Research Article

Characterizations of Orlicz-Sobolev Spaces by Means of Generalized Orlicz-Poincaré Inequalities

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Let Φ be an N -function. We show that a function $u \in L^\Phi(\mathbb{R}^n)$ belongs to the Orlicz-Sobolev space $W^{1,\Phi}(\mathbb{R}^n)$ if and only if it satisfies the (generalized) Φ -Poincaré inequality. Under more restrictive assumptions on Φ , an analog of the result holds in a general metric measure space setting.

1. Introduction

Let $X = (X, d, \mu)$ be a metric measure space, $\Omega \subset X$ open, and Φ a Young function. A pair (u, g) of measurable functions, $u \in L^1_{\text{loc}}(\Omega)$ and $g \geq 0$, satisfy the Φ -Poincaré inequality in Ω , if there is a constant $\tau \geq 1$ such that

$$\int_B |u - u_B| d\mu \leq r_B \Phi^{-1} \left(\int_{\tau B} \Phi(g) d\mu \right), \quad (1.1)$$

for every ball $B = B(x, r_B)$ such that $\tau B \subset \Omega$. Here, $u_B = \int_B u d\mu / \mu(B)$ and $\tau B = B(x, \tau r_B)$. If $\Phi(t) = t^p$, then (1.1) reduces to the familiar p -Poincaré inequality. The Φ -Poincaré inequality was introduced in [1] and further studied in [2–5].

In the euclidean setting, it is well known that $u \in W^{1,1}_{\text{loc}}(\Omega)$ satisfies the 1-Poincaré inequality

$$\int_B |u - u_B| d\mu \leq C_n r_B \int_B |\nabla u| d\mu, \quad (1.2)$$

for every ball $B \subset \Omega$. Thus, by Jensen's inequality, (1.1) holds with $\tau = 1$ and $g = C_n |\nabla u|$. Our first result, Theorem 1.1 below, says that also the converse holds: if $u \in L^\Phi(\Omega)$ and there exists $g \in L^\Phi(\Omega)$ such that (1.1) holds (for the normalized pair), then u belongs to the Sobolev class $W^{1,\Phi}(\Omega)$. More generally, we show that $u \in W^{1,\Phi}(\Omega)$ if and only if the number

$$\|u\|_{A_\tau^{1,\Phi}(\Omega)} = \sup_{B \in \mathcal{B}_\tau(\Omega)} \left\| \sum_{B \in \mathcal{B}} \left(r_B^{-1} \int_B |u - u_B| d\mu \right) \chi_B \right\|_{L^\Phi(\Omega)}, \quad (1.3)$$

where

$$\mathcal{B}_\tau(\Omega) = \{ \{B_i\} : \text{balls } \tau B_i \text{ are disjoint and contained in } \Omega \}, \quad (1.4)$$

is finite. Note that $\|u\|_{A_\tau^{1,\Phi}(\Omega)} \leq 1$ if and only if there is a functional $\nu : \{B \subset \Omega : B \text{ is a ball}\} \rightarrow [0, \infty)$ such that

$$\sum_i \nu(B_i) \leq 1, \quad (1.5)$$

whenever the balls B_i are disjoint, and that the generalized Φ -Poincaré inequality

$$\int_B |u - u_B| d\mu \leq r_B \Phi^{-1} \left(\frac{\nu(\tau B)}{\mu(B)} \right) \quad (1.6)$$

holds whenever $\tau B \subset \Omega$. In particular, if a pair $(u/\|g\|_{L^\Phi(\Omega)}, g/\|g\|_{L^\Phi(\Omega)})$ satisfies the Φ -Poincaré inequality in Ω , then

$$\|u\|_{A_\tau^{1,\Phi}(\Omega)} \leq \|g\|_{L^\Phi(\Omega)}. \quad (1.7)$$

The spaces $A_\tau^{1,\Phi}(\Omega) = \{u \in L_{\text{loc}}^1(\Omega) : \|u\|_{A_\tau^{1,\Phi}(\Omega)} < \infty\}$, for $\Phi(t) = t^p$, were studied in [6].

Theorem 1.1. *Suppose that Φ is an N -function, $\Omega \subset \mathbb{R}^n$ is open, and $u \in L^\Phi(\Omega)$, then the following conditions are equivalent:*

- (a) $u \in W^{1,\Phi}(\Omega)$,
- (b) there exists $g \in L^\Phi(\Omega)$ such that the pair $(u/\|g\|_{L^\Phi(\Omega)}, g/\|g\|_{L^\Phi(\Omega)})$ satisfies the Φ -Poincaré inequality in Ω ,
- (c) $u \in A_\tau^{1,\Phi}(\Omega)$ for some $\tau \geq 1$.

Moreover, if a functional ν satisfies (1.5) and (1.6), one has that

$$|\nabla u(x)| \leq C(C_d, \tau) \limsup_{r \rightarrow 0} \Phi^{-1} \left(\frac{\nu(B(x, r))}{\mu(B(x, r))} \right), \quad (1.8)$$

for a.e. $x \in \Omega$. In particular, (b) implies that $|\nabla u(x)| \leq C(n)g(x)$ for a.e. $x \in \Omega$.

Notice that $\Phi(t) = t$ is not an N -function. In this case, (a) and (b) are still equivalent, but (a) and (c) are not. In fact, it was shown in [6] that $A_r^{1,1}(\Omega)$ coincides with $BV(\Omega)$, the space of functions of bounded variation. The equivalence of (a) and (b) in the case $\Phi(t) = t^p$, $p > 1$ was proved in [7] and in the case $p = 1$ in [8]. A different proof of the case $p \geq 1$ was provided in [9]. The case where both Φ and its conjugate are doubling can be found in [5]. The equivalence of (a) and (c) in the case $\Phi(t) = t^p$, $p > 1$, was proved in [6]. The proof in [6] relies on a reflexivity argument which does not extend to the present setting. Our proof is a modification of the proof from [9].

The rest of our results are partial analogs of Theorem 1.1 in a general metric measure space setting. Let μ be a Borel regular outer measure satisfying $0 < \mu(U) < \infty$, whenever U is nonempty, open, and bounded. Suppose further that μ is doubling, that is, there exists a constant C_d such that

$$\mu(2B) \leq C_d \mu(B), \quad (1.9)$$

whenever $B = B(x, r)$ is a ball and $2B = B(x, 2r)$.

Our substitute for the usual Sobolev class $W^{1,\Phi}$ is based on upper gradients. We call a Borel function $g : X \rightarrow [0, \infty]$ an upper gradient of a function $u : X \rightarrow \overline{\mathbb{R}}$ if

$$|u(\gamma(0)) - u(\gamma(l))| \leq \int_\gamma g \, ds, \quad (1.10)$$

for all rectifiable curves $\gamma : [0, l] \rightarrow X$. Further, g as above is called a Φ -weak upper gradient if (1.10) holds for all curves γ except for a family of Φ -modulus zero, see Section 2.2 below. The concept of an upper gradient was introduced in [10]; also see [7]. The Sobolev space $N^{1,\Phi}(X)$ consists of all functions in $L^\Phi(X)$ that have a Φ -weak upper gradient that belongs to $L^\Phi(X)$.

Theorem 1.2. *Suppose that Φ is a doubling Young function, $\Omega \subset X$ is open, $u, g \in L^\Phi(\Omega)$, and that the pair $(u/\|g\|_{L^\Phi(\Omega)}, g/\|g\|_{L^\Phi(\Omega)})$ satisfies the Φ -Poincaré inequality in Ω , then a representative of u has a Φ -weak upper gradient $g_u \in L^\Phi(\Omega)$ such that $g_u(x) \leq C(C_d)g(x)$ for a.e. $x \in \Omega$.*

In the case $\Phi(t) = t^p$, $p \geq 1$, the result was essentially proved in [8], see [11].

If both Φ and its conjugate are doubling, then a generalization of the proof of [6, Theorem 1.1 (2)] yields the following.

Theorem 1.3. *Let $\Omega \subset X$ be an open set, and let Φ be a doubling Young function whose conjugate is doubling, then a representative of $u \in A_r^{1,\Phi}(\Omega) \cap L^\Phi(\Omega)$ has a Φ -weak upper gradient g with $\|g\|_{L^\Phi(\Omega)} \leq C(C_d, \tau)\|u\|_{A_r^{1,\Phi}(\Omega)}$. Moreover, for a functional v satisfying (1.5) and (1.6), one has that*

$$g(x) \leq C(C_d, \tau) \limsup_{r \rightarrow 0} \Phi^{-1} \left(\frac{v(B(x, r))}{\mu(B(x, r))} \right), \quad (1.11)$$

for a.e. $x \in \Omega$.

We say that a space X supports the Φ -Poincaré inequality if there exist constants C_P and τ such that

$$\int_B |u - u_B| d\mu \leq C_P r_B \Phi^{-1} \left(\int_{\tau B} \Phi(g) d\mu \right), \quad (1.12)$$

whenever $B \subset X$ is a ball, $u \in L^1_{\text{loc}}(X)$, and g is a Φ -weak upper gradient of u . The spaces supporting the Φ -Poincaré inequality include Riemannian manifolds with nonnegative Ricci curvature, Carnot groups, and general Carnot—Carathéodory spaces associated with a system of vector fields satisfying Hörmander's condition; see [11, 12] and the references therein.

Theorem 1.4. *Suppose that Φ is a doubling Young function, X supports the Φ -Poincaré inequality, $\Omega \subset X$ is open, and $u \in L^\Phi(\Omega)$, then the following conditions are equivalent.*

- (a) $u \in N^{1,\Phi}(\Omega)$.
- (b) *There exists $g \in L^\Phi(\Omega)$ such that the pair $(u/\|g\|_{L^\Phi(\Omega)}, g/\|g\|_{L^\Phi(\Omega)})$ satisfies the Φ -Poincaré inequality in Ω .*

If also the conjugate of Φ is doubling, then (a) and (b) are equivalent to

- (c) $u \in A^{1,\Phi}_\tau(\Omega)$ for some $\tau \geq 1$.

2. Preliminaries

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence. By writing $C = C(\lambda_1, \dots, \lambda_n)$, we indicate that the constant depends only on $\lambda_1, \dots, \lambda_n$.

2.1. Young Functions and Orlicz Spaces

In this subsection, we recall the basic facts about Young functions and Orlicz spaces. An exhaustive treatment of the subject is [13]. In the case of N -functions, good expositions are also [14] and [15, Chapter 8].

A function $\Phi : [0, \infty) \rightarrow [0, \infty]$ is called a Young function if it has the form

$$\Phi(t) = \int_0^t \phi(s) ds, \quad (2.1)$$

where $\phi : [0, \infty) \rightarrow [0, \infty]$ is an increasing, left-continuous function, which is neither identically zero nor identically infinite on $(0, \infty)$. If, in addition, $0 < \Phi(t) < \infty$ for $t > 0$, $\lim_{t \rightarrow 0} \Phi(t)/t = 0$, and $\lim_{t \rightarrow \infty} \Phi(t)/t = \infty$, then Φ is called an N -function.

A Young function is convex and, in particular, satisfies

$$\Phi(\varepsilon t) \leq \varepsilon \Phi(t), \quad (2.2)$$

for $0 < \varepsilon \leq 1$ and $0 \leq t < \infty$.

If Φ is a real-valued Young function and $\mu(X) < \infty$, then Jensen's inequality

$$\Phi\left(\int_X u d\mu\right) \leq \int_X \Phi(u) d\mu \quad (2.3)$$

holds for $0 \leq u \in L^1(X)$.

The right-continuous generalized inverse of a Young function Φ is

$$\Phi^{-1}(t) = \inf\{s : \Phi(s) > t\}. \quad (2.4)$$

We have that

$$\Phi(\Phi^{-1}(t)) \leq t \leq \Phi^{-1}(\Phi(t)), \quad (2.5)$$

for $t \geq 0$.

The conjugate of a Young function Φ is the Young function defined by

$$\hat{\Phi}(t) = \sup\{ts - \Phi(s) : s > 0\}, \quad (2.6)$$

for $t \geq 0$.

The conjugate of an N -function is an N -function.

Let Φ be a Young function. The Orlicz space $L^\Phi(X)$ is the set of all measurable functions u for which there exists $\lambda > 0$ such that

$$\int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) < \infty. \quad (2.7)$$

The Luxemburg norm of $u \in L^\Phi(X)$ is

$$\|u\|_{L^\Phi(X)} = \inf\left\{\lambda > 0 : \int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) \leq 1\right\}. \quad (2.8)$$

If $\|u\|_{L^\Phi(X)} \neq 0$, we have that

$$\int_X \Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(X)}}\right) d\mu(x) \leq 1. \quad (2.9)$$

The following generalized Hölder inequality holds for Luxemburg norms:

$$\int_X u(x)v(x) d\mu(x) \leq 2\|u\|_{L^\Phi(X)}\|v\|_{L^{\hat{\Phi}}(X)}. \quad (2.10)$$

A Young function Φ is doubling if there exists a constant $C_\Phi \geq 1$ such that

$$\Phi(2t) \leq C_\Phi \Phi(t), \quad (2.11)$$

for $t \geq 0$. Notice that a doubling Young function is realvalued and $\Phi(x) = 0$ if and only if $x = 0$.

Lemma 2.1. *Let Φ be a doubling Young function.*

- (1) *The space $C_0(X)$ of bounded, boundedly supported continuous functions is dense in $L^\Phi(X)$.*
- (2) *The modular convergence and the norm convergence are equivalent, that is,*

$$\|f_j - f\|_{L^\Phi(X)} \longrightarrow 0 \quad (2.12)$$

if and only if

$$\int_X \Phi(|f_j - f|) d\mu \longrightarrow 0. \quad (2.13)$$

If Φ is doubling, simple functions are dense in $L^\Phi(X)$ [13, Chapter III, Corollary 5]. Hence, the proof of (1.1) is the same as in the L^p -case; see, for example, [11, Theorem 4.2]. For the proof of (1.3), see [13, Chapter III, Theorem 12].

If Φ is doubling, then $(L^\Phi(X))^* = L^{\hat{\Phi}}(X)$, see [13, Chapter IV, Corollary 9]. So, if both Φ and $\hat{\Phi}$ are doubling, $L^\Phi(X)$ is reflexive. Thus, every bounded sequence in $L^\Phi(X)$ admits a weakly converging subsequence. In the proof of Theorem 1.2, we need to extract a weakly converging subsequence also when $\hat{\Phi}$ is not doubling. For this, we need the following lemma.

Lemma 2.2. *Suppose that Φ is a doubling Young function and that $\{g_i\} \subset L^\Phi(X)$ satisfies*

$$\begin{aligned} \sup_i \|g_i\|_{L^\Phi(X)} &< \infty, \\ \lim_{\mu(A) \rightarrow 0} \sup_i \int_A \Phi(|g_i|) d\mu &= 0, \end{aligned} \quad (2.14)$$

then there exists a subsequence (g_{i_j}) of (g_i) and $g \in L^\Phi(X)$ such that $g_{i_j} \rightarrow g$ weakly in $L^\Phi(X)$.

Proof. Since Φ is doubling, the dual of $L^\Phi(X)$ is $L^{\hat{\Phi}}(X)$. By [13, page 144, Corollary 2], a sequence $\{g_i\}$ has a weakly converging subsequence if for each $h \in L^{\hat{\Phi}}(X)$,

$$\begin{aligned} \sup_i \left| \int_X g_i h d\mu \right| &< \infty, \\ \lim_{\mu(A) \rightarrow 0} \sup_i \int_A |g_i h| d\mu &= 0. \end{aligned} \quad (2.15)$$

By the Hölder inequality and Lemma 2.1(2), these follow from (2.14). □

2.2. Sobolev Spaces

The Φ -modulus of a curve family Γ is

$$\text{Mod}_\Phi(\Gamma) = \inf \|g\|_{L^\Phi(X)}, \quad (2.16)$$

where the infimum is taken over all Borel functions $g : X \rightarrow [0, \infty]$ satisfying

$$\int_\gamma g \, ds \geq 1, \quad (2.17)$$

for all locally rectifiable curves $\gamma \in \Gamma$.

The Sobolev space $N^{1,\Phi}(X)$, consisting of the functions $u \in L^\Phi(X)$ having a Φ -weak upper gradient $g \in L^\Phi(X)$, was introduced by Shanmugalingam [16], when $\Phi(t) = t^p$, and extended to the Orlicz case by Tuominen [1]. The space $N^{1,\Phi}(X)$ is a Banach space with the norm

$$\|u\|_{N^{1,\Phi}(X)} = \|u\|_{L^\Phi(X)} + \inf \|g\|_{L^\Phi(X)}, \quad (2.18)$$

where the infimum is taken over Φ -weak upper gradients $g \in L^\Phi(X)$ of u .

Lemma 2.3. *Suppose that $u_i \rightarrow u \in L^\Phi(X)$ and $g_i \rightarrow g \in L^\Phi(X)$ weakly in $L^\Phi(X)$ and that g_i is a Φ -weak upper gradient of u_i , then g is a Φ -weak upper gradient of a representative of u . Moreover, $g(x) \leq \limsup_{i \rightarrow \infty} g_i(x)$ for a.e. $x \in X$.*

Proof. By Mazur's lemma ([17, Page 120, Theorem 2]), there is a sequence $(\tilde{g}_i)$ of convex combinations

$$\tilde{g}_i = \sum_{j=i}^{n_i} \lambda_{i,j} g_j, \quad (2.19)$$

where $\lambda_{i,j} \geq 0$ and $\sum_{j=1}^{n_i} \lambda_{i,j} = 1$, such that $\tilde{g}_i \rightarrow g$ in $L^\Phi(X)$. Hence, a subsequence of $(\tilde{g}_i)$ converges pointwise a.e., which implies that $g(x) \leq \limsup_{i \rightarrow \infty} \tilde{g}_i(x) \leq \limsup_{i \rightarrow \infty} g_i(x)$ for a.e. $x \in X$. The fact that g is a Φ -weak upper gradient of a representative of u was proved in [1, Theorem 4.17]. $\square$

If Φ is doubling and $\Omega \subset \mathbb{R}^n$ is an open set, then $N^{1,\Phi}(\Omega)$ is isomorphic to $W^{1,\Phi}(\Omega)$ [1, Theorem 6.19]. As usual, $W^{1,\Phi}(\Omega)$ is the space of functions $u \in L^\Phi(\Omega)$ having weak partial derivatives in $L^\Phi(\Omega)$. A function $v_i \in L^1_{\text{loc}}(\Omega)$ is a weak partial derivative of u (with respect to x_i) if

$$\int u \frac{\partial \varphi}{\partial x_i} = - \int v_i \varphi, \quad (2.20)$$

for all $\varphi \in C_0^\infty(\Omega)$.

Lemma 2.4. *Let Φ be an N -function. Suppose that the functional $\partial u / \partial x_i : C_0^\infty(\Omega) \rightarrow \mathbb{R}$,*

$$\frac{\partial u}{\partial x_i}[\varphi] := - \int u \frac{\partial \varphi}{\partial x_i}, \quad (2.21)$$

is bounded with respect to the norm $\|\cdot\|_{L^\Phi(\Omega)}$, then u has a weak partial derivative v_i such that $\|v_i\|_{L^\Phi(\Omega)} \leq \|\partial u / \partial x_i\|$.

Proof. Denote by $E^{\hat{\Phi}}(\Omega)$ the closure of the space of bounded, boundedly supported functions in $L^{\hat{\Phi}}(\Omega)$. By [15, Theorem 8.21(d)], $C_0^\infty(\Omega)$ is dense in $E^{\hat{\Phi}}(\Omega)$. Thus, $\partial u / \partial x_i$ extends to a continuous linear functional on $E^{\hat{\Phi}}(\Omega)$. By [15, Theorem 8.19], the dual of $E^{\hat{\Phi}}(\Omega)$ is isomorphic to $L^\Phi(\Omega)$; there exists $v_i \in L^\Phi(\Omega)$ such that

$$\frac{\partial u}{\partial x_i}[\varphi] = \int \varphi v_i, \quad (2.22)$$

for $\varphi \in E^{\hat{\Phi}}(\Omega)$. Moreover, $\|v_i\|_{L^\Phi(\Omega)} \leq \|\partial u / \partial x_i\|$. The claim follows. $\square$

2.3. Lipschitz Functions

A function $u : X \rightarrow \mathbb{R}$ is L -Lipschitz if $|u(x) - u(y)| \leq Ld(x, y)$ for all $x, y \in X$. The (upper) pointwise Lipschitz constant of a locally Lipschitz function u is

$$\text{Lip } u(x) = \limsup_{r \rightarrow 0} r^{-1} \sup_{d(x, y) \leq r} |u(x) - u(y)|. \quad (2.23)$$

It is well known that $\text{Lip } u$ is an upper gradient of u ; see, for example, [18].

3. Proofs

The proof of Theorem 1.1 is a modification of the proof of case $p > 1$ of [9, Lemma 6].

Proof of Theorem 1.1. As noted in the introduction, (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Let us show that (c) $\Rightarrow$ (a). Fix $u \in A_\tau^{1, \Phi}(\Omega)$. We will show that the functional $\partial u / \partial x_i : C_0^\infty(\Omega) \rightarrow \mathbb{R}$,

$$\frac{\partial u}{\partial x_i}[\varphi] := - \int u \frac{\partial \varphi}{\partial x_i}, \quad (3.1)$$

satisfies

$$\left| \frac{\partial u}{\partial x_i}[\varphi] \right| \leq C(n, \tau) \|u\|_{A_\tau^{1, \Phi}(\Omega)} \|\varphi\|_{L^{\hat{\Phi}}(\text{supp } \varphi)}. \quad (3.2)$$

Choose $0 \leq \varphi \in C_0^\infty(B(0,1))$ such that $\int \varphi = 1$, and let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ for $\varepsilon > 0$. Then

$$\frac{\partial u}{\partial x_i}[\varphi] = -\lim_{\varepsilon \rightarrow 0} \int (u * \varphi_\varepsilon) \frac{\partial \varphi}{\partial x_i} = \lim_{\varepsilon \rightarrow 0} \int \left(u * \frac{\partial \varphi_\varepsilon}{\partial x_i} \right) \varphi. \quad (3.3)$$

By the Hölder inequality,

$$\left| \frac{\partial u}{\partial x_i}[\varphi] \right| \leq 2 \liminf_{\varepsilon \rightarrow 0} \left\| u * \frac{\partial \varphi_\varepsilon}{\partial x_i} \right\|_{L^\Phi(\text{supp } \varphi)} \|\varphi\|_{L^\Phi(\text{supp } \varphi)}. \quad (3.4)$$

Since $\int \partial \varphi_\varepsilon / \partial x_i = 0$, we have that

$$\left(u * \frac{\partial \varphi_\varepsilon}{\partial x_i} \right)(x) = \left((u - u_{B(x,\varepsilon)}) * \frac{\partial \varphi_\varepsilon}{\partial x_i} \right)(x). \quad (3.5)$$

Thus,

$$\left| u * \frac{\partial \varphi_\varepsilon}{\partial x_i} \right|(x) \leq C(n)\varepsilon^{-1} \int_{B(x,\varepsilon)} |u(y) - u_{B(x,\varepsilon)}| dy. \quad (3.6)$$

Let $K = \text{supp } \varphi$ and let $0 < \varepsilon < d(K, \Omega^c)/3\tau$. Cover K with balls $B(x_j, 2\varepsilon)$, $x_j \in K$, such that the balls $B(x_j, \varepsilon)$ are disjoint. If $x \in B(x_j, 2\varepsilon)$, then $B(x, \varepsilon) \subset B(x_j, 3\varepsilon)$ and (3.6) implies that

$$\left| u * \frac{\partial \varphi_\varepsilon}{\partial x_i} \right|(x) \leq C(n)(3\varepsilon)^{-1} \int_{B(x_j, 3\varepsilon)} |u(y) - u_{B(x_j, 3\varepsilon)}| dy. \quad (3.7)$$

Thus,

$$\left| u * \frac{\partial \varphi_\varepsilon}{\partial x_i} \right| \leq C(n) \sum_j (3\varepsilon)^{-1} \int_{B(x_j, 3\varepsilon)} |u(y) - u_{B(x_j, 3\varepsilon)}| dy \chi_{B(x_j, 3\varepsilon)}. \quad (3.8)$$

Since the balls $B(x_j, \varepsilon)$ are disjoint, it follows that the family $\mathcal{B} = \{B(x_j, 3\varepsilon)\}$ can be divided into $k = C(n, \tau)$ subfamilies $\mathcal{B}_1, \dots, \mathcal{B}_k$ such that each of the families $\tau \mathcal{B}_j = \{\tau B : B \in \mathcal{B}_j\}$ consists of disjoint balls. Hence,

$$\begin{aligned} \left\| u * \frac{\partial \varphi_\varepsilon}{\partial x_i} \right\|_{L^\Phi(\Omega)} &\leq C(n) \sum_{j=1}^k \left\| \sum_{B \in \mathcal{B}_j} r_B^{-1} \int_B |u - u_B| d\mu \chi_B \right\|_{L^\Phi(\Omega)} \\ &\leq C(n, \tau) \|u\|_{A_\tau^{1,\Phi}(\Omega)}. \end{aligned} \quad (3.9)$$

This, combined with (3.4), yields (3.2). By Lemma 2.4, u has a weak partial derivative $v_i \in L^\Phi(\Omega)$.

Since $|v_i(x)| \leq \limsup_{\varepsilon \rightarrow 0} |(\partial/\partial x_i)(u * \varphi_\varepsilon)(x)|$ for a.e. $x \in \Omega$ and $(\partial/\partial x_i)(u * \varphi_\varepsilon)(x) = u * (\partial\varphi_\varepsilon/\partial x_i)(x)$ for small ε , it follows that

$$|v_i(x)| \leq \limsup_{\varepsilon \rightarrow 0} \left| u * \frac{\partial\varphi_\varepsilon}{\partial x_i}(x) \right|, \quad (3.10)$$

for a.e. $x \in \Omega$. Let v be a functional satisfying (1.5) and ((1.6), then, by (3.6),

$$\left| u * \frac{\partial\varphi_\varepsilon}{\partial x_i}(x) \right| \leq C(n)\Phi^{-1}\left(\frac{v(B(x, \tau\varepsilon))}{\mu(B(x, \varepsilon))}\right) \leq C(n, \tau)\Phi^{-1}\left(\frac{v(B(x, \tau\varepsilon))}{\mu(B(x, \tau\varepsilon))}\right). \quad (3.11)$$

Thus, (1.8) holds for a.e. $x \in \Omega$. If condition (b) is satisfied, we have

$$\left| u * \frac{\partial\varphi_\varepsilon}{\partial x_i}(x) \right| \leq C(n)\|g\|_{L^\Phi(\Omega)}\Phi^{-1}\left(\int_{B(x, \tau\varepsilon)} \Phi\left(\frac{g(y)}{\|g\|_{L^\Phi(\Omega)}}\right)dy\right), \quad (3.12)$$

which implies that $|v_i(x)| \leq C(n)g(x)$ for a.e. $x \in \Omega$. This completes the proof. $\square$

The proofs of Theorems 1.2 and 1.3 are based on approximation by Lipschitz convolutions. The same technique was employed in [6–8]. The proof of Theorem 1.3 is a generalization of the proof of [6, Theorem 1.1]. Using a partition of unity and averages of u on balls, we construct a sequence of locally Lipschitz functions u_j so that $u_j \rightarrow u$ in $L^\Phi(\Omega)$ and that

$$\|\text{Lip } u_j\|_{L^\Phi(\Omega)} \leq C\|u\|_{A_\tau^{1,\Phi}(\Omega)}. \quad (3.13)$$

Since $L^\Phi(\Omega)$ is reflexive, a subsequence of $(\text{Lip } u_j)$ converges weakly, and the claim follows from Lemma 2.3.

Under the assumptions of Theorem 1.2, $L^\Phi(\Omega)$ may not be reflexive, but using the Poincaré inequality, we can show that $(\text{Lip } u_j)$ is uniformly Φ -integrable. The existence of a weakly converging subsequence then follows from Lemma 2.2. A similar argument was used in [8].

We need a couple of standard lemmas. For the proofs, see [19, Theorem III.1.3] and [20, Lemmas 2.9 and 2.16].

Lemma 3.1. *Let $\Omega \subset X$ be open. Given $\varepsilon > 0$, $\lambda \geq 1$, there is a cover $\{B_i = B(x_i, r_i)\}$ of Ω with the following properties:*

- (1) $r_i \leq \varepsilon$ for all i ,
- (2) $\lambda B_i \subset \Omega$ for all i ,
- (3) if λB_i meets λB_j , then $r_i \leq 2r_j$,
- (4) each ball λB_i meets at most $C = C(C_d, \lambda)$ balls λB_j .

A collection $\{B_i\}$ as above is called an (ε, λ) -covering of Ω . Clearly, an (ε, λ) -cover is an (ε', λ') -cover provided $\varepsilon' \geq \varepsilon$ and $\lambda' \leq \lambda$.

Lemma 3.2. Let $\Omega \subset X$ be open, and let $\mathcal{B} = \{B_i = B(x_i, r_i)\}$ be an $(\infty, 2)$ -cover of Ω , then there is a collection $\{\varphi_i\}$ of functions $\Omega \rightarrow \mathbb{R}$ such that

- (1) each φ_i is $C(C_d)r_i^{-1}$ -Lipschitz,
- (2) $0 \leq \varphi_i \leq 1$ for all i ,
- (3) $\varphi_i(x) = 0$ for $x \in X \setminus 2B_i$ for all i ,
- (4) $\sum_i \varphi_i(x) = 1$ for all $x \in \Omega$.

A collection $\{\varphi_i\}$ as above is called a partition of unity with respect to $\mathcal{B}$.

Let $\mathcal{B} = \{B_i\}$ be as in the lemma above, and let $\{\varphi_i\}$ be a partition of unity with respect to $\mathcal{B}$. For a locally integrable function u on Ω , define

$$u_{\mathcal{B}}(x) = \sum_i u_{B_i} \varphi_i(x). \quad (3.14)$$

The following lemma describes the most important properties of $u_{\mathcal{B}}$.

Lemma 3.3. (1) The function $u_{\mathcal{B}}$ is locally Lipschitz. Moreover, for each $x \in B_i$,

$$\text{Lip } u_{\mathcal{B}}(x) \leq C(C_d)r_{B_i}^{-1} \int_{5B_i} |u - u_{5B_i}| d\mu. \quad (3.15)$$

(2) Let Φ be a doubling Young function, and let $u \in L^{\Phi}(\Omega)$. If $\mathcal{B}_k$ is an $(\varepsilon_k, 2)$ -cover of Ω and $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, then $u_{\mathcal{B}_k} \rightarrow u$ in $L^{\Phi}(\Omega)$.

Proof. (1) See the proof of [6, Lemma 5.3(1)].

(2) We begin by showing that, for every $w \in L^{\Phi}(\Omega)$,

$$\|w_{\mathcal{B}}\|_{L^{\Phi}(\Omega)} \leq C(C_d)\|w\|_{L^{\Phi}(\Omega)}. \quad (3.16)$$

We may assume that $\|w\|_{L^{\Phi}(\Omega)} = 1$. By Jensen's inequality, $\Phi(|w_{\mathcal{B}}|) \leq (\Phi(|w|))_{\mathcal{B}}$. Hence, by the properties of the functions φ_i ,

$$\begin{aligned} \int_{\Omega} \Phi(|w_{\mathcal{B}}|) d\mu &\leq \int_{\Omega} (\Phi(|w|))_{\mathcal{B}} d\mu \leq \sum_i \int_{\Omega} (\Phi(|w|))_{B_i} \varphi_i d\mu \\ &\leq \sum_i \int_{2B_i} \Phi(|w|)_{B_i} d\mu \leq C_d \sum_i \int_{B_i} \Phi(|w|) d\mu \\ &= C_d \int_{\Omega} \Phi(|w|) \sum_i \chi_{B_i} d\mu \leq C(C_d) \int_{\Omega} \Phi(|w|) d\mu \\ &\leq C(C_d). \end{aligned} \quad (3.17)$$

Thus, by (2.2), we obtain (3.16).

Let $u \in L^\Phi(\Omega)$ and $\varepsilon > 0$. By Lemma 2.1(1), there exists $v \in C_0(\Omega)$ such that $\|u - v\|_{L^\Phi(\Omega)} < \varepsilon$. Then, by (3.16), we obtain

$$\|u_{\mathcal{B}} - v_{\mathcal{B}}\|_{L^\Phi(\Omega)} = \|(u - v)_{\mathcal{B}}\|_{L^\Phi(\Omega)} \leq C(C_d)\|u - v\|_{L^\Phi(\Omega)} < C(C_d)\varepsilon, \quad (3.18)$$

and so

$$\begin{aligned} \|u_{\mathcal{B}} - u\|_{L^\Phi(\Omega)} &\leq \|u_{\mathcal{B}} - v_{\mathcal{B}}\|_{L^\Phi(\Omega)} + \|v_{\mathcal{B}} - v\|_{L^\Phi(\Omega)} + \|v - u\|_{L^\Phi(\Omega)} \\ &< \|v_{\mathcal{B}} - v\|_{L^\Phi(\Omega)} + C(C_d)\varepsilon. \end{aligned} \quad (3.19)$$

Therefore, it suffices to show that $\|v_{\mathcal{B}_k} - v\|_{L^\Phi(\Omega)} \rightarrow 0$ as $\varepsilon_k \rightarrow 0$. Now, $|v_{\mathcal{B}_k} - v| \leq 2 \sup |v|$, and for all x , we have that

$$\begin{aligned} |v_{\mathcal{B}_k}(x) - v(x)| &\leq \sum_{2\mathcal{B}_i \ni x} \int_{\mathcal{B}_i} |v(y) - v(x)| d\mu(y) \\ &\leq C(C_d) \int_{B(x, 5\varepsilon_k)} |v(y) - v(x)| d\mu(y), \end{aligned} \quad (3.20)$$

which converges to 0 as $\varepsilon_k \rightarrow 0$ by the continuity of v . Thus, by the dominated convergence theorem,

$$\int_{\Omega} \Phi(|v_{\mathcal{B}_k} - v|) d\mu \rightarrow 0, \quad (3.21)$$

and so, by Lemma 2.1(2), $\|v_{\mathcal{B}_k} - v\|_{L^\Phi(\Omega)} \rightarrow 0$. □

Proof of Theorem 1.3. Let $u \in A_\tau^{1,\Phi}(\Omega) \cap L^\Phi(\Omega)$. For $j \in \mathbb{N}$, let $\mathcal{B}_j$ be a $(j^{-1}, 5\tau)$ -cover (and hence also a $(j^{-1}, 2)$ -cover) of Ω , then, by Lemma 3.3(2), $u_j := u_{\mathcal{B}_j} \rightarrow u$ in $L^\Phi(\Omega)$. Let us show that

$$\|\text{Lip } u_j\|_{L^\Phi(\Omega)} \leq C(C_d, \tau)\|u\|_{A_\tau^{1,\Phi}(\Omega)}. \quad (3.22)$$

By Lemma 3.3(1),

$$\text{Lip } u_j \leq C(C_d) \sum_{B \in \mathcal{B}_j} r_B^{-1} \int_{5B} |u - u_{5B}| d\mu \chi_B. \quad (3.23)$$

It follows from Lemma 3.1(4) that $\mathcal{B}_j$ can be divided into $k = C(C_d, \tau)$ subfamilies $\mathcal{B}_{j,1}, \dots, \mathcal{B}_{j,k}$ so that each of the families $5\tau\mathcal{B}_{j,l}$ consists of disjoint balls. Since the families $5\mathcal{B}_{j,1}, \dots, 5\mathcal{B}_{j,k}$ belong to $\mathcal{B}_\tau(\Omega)$, we have that

$$\begin{aligned} \|\text{Lip } u_j\|_{L^\Phi(\Omega)} &\leq C(C_d) \sum_{l=1}^k \left\| \sum_{B \in \mathcal{B}_{j,l}} r_B^{-1} \int_{5B} |u - u_{5B}| d\mu \chi_B \right\|_{L^\Phi(\Omega)} \\ &\leq C(C_d) \sum_{l=1}^k \left\| \sum_{B \in 5\mathcal{B}_{j,l}} r_B^{-1} \int_B |u - u_B| d\mu \chi_B \right\|_{L^\Phi(\Omega)} \\ &\leq C(C_d, \tau) \|u\|_{A_\tau^{1,\Phi}(\Omega)}. \end{aligned} \quad (3.24)$$

Since Φ and $\widehat{\Phi}$ are doubling, $L^\Phi(\Omega)$ is reflexive. Thus, the bounded sequence $(\text{Lip } u_j)$ has a subsequence that converges weakly to some $g \in L^\Phi(\Omega)$. By Lemma 2.3, g is a Φ -weak upper gradient of a representative of u . As a weak limit, g satisfies

$$\|g\|_{L^\Phi(\Omega)} \leq \limsup_{j \rightarrow \infty} \|\text{Lip } u_j\|_{L^\Phi(\Omega)} \leq C(C_d, \tau) \|u\|_{A_\tau^{1,\Phi}(\Omega)}. \quad (3.25)$$

Let ν be a functional satisfying (1.5) and (1.6). Using Lemma 3.3(1), we obtain

$$\begin{aligned} \text{Lip } u_j(x) &\leq C(C_d) \left(10j^{-1}\right)^{-1} \int_{B(x, 10j^{-1})} |u - u_{B(x, 10j^{-1})}| d\mu \\ &\leq C(C_d) \Phi^{-1} \left(\frac{\nu(B(x, 10\tau j^{-1}))}{\mu(B(x, 10j^{-1}))} \right) \\ &\leq C(C_d, \tau) \Phi^{-1} \left(\frac{\nu(B(x, 10\tau j^{-1}))}{\mu(B(x, 10\tau j^{-1}))} \right). \end{aligned} \quad (3.26)$$

Since, by Lemma 2.3, $g(x) \leq \limsup_{j \rightarrow \infty} \text{Lip } u_j(x)$ for a.e. $x \in \Omega$, the pointwise inequality (1.11) follows. $\square$

Proof of Theorem 1.2. We may assume that $\|g\|_{L^\Phi(\Omega)} = 1$. Define the functions u_j as in the proof of Theorem 1.3. By (3.22) and (1.7), we have that

$$\|\text{Lip } u_j\|_{L^\Phi(\Omega)} \leq C(C_d, \tau). \quad (3.27)$$

Let us show that

$$\lim_{\mu(E) \rightarrow 0} \sup_j \int_E \Phi(\text{Lip } u_j) d\mu = 0. \quad (3.28)$$

By Lemma 3.3(1) and by the Φ -Poincaré inequality,

$$\begin{aligned} \text{Lip } u_j &\leq C(C_d) \sum_{B \in \mathcal{B}_j} r_B^{-1} \int_{5B} |u - u_{5B}| d\mu \chi_B \\ &\leq C(C_d) \sum_{B \in \mathcal{B}_j} \Phi^{-1} \left(\int_{5\tau B} \Phi(g) d\mu \right) \chi_B. \end{aligned} \quad (3.29)$$

Thus,

$$\int_E \Phi(\text{Lip } u_j) d\mu \leq C(C_d, C_\Phi) \sum_{B \in \mathcal{B}_j} \frac{\mu(E \cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g) d\mu. \quad (3.30)$$

Since $\mathcal{B}_j$ can be divided into $k = C(C_d, \tau)$ subfamilies $\mathcal{B}_{j,1}, \dots, \mathcal{B}_{j,k}$ so that each of the families $5\tau\mathcal{B}_{j,l}$ consists of disjoint balls, it suffices to show that, for $1 \leq l \leq k$,

$$\lim_{\mu(E) \rightarrow 0} \sum_{B \in \mathcal{B}_{j,l}} \frac{\mu(E \cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g) d\mu = 0. \quad (3.31)$$

Fix $\varepsilon > 0$. Then, there exists $\delta > 0$ such that $\int_A \Phi(g) < \varepsilon$ whenever $\mu(A) < \delta$. Denote by $\mathcal{B}$ the family of those balls B in $\mathcal{B}_{j,l}$ for which

$$\frac{\mu(E \cap B)}{\mu(5\tau B)} < \varepsilon. \quad (3.32)$$

Also, let $\mathcal{B}' = \mathcal{B}_{j,l} \setminus \mathcal{B}$. Now, if $\mu(E) < \varepsilon\delta$, we have that $\mu(\bigcup_{B \in \mathcal{B}'} 5\tau B) \leq \varepsilon^{-1} \mu(E) < \delta$. Thus,

$$\begin{aligned} \sum_{B \in \mathcal{B}_{j,l}} \frac{\mu(E \cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g) d\mu &= \sum_{B \in \mathcal{B}} \frac{\mu(E \cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g) d\mu + \sum_{B \in \mathcal{B}'} \frac{\mu(E \cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g) d\mu \\ &\leq \varepsilon \int_{\Omega} \Phi(g) d\mu + \int_{\bigcup_{B \in \mathcal{B}'} 5\tau B} \Phi(g) d\mu \\ &\leq 2\varepsilon. \end{aligned} \quad (3.33)$$

This completes the proof of (3.28).

By Lemma 2.2, a subsequence of $(\text{Lip } u_j)$ converges weakly to some $g_u \in L^\Phi(\Omega)$, which, by Lemma 2.3, is a Φ -weak upper gradient of a representative of u . Moreover, $g_u(x) \leq \limsup_{j \rightarrow \infty} \text{Lip } u_j(x)$ for a.e. $x \in \Omega$. It follows from Lemma 3.3(1) and from the Φ -Poincaré inequality that

$$\text{Lip } u_j(x) \leq C(C_d) \Phi^{-1} \left(\int_{B(x, 10\tau j^{-1})} \Phi(g) d\mu \right). \quad (3.34)$$

Thus, $g_u(x) \leq C(C_d)g(x)$ for a.e. $x \in \Omega$. □

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