A Path-Integral Approach to the
Cameron-Martin-Maruyama-Girsanov Formula
Associated to a Bilaplacian

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We define the Wiener product on a bosonic Connes space associated to a Bilaplacian and we introduce formal Wiener chaos on the path space. We consider the vacuum distribution on the bosonic Connes space and show that it is related to the heat semigroup associated to the Bilaplacian. We deduce a Cameron-Martin quasi-invariance formula for the heat semigroup associated to the Bilaplacian by using some convenient coherent vector. This paper enters under the Hida-Streit approach of path integral.

1. Introduction

Let us recall some basic tools of Wiener analysis. Let $B_t$ be a one-dimensional Brownian motion starting from 0. It is classically related to the heat equation on $\mathbb{R}$:

$$\frac{\partial}{\partial t} E[f(B_t)] = \frac{1}{2} E[\Delta f(B_t)], \tag{1.1}$$

where $\Delta = \partial^2/\partial x^2$ is the standard Laplacian and $f$ is a smooth function with bounded derivatives at each order. Associated to the heat equation there is a convenient probability measure on a convenient path space. Almost surely, the trajectory of $B$ is continuous. We construct by this way the Wiener measure $dP$ on the continuous path space endowed with its Borelian $\sigma$-algebra. Let $H$ be the Hilbert space $L^2([0,1];\mathbb{R})$. We consider the symmetric tensor
product $\mathbb{H}^\otimes n$ of this Hilbert space. It is constituted of maps $h_n(s_1, \ldots, s_n)$ symmetric in $s_i$ such that

$$\|h_n\|^2 = \int_{[0,1]^n} h_n^2(s_1, \ldots, s_n) ds_1 \cdots ds_n < \infty. \quad (1.2)$$

We consider the symmetric Fock space $F(\mathbb{H})$ of set $\sigma = \sum_{n=0}^{\infty} h_n$ such that

$$\|\sigma\|^2 = \sum_{n!}^n \|h_n\|^2 < \infty. \quad (1.3)$$

We consider the vacuum expectation.

$$\mu(\sigma) = h_0. \quad (1.4)$$

With an element $h_n$ of $\mathbb{H}^\otimes n$ is associated the Wiener chaos

$$\Psi(h_n) = \int_{[0,1]^n} h_n(s_1, \ldots, s_n) dB_{s_1} \cdots dB_{s_n}. \quad (1.5)$$

The mat $\Psi$ realizes a isomorphism between $F(\mathbb{H})$ and $L^2(dP)$. On the level of the Fock space some important elements are constituted by coherent vectors:

$$\sigma = \sum \frac{h_n^\otimes n}{n!}. \quad (1.6)$$

The functional associated to such a coherent vector is a so-called exponential martingale

$$\Psi(\sigma) = \exp \left[ \int_0^1 h_s dB_s - \frac{\|h\|^2}{2} \right]. \quad (1.7)$$

We refer to the books of Hida et al. [1], to the book of Obata [2], and to the book of Meyer [3] for an extensive study on that subject. Especially on the Fock space, we can define the Wiener product:

$$\Psi(\sigma_1 \cdot \sigma_2) = \Psi(\sigma_1)\Psi(\sigma_2), \quad (1.8)$$

where we consider the ordinary product of the two $\Psi(\sigma_i)$. For that, we use the Itô table for the Laplacian

$$dB_s \cdot dB_s = \frac{1}{2} ds,$$

$$dB_s \cdot ds = ds \cdot ds = 0 \quad (1.9)$$
which reflect algebraically the Itô formula for the Brownian motion. From this Itô table, we
deduce classically that if $\sigma$ is an exponential vector, $\Psi(\sigma) = \exp[\int_0^1 h_s dB_s - \|h\|^2/2]$ and not
\[ \exp[\int_0^1 h_s dB_s - \|h\|^2/2]. \]
The law of $B_t + \int_0^t h_s ds$ is absolutely continuous with respect of the law of $B_t$, and the
Radon-Nikodym derivative between these two laws is $\Psi(\sigma) = \exp[-\int_0^1 h_s dB_s - \|h\|^2/2]$. It is
the subject of the Cameron-Martin formula.

The construction of a full path probability measure associated to a semi-group is
related to Hunt theory: the generator $L$ of the semi-group has to satisfy maximum principle.
We are motivated where we take others type of generator. To simplify the computations
we take the simplest of such operators $L = -\partial^4/\partial x^4$. We have implemented recently some
stochastic tools for semi-groups whose generators do not simplify maximum principle ([4–
10]). We construct in [8, 9] the Wiener distribution associated to a Bilaplacian using the
Hida-Streit approach of path integrals as distribution. We refer to the works of Funaki [11],
Hochberg [12], Krylov [13], and the review paper of Mazzucchi [14] for other approaches.
We refer to the review paper of Albeverio [15] for various approach of path integrals.

In the Hida-Streit approach of path integral, there are basically 3 objects:

(i) an algebraic space, generally a kind of Fock space;

(ii) a map $\Psi$ from this algebraic space into a set of functionals on a mapping space;

(iii) the path integral is continuous on the level of the algebraic set. We say that it is an
Hida-type distribution.

Generally, people were considering map $\Psi$ as the map Wiener chaos. A breakdown
was performed by Getzler [16] motivated by the works of Atiyah-Bismut-Witten relating the
structure of the free loop space and the Index theory. Developments were done by Léandre
in [17, 18]. Especially, in [8, 9] we were using map $\Psi$ as related to cylindrical functional to
define a path integral associated to the Bilaplacian and to state some properties related to this
path integral.

In this paper, we come back to the original map $\Psi$ of Wiener, by using Wiener chaos.
But we use formal Wiener chaos. We consider a continuous path $\omega$. We consider a map
$h_n^{i_1,\ldots,i_n}(s_1,\ldots,s_n)$ $s_1 < s_2 < \cdots < s_n < 1$ with values in $\mathbb{R}$. We consider the formal Wiener
chaos:

$$\Psi(h_n) = \int_{0<s_1<\cdots<s_n<1} h_n^{i_1,\ldots,i_n}(s_1,\ldots,s_n) d\omega_{s_1}^{i_1} \cdots d\omega_{s_n}^{i_n}. \quad (1.10)$$

We put

$$d\omega_{s}^4 = 24 ds. \quad (1.11)$$

If $i > 4$, $d\omega_{s}^i = 0$. We use in order to define the Wiener product on formal chaos associated to
the Bilaplacian $L$ the Itô table for the Bilaplacian:

$$d\omega_{s}^i d\omega_{s}^j = d\omega_{s}^{i+j}. \quad (1.12)$$
In order to simplify the exposition, we use in the sequel Connes space and not a Hida Fock space. We consider $L_\infty$ the set of maps $h$ from $[0,1]$ into $\mathbb{R}^3$ such that
\begin{equation}
\sup_s |h(s)| = \|h\|_\infty. \tag{1.13}
\end{equation}

We introduce the bosonic Connes space $\text{CO}_\infty(L_\infty)$ (a refinement of the traditional bosonic Fock space). To $\sigma \in \text{CO}_\infty(L_\infty)$, we associate a formal Wiener chaos $\Psi(\sigma)$. We use the Itô table for the Bilaplacian in order to define a Wiener product on the bosonic Connes space:
\begin{equation}
\Psi(\sigma_1 \cdot \sigma_2) = \Psi(\sigma_1)\Psi(\sigma_2). \tag{1.14}
\end{equation}

The bosonic Connes space becomes a commutative topological algebra for the Wiener product (For similar consideration for the case of the standard Laplacian, we refer to the book of Meyer [3]).

We consider as classical the vacuum expectation on the bosonic Connes space, and we state a kind of Itô-Segal-Bargmann-Wiener isomorphism, but in this case there is no Hilbert space involved. We show that for the vacuum expectation $\psi_s$ has in some sense independent increments. We consider a type of generalization of the exponential martingale of the Brownian motion:
\begin{equation}
\Psi(\sigma_t) = \sum \int_{0 < s_1 < \cdots < s_n < t} h_{s_1}dw_{s_1} \cdots h_{s_n}dw_{s_n}. \tag{1.15}
\end{equation}

We suppose that $h$ is continuous. Let $f$ be a polynomial on $\mathbb{R}$. We put
\begin{equation}
Q^h_t[f] = \mu \left[ f \left( w^t_1 \right) \Psi(\sigma_t) \right]. \tag{1.16}
\end{equation}

We show the following Cameron-Martin-Maruyama-Girsanov type formula:
\begin{equation}
\frac{\partial}{\partial t} Q^h_t[f] = Q^h_t[L_h,tf], \tag{1.17}
\end{equation}

where
\begin{equation}
L_{h,t} = L + \text{lowerterm}. \tag{1.18}
\end{equation}

### 2. Formal Wiener Chaos Associated to a Bilaplacian

We consider the set $L_\infty$. $(L_\infty)^\otimes n$ is constituted of maps:
\begin{equation}
\sum_{h_{i_1}, \ldots, h_{i_n}} h_{i_1, \ldots, i_n}(s_1, \ldots, s_n)e_i_1 \otimes \cdots \otimes e_i_n = h_n(s_1, \ldots, s_n), \tag{2.1}
\end{equation}

where $e_i$ is the standard basis of $\mathbb{R}^3$. On $(L_\infty)^\otimes n$, we consider the natural supremum norm $\|h_n\|_\infty$. Moreover, there is a natural action of the symmetric group on $(L_\infty)^\otimes n$. Elements which
are invariant under this action of the symmetric group are called elements of the symmetric tensor product \((L^\infty)^{\otimes n}\). \(CO_{C^r}(L^\infty)\) \((r > 0, C > 0)\) is constituted of formal series \(\sigma = \sum h_n\) where \(h_n\) belongs to \((L^\infty)^{\otimes n}\) such that

\[
\|\sigma\|_{C^n} = \sum C^n n!\|h_n\|_\infty < \infty.
\] (2.2)

**Definition 2.1.** The intersection of all \(CO_{C}(L^\infty)\) is called the bosonic Connes space \(CO_{\infty_1}(L^\infty)\).

**Remark 2.2.** In the sequel we could choose an Hida Fock space.

**Definition 2.3.** The vacuum expectation \(\mu\) on \(CO_{\infty_1}(L^\infty)\) is defined by

\[
\mu(\sigma) = h_0.
\] (2.3)

If \(h_n\) belongs to \((L^\infty)^{\otimes n}\), we consider the formal Wiener chaos:

\[
\Psi(h_n) = \sum \int_{0 < s_1 < < s_n < 1} h_{n}^{s_1,\ldots,s_n} (s_1, \ldots, s_n) dw_{s_1}^{s_2} \cdots dw_{s_n}^{s_2}.
\] (2.4)

We could do the same expression if \(h_n\) belongs to \((L^\infty)^{\otimes n}\).

**Definition 2.4.** The map \(\Psi\) defined on \(CO_{\infty_1}(L^\infty)\) is called the map formal Wiener chaos.

Let \(\{1, \ldots, n\}, \{n + 1, \ldots, n + m\}\). Let \(l\) be a concatenation (or pairing). It is an increasing injective map from a set with \(l\) element in \(\{1, \ldots, n\}\) into \(\{n + 1, \ldots, n + m\}\). There is at most \(C^{n+m}\) pairing of length \(l\). We consider \(h_{l_1}^{l_2} \otimes |l|, sh_{l_1}, h_{m_1}^{m_2}\) where we concatenate the time in \(h_n\) and in \(h_m\) according the pairing, and we shuffle according to the shuffle \(sh_{l}\) and the time in \(h_{n}^{l_2}\) and \(h_{m}^{m_2}\) between two continuous times in the pairing. When we concatenate two times, we use the It\(\tilde{o}\) table for the Bilaplacian, and we symmetrized the expression in the time.

The classical product of \(\Psi(h_{l_1}^{l_2})\Psi(h_{m}^{m_2})\) is equal to \(\sum_{|l|, sh_{l_1}} \Psi(h_{l_1}^{l_2} \otimes |l|, sh_{l_1}, h_{m_1}^{m_2})\) and generalized with this new It\(\tilde{o}\) table the standard formula which gives the product of two Wiener chaos in the Brownian case. There are at most \(C^{n+m}C^l_{n}C^l_{m}\) pairing \(|l|\) and shuffle according to the pairing \(|l|\).

**Definition 2.5.** The Wiener product of \(h_{n}^{l_1}\) and \(h_{m}^{m_2}\) is defined by

\[
\Psi\left(h_{n}^{l_1} \cdot h_{m}^{m_2}\right) = \Psi(h_{n}^{l_1}) \Psi(h_{m}^{m_2}).
\] (2.5)

**Theorem 2.6.** The Wiener product endows the symmetric Connes space with a structure of topological commutative algebra.

**Proof.** Let us show first of all that the Wiener product is continuous. We have

\[
\left\|h_{\otimes (|l|, sh_{l_1}, h_{m_1}^{m_2})}\right\|_\infty \leq C^{n+m} \left\|h_{n}^{l_1}\right\|_\infty \left\|h_{m}^{m_2}\right\|_\infty.
\] (2.6)
Therefore,
\[
\| h_n \cdot h_m^2 \|_C \leq C_1^{n+m} C_{n+m} \| h_n \|_\infty \| h_m^2 \|_\infty \sum_{[l],sh[l]} C^{-l}((n + m - 2l)!).
\] (2.7)

But
\[
\sum_{[l],sh[l]} C^{-l} \leq \sum_l C_l^l C_m^m C_{n+m}^l C^{-l} \leq C_2^{n+m} (1 + C^{-1})^{n+m} \leq C_4^{n+m}.
\] (2.8)

On the other hand, by the Stirling formula,
\[
(n!)^{-1} (m!)^{-1} (n + m - 2l)! \leq C_3^{n+m}.
\] (2.9)

We deduce that
\[
\| \sigma_1 \cdot \sigma_2 \|_C \leq K \| \sigma_1 \|_C \| \sigma_2 \|_C
\] (2.10)

and therefore the Wiener product is continuous on the bosonic Connes space.

Let \( h_n, h_m, \) and \( h_{n_i} \) be 3 elements of the bosonic Connes space.

Let \( sh_{1,2,3} \) be a shuffle between the 3 sets \( \{1, n_1\}, \{n_1 + 1, n_1 + n_2\}, \) and \( \{n_1 + n_2 + 1, n_1 + n_2 + n_3\} \).

We perform two concatenations between the times when the shuffle is done:

(i) either we concatenate 2 contiguous times in \( \{1, n_1\} \) and in \( \{n_1 + 1, n_1 + n_2\} \) and two contiguous time in \( \{1, n_1\} \) and in \( \{n_1 + n_2 + 1, n_1 + n_2 + n_3\} \);

(ii) either we concatenate 2 contiguous times in \( \{n_1 + 1, n_1 + n_2\} \) and in \( \{1, n_1\} \) and two contiguous times in \( \{n_1 + n_2 + 1, n_1 + n_2 + n_3\} \);

(iii) either we concatenate 2 contiguous times in \( \{n_1 + n_2 + 1, n_1 + n_2 + n_3\} \) and in \( \{1, n_1\} \) and two contiguous times in \( \{n_1 + n_2 + 1, n_1 + n_2 + n_3\} \) and in \( \{n_1 + 1, n_1 + n_2\} \);

(iv) or we concatenate 3 contiguous times in \( \{1, n_1\}, \) in \( \{n_1 + 1, n_1 + n_2\} \) and in \( \{n_1 + n_2 + 1, n_1 + n_2 + n_3\} \).

When we concatenate time, we use the iterated Itô rule:
\[
(d w_t^i \cdot d w_t^j) \cdot d w_{t+s}^{ij} = d w_{t+s}^{i+j+ij}.
\] (2.11)

Such a concatenation is called \( I_{1,2,3} \) and the final result is called \( h_n \otimes_{sh_{1,2,3}} I_{1,2,3} h_m \otimes_{sh_{1,2,3}} I_{1,2,3} h_{n_3}. \) We deduce the formula
\[
(h_n \cdot h_{n_2}) \cdot h_{n_3} = \sum_{I_{1,2,3},sh_{1,2,3}} h_n \otimes_{sh_{1,2,3}} I_{1,2,3} h_{n_2} \otimes_{sh_{1,2,3}} I_{1,2,3} h_{n_3}.
\] (2.12)

From this formula we deduce the associativity of the Wiener product.

From the product formula, we deduce easily the next theorem.
**Theorem 2.7** (Itô-Bargmann-Wiener-Segal). Let \( h_{n_1}^{h_1,\ldots,h_n} \) and \( h_{n_2}^{h_1,\ldots,h_n} \) be elements of the bosonic Connes space. They are seen as a function on the involved simplices. Then

\[
\mu[\Psi(h_{n_1})\Psi(h_{n_2})] = \delta_{n_1,n_2} \prod_1^{n_1} \delta_{h_i,h_j} = 24^n i \int_0^{s_1<\cdots<s_n<1} h_{n_1}^{h_1,\ldots,h_n}(s_1,\ldots,s_n)h_{n_2}^{h_1,\ldots,h_n}(s_1,\ldots,s_n)ds_1 \cdots ds_n.
\]

(2.13)

**Remark 2.8.** In the case of the classical Laplacian, this formula justifies the choice of \( \mathbb{H} \) instead of \( L^\infty \). But in the previous formula, only a prehilbert space appears. So it is not obviously justified to choose \( \mathbb{H} \) instead of \( L^\infty \) to perform our computations. We have chosen \( L^\infty \) because the estimates are simpler with this space.

We say that \( h_n \) belongs to \( CO_{\infty,-t}(L^\infty) \) if \( h_n \) vanishes as soon as one of the \( s_i \geq t \). We say that \( h_n \) belongs to \( CO_{\infty,-t}(L^\infty) \) if \( h_n \) vanishes as soon as one of the \( s_i \leq t \). We get the next theorem whose proof is obvious.

**Theorem 2.9.** \( CO_{\infty,-t}(L^\infty) \) and \( CO_{\infty,-t}(L^\infty) \) are subalgebras of \( CO_{\infty,-t}(L^\infty) \) for the Wiener product. Moreover, if \( \sigma_1 \in CO_{\infty,-t}(L^\infty) \) and if \( \sigma_2 \in CO_{\infty,-t}(L^\infty) \),

\[
\mu[\Psi(\sigma_1)\Psi(\sigma_2)] = \mu[\Psi(\sigma_1)]\mu[\Psi(\sigma_2)].
\]

(2.14)

**Remark 2.10.** Let us justify heuristically this part. Let \( Q^0_t \) be the semi-group generated by \( L \). Let us suppose that there is a formal measure \( d\mu \) on a path space \( t \rightarrow w_t \) such that

\[
Q^0_t[f] = \int f(w_t)d\mu.
\]

(2.15)

(In the case of the standard Laplacian it is the measure of the Brownian motion). We refer to [19] for a physicist way to construct this measure. We have

\[
Q^0_t[x^1] = 24t
\]

(2.16)

So the infinitesimal increment \( (d\omega_t)^i \) of \( \omega_t \) should satisfy the Itô table (1.12) and the formal Wiener chaos should be an extension of the classical Wiener chaos in the Brownian case.

### 3. A Cameron-Martin-Maruyama-Girsanov Formula Associated to a Bilaplacian

We put if \( f \) is a polynomial,

\[
Q^h_t[f] = \mu[f(\omega_t^1)\Psi(\sigma_t)]
\]

(3.1)
where
\[ \Psi(\sigma_t) = \sum \int_{0 < s_1 < \ldots < s_n < t} h_{s_1} d\omega_{s_1} \cdots h_{s_n} d\omega_{s_n}. \] (3.2)

We suppose that \( h \) is continuous. In this formula, only finite sums appear due to (2.13). We get the following.

**Theorem 3.1** (Cameron-Martin-Maruyama-Girsanov). If \( f \) is a polynomial,
\[ \frac{\partial}{\partial t} Q^h_t[f] = Q^h_t[L_h f], \] (3.3)
where
\[ L_h = -\frac{\partial^4}{\partial x^4} + a h_t \frac{\partial^3}{\partial x^3}. \] (3.4)

**Proof.** Let us consider the case where \( f(x) = x^n \). We use \( \omega_t^1 = \int_0^t d\omega_s^1 \) and the fact that the Wiener product is associative. We get
\[ \left( \omega^1 + \omega_{t+\Delta t}^1 - \omega_t^1 \right)^n = \sum C_n^k \left( \omega_t^1 \right)^{n-k} \left( \omega_{t+\Delta t}^1 - \omega_t^1 \right)^k. \] (3.5)

We put
\[ \sigma_{\Delta t} = \sum \frac{[\omega_{t+\Delta t}^1]}{n!} \] (3.6)
such that by the Itô rules on \([t, t+\Delta t]\) for \( \Delta t > 0 \):
\[ \sigma_{t+\Delta t} = \sigma_t \cdot \sigma_{\Delta t}. \] (3.7)

We use Theorem 2.9 and the Itô table on \([t, t+\Delta t]\). We deduce that
\[ \mu \left[ \left( \omega_{t+\Delta t}^1 \right)^n \Psi(\sigma_{t+\Delta t}) \right] = \mu \left[ \left( \omega_t^1 \right)^n \Psi(\sigma_t) \right] + n(n-1)(n-2)(n-3) \mu \left[ \left( \omega_t^1 \right)^{n-4} \Psi(\sigma_t) \right] \Delta t \]
\[ \Delta t + o(\Delta t). \] (3.8)

Therefore, the result is obtained. \( \square \)
References


