \in L_{\alpha}^2$ then $fg \in L_{\alpha}^1$ and

$$\|fg\|_{1,\alpha} \leq \|f\|_{2,\alpha} \|g\|_{2,\alpha} \quad (2.5)$$

and its *converse* if for each $h \in L_{\alpha}^2$, $\|Gh\|_{1,\alpha} \leq A\|h\|_{2,\alpha}$ then $G \in L_{\alpha}^2$ and

$$\|G\|_{2,\alpha} \leq A. \quad (2.6)$$

The Bessel transform of order $\alpha > -1/2$ of a function $f \in L_{\alpha}^1$ is defined by

$$\mathcal{B}_{\alpha}(f)(\lambda) = \int_0^{\infty} j_{\alpha}(\lambda x) f(x) d\mu_{\alpha}(x) \quad \text{for } \lambda \geq 0, \quad (2.7)$$

where j_{α} is the normalized Bessel function defined by

$$j_{\alpha}(\lambda x) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+k+1)} \left(\frac{\lambda x}{2} \right)^{2k}. \quad (2.8)$$

The following properties are fundamental and are used to prove the main results of this paper.

Theorem 2.1. *Let $\alpha > -1/2$. Then*

- (i) *if $f \in L_{\alpha}^1$, then $\mathcal{B}_{\alpha}(f)$ is a continuous function vanishing at infinity.*
- (ii) *Let f be such that f and $\mathcal{B}_{\alpha}(f) \in L_{\alpha}^1$. Then*

$$f(x) = \int_0^{\infty} \mathcal{B}_{\alpha}(f)(t) j_{\alpha}(xt) d\mu_{\alpha}(t) \quad \text{a.e.} \quad (2.9)$$

- (iii) The Bessel transform $\mathcal{B}_\alpha$ is an isomorphism from $\mathcal{S}_*(\mathbb{R})$ onto itself and its inverse denoted $\mathcal{B}_\alpha^{-1} = \mathcal{B}_\alpha$, where $\mathcal{S}_*(\mathbb{R})$ is the space of even infinitely differentiable functions on $\mathbb{R}$, rapidly decreasing together with all their derivatives equipped with its usual topology.

Theorem 2.2 (Plancherel). Let $\alpha > -1/2$. Then there exists a unique isomorphism

$$\Phi_\alpha : L_\alpha^2 \longrightarrow L_\alpha^2, \quad (2.10)$$

satisfying

- (i) for all $f \in \mathcal{S}_*(\mathbb{R})$, $\Phi_\alpha(f) = \mathcal{B}_\alpha(f)$,
(ii) for all $f \in L_\alpha^2$,

$$\|\Phi_\alpha(f)\|_{2,\alpha} = \|f\|_{2,\alpha}. \quad (2.11)$$

The inverse of Φ_α denoted $\Phi_\alpha^{-1} = \Phi_\alpha$ almost everywhere. The transform Φ_α is called Bessel Plancherel transform.

Proposition 2.3. (i) For all $f \in L_\alpha^1 \cap L_\alpha^2$ one has $\mathcal{B}_\alpha = \Phi_\alpha$ almost everywhere.
(ii) Let $f \in L_\alpha^2$ then

$$\Phi_\alpha(f)(x) = \lim_{R \rightarrow \infty} \int_0^R f(y) j_\alpha(xy) d\mu_\alpha(y), \quad (2.12)$$

where $\lim$ stands for $\lim$ in the L_α^2 mean. That is,

$$\lim_{R \rightarrow +\infty} \int_0^\infty \left| \Phi_\alpha(f)(x) - \int_0^R f(y) j_\alpha(xy) d\mu_\alpha(y) \right|^2 d\mu_\alpha(x) = 0. \quad (2.13)$$

For more details of the previous results see [3, 16, 17].

3. A Class of Bounded Operators on L_α^2

In this section, we will address the boundedness and some properties on L_α^2 of certain class of integral operators.

Theorem 3.1. Let φ be nonnegative and measurable function such that

$$\int_0^\infty \frac{\varphi(y)}{y^{\alpha+1}} d\mu_\alpha(y) = A_\alpha < \infty. \quad (3.1)$$

Then the linear operator T_φ defined on L_α^2 by

$$T_\varphi(f)(x) = \frac{1}{x^{2\alpha+2}} \int_0^\infty \varphi\left(\frac{y}{x}\right) f(y) d\mu_\alpha(y) \quad (3.2)$$

is a bounded operator on L_α^2 and one has

$$\|T_\varphi\| \leq A_\alpha. \quad (3.3)$$

Proof. For any function $h \in L_\alpha^2$ we have

$$\int_0^\infty |T_\varphi(f)(x)h(x)|d\mu_\alpha(x) \leq \int_0^\infty \frac{|h(x)|}{x^{2\alpha+2}} \left(\int_0^\infty \varphi\left(\frac{y}{x}\right) |f(y)|d\mu_\alpha(y) \right) d\mu_\alpha(x). \quad (3.4)$$

By change of variable $y = tx$ we obtain

$$\int_0^\infty |T_\varphi(f)(x)h(x)|d\mu_\alpha(x) \leq \int_0^\infty |h(x)| \left(\int_0^\infty \varphi(t) |f(xt)|d\mu_\alpha(t) \right) d\mu_\alpha(x). \quad (3.5)$$

So, Fubini-Tonelli theorem yields

$$\int_0^\infty |T_\varphi(f)(x)h(x)|d\mu_\alpha(x) \leq \int_0^\infty \varphi(t) \left(\int_0^\infty |h(x)| |f(xt)|d\mu_\alpha(x) \right) d\mu_\alpha(t). \quad (3.6)$$

But, by Schwarz inequality we have

$$\int_0^\infty |h(x)f(xt)|d\mu_\alpha(x) \leq \left(\int_0^\infty |h(x)|^2 d\mu_\alpha(x) \right)^{1/2} \left(\int_0^\infty |f(xt)|^2 d\mu_\alpha(x) \right)^{1/2}. \quad (3.7)$$

Furthermore

$$\left(\int_0^\infty |f(xt)|^2 d\mu_\alpha(x) \right)^{1/2} = \frac{1}{t^{\alpha+1}} \left(\int_0^\infty |f(x)|^2 d\mu_\alpha(x) \right)^{1/2}. \quad (3.8)$$

Thus, (3.6) and (3.7) yield

$$\int_0^\infty |T_\varphi(f)(x)h(x)|d\mu_\alpha(x) \leq A_\alpha \|h\|_{2,\alpha} \|f\|_{2,\alpha}. \quad (3.9)$$

This proves that the last integral converges absolutely and we obtain

$$\|T_\varphi(f)h\|_{1,\alpha} \leq A_\alpha \|h\|_{2,\alpha} \|f\|_{2,\alpha}. \quad (3.10)$$

Moreover the converse of Schwarz inequality allows us to get $T_\varphi(f) \in L_\alpha^2$ and we have

$$\|T_\varphi(f)\|_{2,\alpha} \leq A_\alpha \|f\|_{2,\alpha}, \quad \forall f \in L_\alpha^2. \quad (3.11)$$

So, T_φ is a bounded linear transformation on L_α^2 into itself and

$$\|T_\varphi\| \leq A_\alpha. \quad (3.12)$$

The theorem is thus established. $\square$

Now, choose any $f, h \in L_\alpha^2$. Then with $\langle f, h \rangle$ defined as

$$\langle f, h \rangle = \int_0^\infty f(x)h(x)d\mu_\alpha(x) \quad (3.13)$$

the inner product in L_α^2 . Then for all nonnegative measurable function φ such that

$$\int_0^\infty \frac{\varphi(t)}{t^{\alpha+1}} d\mu_\alpha(t) \quad (3.14)$$

is finite, we have

$$\begin{aligned} \langle T_\varphi(f), h \rangle &= \int_0^\infty T_\varphi(f)(x)h(x)d\mu_\alpha(x) \\ &= \int_0^\infty \frac{1}{x^{2\alpha+2}} \left(\int_0^\infty \varphi\left(\frac{y}{x}\right) f(y) d\mu_\alpha(y) \right) h(x) d\mu_\alpha(x). \end{aligned} \quad (3.15)$$

Using Fubini theorem we get

$$\langle T_\varphi(f), h \rangle = \int_0^\infty \left(\int_0^\infty \frac{1}{x^{2\alpha+2}} \varphi\left(\frac{y}{x}\right) h(x) dm_\alpha(x) \right) f(y) dm_\alpha(y). \quad (3.16)$$

That is by definition of adjoint shows that

$$\langle T_\varphi(f), h \rangle = \langle f, T_\varphi^*(h) \rangle, \quad (3.17)$$

where T_φ^* is the adjoint operator of T_φ defined on L_α^2 by

$$T_\varphi^*(f)(y) = \int_0^\infty \varphi\left(\frac{y}{x}\right) \frac{h(x)}{x^{2\alpha+2}} d\mu_\alpha(x). \quad (3.18)$$

So, by Theorem 3.1 we get the following.

Theorem 3.2. *Let φ be nonnegative and measurable function such that*

$$\int_0^\infty \frac{\varphi(y)}{y^{\alpha+1}} d\mu(y) = A_\alpha < \infty. \quad (3.19)$$

Then the linear adjoint operator T_φ^* of the operator T_φ defined on L_α^2 by

$$T_\varphi^*(f)(y) = \int_0^\infty \varphi\left(\frac{y}{x}\right) \frac{h(x)}{x^{2\alpha+2}} d\mu_\alpha(x) \quad (3.20)$$

is a bounded operator on L_α^2 and one has

$$\|T_\varphi^*\| \leq A_\alpha. \quad (3.21)$$

Remark 3.3. The integrals defining T_φ and T_φ^* in Theorems 3.1 and 3.2 exist almost everywhere.

4. Bessel Plancherel Transform and Operators T_φ , and Its Adjoint

This section deals with the relation between Bessel Plancherel transform and a class of integral operators. In particular we construct a class of integral operators which commute with the Bessel Plancherel transform on L_α^2 .

Theorem 4.1. *Let $\alpha > -1/2$ and φ a measurable nonnegative function such that $\varphi \in L_\alpha^1$ and*

$$\int_0^\infty \frac{\varphi(y)}{y^{\alpha+1}} d\mu_\alpha(y) < \infty. \quad (4.1)$$

Then

$$T_\varphi \Phi_\alpha = \Phi_\alpha T_\varphi^*, \quad (4.2)$$

where T_φ and T_φ^ are as in Theorems 3.1 and 3.2 and Φ_α is Bessel Plancherel transform as in Theorem 2.2 and Proposition 2.3.*

Proof. Using Proposition 2.3 and since $L_\alpha^1 \cap L_\alpha^2$ is dense in L_α^2 and T_φ and T_φ^* are continuous on L_α^2 then to prove the theorem it is sufficient to prove that

$$T_\varphi \mathcal{B}_\alpha = \mathcal{B}_\alpha T_\varphi^*, \quad \text{for } f \in L_\alpha^1 \cap L_\alpha^2, \quad (4.3)$$

where $\mathcal{B}_\alpha$ is the Bessel transform defined in Section 2. Accordingly, choose any $f \in L_\alpha^1 \cap L_\alpha^2$

$$T_\varphi(\mathcal{B}_\alpha(f))(x) = \frac{1}{x^{2\alpha+2}} \int_0^\infty \varphi\left(\frac{y}{x}\right) \mathcal{B}_\alpha(f)(y) d\mu_\alpha(y). \quad (4.4)$$

Making the change of variable $y = sx$ yields

$$\begin{aligned} T_\varphi(\mathcal{B}_\alpha(f))(x) &= \int_0^\infty \varphi(s) \mathcal{B}_\alpha(f)(sx) d\mu_\alpha(s) \\ &= \int_0^\infty \varphi(s) \left(\int_0^\infty j_\alpha(sxt) f(t) d\mu_\alpha(t) \right) d\mu_\alpha(s). \end{aligned} \quad (4.5)$$

Interchanging the order of integration and making the change of variable $u = st$ yields

$$T_\varphi(\mathcal{B}_\alpha(f))(x) = \int_0^\infty \frac{f(t)}{t^{2\alpha+2}} \left(\int_0^\infty \varphi\left(\frac{u}{t}\right) j_\alpha(xu) d\mu_\alpha(u) \right) d\mu_\alpha(t). \quad (4.6)$$

Interchanging again the order of integration, so

$$\begin{aligned} T_\varphi(\mathcal{B}_\alpha(f))(x) &= \int_0^\infty j_\alpha(xu) \left(\int_0^\infty \frac{f(t)}{t^{2\alpha+2}} \varphi\left(\frac{u}{t}\right) d\mu_\alpha(t) \right) d\mu_\alpha(u) \\ &= \int_0^\infty j_\alpha(xu) T_\varphi^*(f)(u) d\mu_\alpha(u). \end{aligned} \quad (4.7)$$

The integral (4.5) converges absolutely since $\varphi, f \in L_\alpha^1$ and $|j_\alpha(u)| \leq 1$. This justifies the changes in order of integration and also shows that $T_\varphi^*(f)$ belongs to L_α^1 . Thus from (4.7) we get

$$T_\varphi \mathcal{B}_\alpha = \mathcal{B}_\alpha T_\varphi^*, \quad (4.8)$$

which is what we wanted to show. $\square$

A more general result, we may drop the hypothesis that $\varphi \in L_\alpha^1$ in Theorem 4.1.

Theorem 4.2. *If φ is measurable nonnegative function such that*

$$\int_0^\infty \frac{\varphi(y)}{y^{\alpha+1}} d\mu(y) < \infty. \quad (4.9)$$

Then

$$T_\varphi \Phi_\alpha = \Phi_\alpha T_\varphi^*. \quad (4.10)$$

Proof. Choose any nonnegative function such that $\int_0^\infty \varphi(y) y^\alpha dy = A_\alpha < \infty$ (but not necessarily such that $\varphi \in L_\alpha^1$). For $n = 1, 2, \dots$ define

$$\varphi_n(y) = \varphi(y), \quad \text{if } y \in \left[\frac{1}{n}, n \right], \quad \varphi_n(y) = 0, \quad \text{if } y \in \left[0, \frac{1}{n} \right[\cup]n, \infty[. \quad (4.11)$$

Then

$$\int_0^\infty \varphi_n(y) y^\alpha dy = \int_{1/n}^n \varphi_n(y) y^\alpha dy = A_{n,\alpha} < \infty, \quad (4.12)$$

and by the Lebesgue convergence theorem

$$\lim_{n \rightarrow \infty} A_{n,\alpha} = A_\alpha. \quad (4.13)$$

Moreover if T_φ, T_{φ_n} are defined by φ in Theorem 3.2 then

$$T_\varphi - T_{\varphi_n} = T_{\varphi - \varphi_n}, \quad (4.14)$$

and thus by Theorem 3.2

$$\|T_\varphi - T_{\varphi_n}\| \leq A_\alpha - A_{n,\alpha}. \quad (4.15)$$

But T_{φ_n} obeys the hypothesis of Theorem 4.1. Hence

$$T_{\varphi_n} \mathcal{B}_\alpha = \mathcal{B}_\alpha T_{\varphi_n}^*. \quad (4.16)$$

Letting $n \rightarrow \infty$ and using (4.15) we get

$$T_\varphi \mathcal{B}_\alpha = \mathcal{B}_\alpha T_\varphi^*. \quad (4.17)$$

We have thus shown the result. $\square$

In the following we will construct a class of self-adjoint bounded operators which commute with the Bessel Plancherel transform.

Using the same techniques as those used by Goldberg [2], we get the following result.

Theorem 4.3. *Let φ be nonnegative on $(0, 1)$ with*

$$\int_0^1 \frac{\varphi(t)}{t^{\alpha+1}} d\mu_\alpha(t) \quad (4.18)$$

is finite. Define

$$\varphi(t) = \frac{1}{t^{2\alpha+2}} \varphi\left(\frac{1}{t}\right), \quad \text{for } t > 1. \quad (4.19)$$

Then the operator T_φ is as in Theorem 3.1 commutes with the Bessel Plancherel transform Φ_α , that is

$$T_\varphi \Phi_\alpha = \Phi_\alpha T_\varphi. \quad (4.20)$$

5. Applications

Consider the differential operator l_α defined on $]0, \infty[$ by

$$l_\alpha u = u'' + \frac{2\alpha + 1}{x} u' = \frac{1}{x^{2\alpha+1}} \frac{d}{dx} \left(x^{2\alpha+1} \frac{d}{dx} \right) (u). \quad (5.1)$$

Then for $\alpha > -1/2$ the system

$$\begin{aligned} l_\alpha u(x) &= -\lambda^2 u(x), \\ u(0) &= 1, \quad u'(0) = 0, \end{aligned} \quad (5.2)$$

admits the unique solution given by the function

$$x \longrightarrow j_\alpha(\lambda x), \quad (5.3)$$

defined in Section 1 [12, 17–19].

In the following we introduce some function spaces, namely,

- (i) $C_*(\mathbb{R})$ the space of even continuous function on $\mathbb{R}$,
- (ii) $\xi_*(\mathbb{R})$ the space of even infinitely differentiable functions on $\mathbb{R}$,
- (iii) $\mathfrak{D}_*(\mathbb{R})$ the space of even infinitely differentiable function with compact support.

Each of these space is equipped with usual topology.

Definition 5.1. (1) The Riemann-Liouville transform $\mathcal{R}_\alpha$, $\alpha > -1/2$ associated with the differential operator l_α is the integral transform defined on $\xi_*(\mathbb{R})$ by

$$\mathcal{R}_\alpha(f)(x) = \begin{cases} C_\alpha x^{-2\alpha} \int_0^x (x^2 - y^2)^{\alpha-1/2} f(y) dy, & \text{if } x > 0, \\ f(0) & \text{if } x = 0. \end{cases} \quad (5.4)$$

(2) The Weyl transform $\mathcal{W}_\alpha$ associated with l_α is defined on $\mathfrak{D}_*(\mathbb{R})$ by

$$\mathcal{W}_\alpha(f)(x) = C_\alpha \int_x^\infty (y^2 - x^2)^{\alpha-1/2} f(y) y dy, \quad (5.5)$$

where

$$C_\alpha = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + 1/2)}. \quad (5.6)$$

We note that the relation between the normalized Bessel function and the Riemann-Liouville transform is the following

$$j_\alpha(\lambda x) = \mathcal{R}_\alpha(\cos(\lambda \cdot))(x). \quad (5.7)$$

Trimèche [3, 16] has studied properties of $\mathcal{R}_\alpha$ on $\dot{\mathcal{S}}_*(\mathbb{R})$ and $\mathcal{W}_\alpha$ on $\mathfrak{D}_*(\mathbb{R})$ and gave the inversion formula of these operators.

The objective here is to give a relation between these operators and Bessel Plancherel transform on L_α^2 .

Theorem 5.2. *Let $-1/2 < \alpha < 0$. Then*

(i) *there exists a positive constant A_α such that*

$$\begin{aligned} \|\mathcal{R}_\alpha(f)\|_{2,\alpha} &\leq A_\alpha \|f\|_{2,\alpha}, \\ \left\| \frac{1}{x^{2\alpha+1}} \mathcal{W}_\alpha(f) \right\|_{2,\alpha} &\leq A_\alpha \|f\|_{2,\alpha}. \end{aligned} \quad (5.8)$$

(ii) *And*

$$\mathcal{R}_\alpha \Phi_\alpha = \Phi_\alpha \left(\frac{1}{x^{2\alpha+1}} \mathcal{W}_\alpha \right). \quad (5.9)$$

Proof. An easy calculation gives

$$\mathcal{R}_\alpha(f)(x) = C_{1,\alpha} T_\varphi(f)(x), \quad \frac{1}{x^{2\alpha+1}} \mathcal{W}_\alpha(f)(x) = C_{2,\alpha} T_\varphi^*(f)(x), \quad (5.10)$$

where

$$\varphi(t) = t^{-2\alpha-1} (1-t^2)^{\alpha-1/2} \cdot \chi_{]0,1[}(t), \quad (5.11)$$

with $\chi_{]0,1[}$ is the characteristic function of $]0,1[$. To obtain the theorem it suffices to verify the hypothesis of Theorems 3.1, 3.2, and 4.2. Indeed it is clear that φ is a measurable nonnegative function. Furthermore

$$\int_0^\infty \frac{\varphi(t)}{t^{\alpha+1}} d\mu_\alpha(t) = k_\alpha \int_0^1 \frac{(1-t^2)^{\alpha-1/2}}{t^{\alpha+1}} dt. \quad (5.12)$$

But this integral is finite if $-1/2 < \alpha < 0$. Thus the theorem is proved. $\square$

In the following we define and study another important integral transform associated with the differential operator l_α , namely, Sonine transform.

Definition 5.3. Let $\alpha, \beta \geq -1/2$, $\beta - \alpha > 0$. The Sonine transform associated with l_α is defined on $C_*(\mathbb{R})$ by

$$\mathcal{S}_{\beta,\alpha}(f)(x) = \begin{cases} C_{\beta,\alpha} x^{-2\beta} \int_0^x (x^2 - y^2)^{\beta-\alpha-1} f(y) d\mu_\alpha(y), & \text{if } x > 0, \\ f(0) & \text{if } x = 0, \end{cases} \quad (5.13)$$

where

$$C_{\beta,\alpha} = \frac{2\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}. \quad (5.14)$$

We note that

$$\mathcal{S}_{\beta,\alpha}(f)(x) = C_{\beta,\alpha} T_\varphi(f)(x), \quad (5.15)$$

where

$$\varphi(t) = (1-t^2)^{\beta-\alpha-1} \chi_{]0,1[}(t), \quad (5.16)$$

with $C_{\beta,\alpha}$ is given by relation (5.14). Moreover the function φ is measurable and nonnegative function, also for $\beta - \alpha > 0$ and $\alpha, \beta > -1/2$

$$\int_0^\infty \frac{\varphi(t)}{t^{\alpha+1}} d\mu_\alpha(t) = k(\alpha, \beta) \int_0^1 (1-t^2)^{\beta-\alpha-1} t^\alpha dt < \infty, \quad (5.17)$$

where $k(\alpha, \beta)$ is a nonnegative constant. This allows us to obtain the following result.

Theorem 5.4. *For $\alpha, \beta > -1/2$, $\beta - \alpha > 0$ one has the following.*

(i) *There exists a nonnegative constant $C_{1,\alpha,\beta}$ such that*

$$\begin{aligned} \|\mathcal{S}_{\beta,\alpha}(f)\|_{2,\alpha} &\leq C_{1,\alpha,\beta} \|f\|_{2,\alpha}, \\ \|\mathcal{S}_{\beta,\alpha}^*(f)\|_{2,\alpha} &\leq C_{1,\alpha,\beta} \|f\|_{2,\alpha}. \end{aligned} \quad (5.18)$$

(ii) *And*

$$\mathcal{S}_{\beta,\alpha} \Phi_\alpha = \Phi_\alpha \mathcal{S}_{\beta,\alpha}^*, \quad (5.19)$$

where

$$\mathcal{S}_{\beta,\alpha}^*(f)(x) = C_{\beta,\alpha} \int_x^\infty (y^2 - x^2)^{\beta-\alpha-1} \frac{f(y)}{y^{2\beta+1}} d\mu_\alpha(y). \quad (5.20)$$

Finally we give a particular integral transform which commutes with Bessel Plancherel transform. Let $\alpha > -1/2$ and

$$\varphi(t) = \frac{1}{(1+t^2)^{\alpha+1}}, \quad \text{for } t > 0. \quad (5.21)$$

It is clear that φ satisfies the hypothesis of Theorem 4.3. Thus the integral operator I_α defined by

$$I_\alpha(f)(x) = \int_0^\infty \frac{f(y)}{(x^2 + y^2)^{\alpha+1}} d\mu_\alpha(y) \quad (5.22)$$

is a self adjoint operator and we have the following.

Proposition 5.5. *Let $\alpha > -1/2$. Then*

(i) *there exists a nonnegative constant $C_{1,\alpha}$ such that*

$$\|I_\alpha(f)\|_{2,\alpha} \leq C_{1,\alpha} \|f\|_{2,\alpha'} \quad (5.23)$$

(ii) *And*

$$I_\alpha \Phi_\alpha = \Phi_\alpha I_\alpha. \quad (5.24)$$

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