Research Article

Reproducing Kernels and Variable Bandwidth

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We show that a modulation space of type $M_m^2(\mathbb{R})$ is a reproducing kernel Hilbert space (RKHS). In particular, we explore the special cases of variable bandwidth spaces Aceska and Feichtinger (2011) with a suitably chosen weight to provide strong enough decay in the frequency direction. The reproducing kernel property is valid even if $M_m^2(\mathbb{R})$ does not coincide with any of the classical Sobolev spaces because unbounded bandwidth (globally) is allowed. The reproducing kernel will be described explicitly.

1. Motivation and Preliminaries

Classical Sobolev spaces $\mathcal{H}_s^2(\mathbb{R})$ [1, 2] are defined by weighting on the frequency side with

$$v_s^o(\omega) = (1 + \omega^2)^{s/2}. \quad (1.1)$$

It is submultiplicative for $s \geq 0$ and defines a reproducing kernel Hilbert space (RKHS), whenever $s > 1/2$ [3]. Knowing that modulation spaces $M_{v_s^o}^2(\mathbb{R})$, defined with respect to (1.1), coincide with $\mathcal{H}_s^2(\mathbb{R})$ (Prop. 11.3.1, [4]) and are therefore RKHSs, it is reasonable to ask if a reproducing kernel (RK) for a modulation space $\mathcal{M} = M_m^2(\mathbb{R})$ with respect to a more general weight m exists. Even for the classical case, we have the Sobolev embedding condition $s > 1/2$ to be satisfied, so m cannot be arbitrary.

Further on, we will explore what happens to the RK when the weight mildly varies from (1.1). Or, more precisely, we check if spaces of variable bandwidth [5] (VB-spaces for short) are in fact RKHSs and what is the specific design of the respective RK.

By the Riesz representations theorem [6], in a Hilbert space $\mathcal{M}$ with inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}}$, we seek for the existence of a function $\Phi_y \in \mathcal{M}$, $y \in \mathbb{R}$, that satisfies the *reproducing property*

$$f(y) = \langle f, \Phi_y \rangle_{\mathcal{M}} \quad (1.2)$$

for each $f \in \mathcal{M}$. Then it also holds $\Phi_t(y) = \langle \Phi_t, \Phi_y \rangle_{\mathcal{M}}$, $t, y \in \mathbb{R}$.

The work is here organized as follows: after reviewing basic time-frequency tools and modulation spaces, in the next section, we adapt the VB concept to the Sobolev-type weights. Further on, we calculate the RK for a general modulation space. Then we observe the special examples with respect to the standard weight (1.1) and the respective inner product and a VB weight with respect to a continuous and a constant bandwidth.

Because it simplifies our proofs, we use a normalized Schwartz window $g \in \mathcal{S}(\mathbb{R})$, that is, $\|g\|_2 = 1$. We denote by $\mathcal{S}'$ the space of tempered distributions, while $\mathfrak{D}$ and $\mathfrak{D}'$ denote the space of test functions with compact support and its dual space.

1.1. Fourier Transform, Sobolev Space

Classical Fourier analysis employs two complementary representations to describe functions: the function f itself (temporal behavior) and its Fourier transform

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} dt \quad (1.3)$$

defined on $L^1(\mathbb{R})$, which describes the frequency behavior. Plancherel's theorem gives the possibility of extending $\mathcal{F}$ to a unitary operator on $L^2(\mathbb{R})$ and satisfies Parseval's formula $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$. Its inverse transform equals its adjoint operator and $\mathcal{F}^{-1} = \mathcal{O}\mathcal{F}$, where $\mathcal{O}$ is the reflection $\mathcal{O}f(t) = f(-t)$. It holds

$$T_y \mathcal{F}^{-1}(h) = \mathcal{F}^{-1}(h e^{-2\pi i y \cdot}), \quad (1.4)$$

where T_y is the translation operator.

As a rule of thumb, smoothness of f implies decay of $\hat{f}$ and vice versa. Thus it makes sense to work with the following class of Sobolev spaces, the so-called *Bessel potential spaces*:

$$\mathcal{H}_s^2(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) : \int_{\mathbb{R}} |\hat{f}|^2 (1 + |\omega|^2)^s d\omega < \infty \right\}. \quad (1.5)$$

For $s \geq 0$, this is a subspace of $L^2(\mathbb{R})$ consisting of smooth functions. For $s < 0$ (negative smoothness), one has $L^2 \subseteq \mathcal{H}_s^2$ and such spaces may include discrete measures and distributions. The respective inner product is of form

$$\langle f, h \rangle_{\mathcal{H}_s^2} = \langle \hat{f} \cdot v, \hat{h} \cdot v \rangle, \quad (1.6)$$

where $\langle \cdot, \cdot \rangle$ is the L^2 -inner product and $v = v_s^0$ is as in (1.1).

In terms of a generalized inner product, the inverse Fourier transform can be written as $f(y) = \langle \hat{f}, e^{-2\pi i y} \rangle$. If we want the representation of f via (1.6), we have

$$f(y) = \langle f, \Phi_y \rangle_{\mathcal{L}_s^2} = \langle \hat{f} \cdot v, \mathcal{F}(\Phi_y) \cdot v \rangle = \langle \hat{f}, \mathcal{F}(\Phi_y) \cdot v^2 \rangle. \quad (1.7)$$

Thus $\mathcal{F}(\Phi_y)v^2 = e^{-2\pi i y}$, so $\Phi_y = \mathcal{F}^{-1}(v^{-2}e^{-2\pi i y})$. After applying (1.4), we derive the reproducing kernel for the Sobolev space $\mathcal{L}_s^2(\mathbb{R})$ with respect to the classic inner product (1.6) to be

$$\Phi_y(t) = \Phi(t, y) = T_y \mathcal{F}^{-1}(v^{-2})(t) \quad (1.8)$$

provided $s > 1/2$ [3].

The classical Sobolev spaces as defined above in (1.5) have an alternative description in the context of modulation spaces, where one has to use the same weight as a function of two variables (Example 1.1). Thus, it makes sense to calculate the RK in the general case.

1.2. Weights

A *weight* is a positive function on $\mathbb{R}^2$. By v , we denote a positive, even, *submultiplicative* weight such that $v(0) = 1$, which satisfies the inequality $v(z_1 + z_2) \leq v(z_1)v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^2$. A positive, even weight function m on $\mathbb{R}^2$ is called *v-moderate* if it satisfies for some $C > 0$

$$m(z_1 + z_2) \leq C v(z_1) m(z_2), \quad \forall z_1, z_2 \in \mathbb{R}^2. \quad (1.9)$$

Two weights m_1 and m_2 are called *equivalent* if one has for some $C > 0$

$$C^{-1} m_1(z) \leq m_2(z) \leq C m_1(z), \quad \text{for any } z \in \mathbb{R}^2. \quad (1.10)$$

Commonly used weights are the (equivalent) submultiplicative weights of polynomial type $(1 + |\omega|)^s$ and $(1 + \omega^2)^{s/2}$, $s \geq 0$ because one has

$$(1 + \omega^2)^{s/2} \leq (1 + |\omega|)^s \leq 2^{s/2} (1 + \omega^2)^{s/2}. \quad (1.11)$$

It is easy to show that, if the moderate weight m_o (with respect to v) is equivalent to a weight m , then m is also moderate with respect to the same weight v [5]. For more information on the use of weights in time-frequency analysis, we refer the reader to [4, 7].

1.3. Short-Time Fourier Transform

By a time-frequency shift of a function f , we mean $M_\omega T_x f(t) := e^{2\pi i t \cdot \omega} f(t - x)$, for any pair $(x, \omega) \in \mathbb{R}^2$. For convenience, we will sometimes use the following notation for a function (depending on g and $y \in \mathbb{R}$) on the TF-plane: $\pi^* g(y)(x, \omega) := \overline{M_\omega T_x g(y)}$, $x, \omega \in \mathbb{R}$.

The *short-time Fourier transform* (STFT for short notation) of a function $f \in L^2(\mathbb{R})$ with respect to a *window* L^2 -function g is

$$V_g f(x, \omega) := \int_{\mathbb{R}} f(t) \overline{g(t-x)} e^{-2\pi i \omega t} dt = \langle f, M_\omega T_x g \rangle \quad (1.12)$$

for all $(x, \omega) \in \mathbb{R}^2$ and it holds $\|V_g f\|_2 = \|f\|_2 \|g\|_2$. Compared to (1.3), the STFT is a significant improvement as it is a joint time-frequency representation; by its structure it is in fact a localized Fourier transform.

The definition of $V_g f$ can be generalized to larger classes, whenever the inner product in (1.12) is well defined (for instance: $f \in \mathcal{S}'(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$; as this choice gives us greatest freedom in choosing the function at hand, we will only use Schwartz atoms in our work). In fact, it is enough that g and f belong to time-frequency shift-invariant, mutually dual spaces. As an example, if $g \in \mathfrak{D}$, then it is possible to calculate the STFT of the delta distribution $\delta_y \in \mathfrak{D}'$ by

$$V_g(\delta_y)(x, \omega) = \langle \delta_y, M_\omega T_x g \rangle = \overline{M_\omega T_x g(y)}, \quad \text{or} \quad V_g(\delta_y) = \pi^* g(y). \quad (1.13)$$

The *inversion formula* for the STFT

$$f = \frac{1}{\langle \gamma, g \rangle} \iint V_g f(x, \omega) M_\omega T_x \gamma dx d\omega = \frac{1}{\langle \gamma, g \rangle} \langle V_g f, \pi^* \gamma \rangle \quad (1.14)$$

is well defined in the weak sense for all $f \in L^2(\mathbb{R})$ and windows γ, g such that $\langle \gamma, g \rangle \neq 0$. Its calculation via Riemannian sums is explored in [8]. Written as vector-valued integrals, we have the adjointness relation $\langle V_\gamma^* F, h \rangle = \langle F, V_\gamma h \rangle$ for $F \in \mathcal{S}'(\mathbb{R}^2)$ and $h, \gamma \in \mathcal{S}(\mathbb{R})$. V_γ^* is a well-defined operator from a weighted (mixed-norm) space over the TF-plane to the corresponding modulation space.

1.4. Modulation Spaces

Modulation spaces theory is a special example of the much wider coorbit theory [9–11], which covers the case of solid, translation-invariant spaces. We comprise here known properties and facts about modulation space, following mostly [4]. Given a fixed non-zero window $g \in \mathcal{S}$, a v -moderate weight m on $\mathbb{R}^2$ and $1 \leq p, q < \infty$, the *modulation space* $M_m^{p,q}(\mathbb{R})$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ for which $V_g f$ has finite weighted mixed $L^{p,q}$ -norm

$$\|f\|_{M_m^{p,q}} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |V_g f(x, \omega)|^p m(x, \omega)^p dx \right)^{q/p} d\omega \right)^{1/q} < \infty \quad (1.15)$$

(with the usual adjustment for $p, q = \infty$). If $p = q$, we write M_m^p instead of $M_m^{p,p}$ and if $m = 1$ then we write $M^{p,q}$ instead of $M_1^{p,q}$. A special example is Feichtinger's algebra $S_o = M^1$. It is sometimes useful to calculate $V_g f$ with $f \in \mathcal{S}'_o$ and $g \in S_o$, in particular when we want to go beyond L^2 -functions.

Example 1.1. Interesting examples of modulation spaces ([4], page 232) are

- (a) $M_m^2(\mathbb{R}) = L_m^2(\mathbb{R})$ with $m(x, \omega) = m(x)$
- (b) in particular $M^2(\mathbb{R}) = L^2(\mathbb{R})$
- (c) $M_m^2(\mathbb{R}) = \mathcal{F}L_m^2(\mathbb{R})$ when $m(x, \omega) = m(\omega)$
- (d) $M_{v_s^0}^2(\mathbb{R})$, which coincides with $\mathcal{H}_s^2(\mathbb{R})$, $s > 1/2$.

Given any moderate weight m , the space $M_m^2(\mathbb{R})$ is a time-frequency shift-invariant Hilbert space, with inner product

$$\langle f, h \rangle_{M_m^2} := \langle V_g f \cdot m, V_g h \cdot m \rangle_{L^2(\mathbb{R})}. \quad (1.16)$$

The norm deriving from this inner product is independent of the choice of the used window that is, two nonzero windows g and g_0 produce equivalent norms, and there exists a constant $c > 0$ such that for all $f \in M_m^2$ it holds $c^{-1} \|V_g f\|_{L_m^2} \leq \|V_{g_0} f\|_{L_m^2} \leq c \|V_g f\|_{L_m^2}$. Similarly, equivalent weights m_1 and m_2 define the same modulation space.

2. Reproducing Kernel with respect to a v -Moderate Weight m

Given a v -moderate weight m , we seek a two-dimensional function $\Phi_y(t) := \Phi(t, y)$ such that for all $f \in M_m^2(\mathbb{R})$, $y \in \mathbb{R}$, it holds $f(y) = \langle f, \Phi_y \rangle_{M_m^2(\mathbb{R})}$, that is

$$f(y) = \langle V_g f \cdot m, V_g \Phi_y \cdot m \rangle_{L^2(\mathbb{R}^2)} = \left\langle V_g f, m^2 \cdot V_g \Phi_y \right\rangle_{L^2(\mathbb{R}^2)}. \quad (2.1)$$

We compare to (1.14) and conclude that $\pi^* g(y) = m^2 \cdot V_g \Phi_y$, that is,

Theorem 2.1. *The reproducing kernel for $M_m^2(\mathbb{R})$ defined with respect to a v -moderate weight m is*

$$\Phi_y = V_g^* \left(m^{-2} \pi^* g(y) \right). \quad (2.2)$$

For y -fixed,

$$\|\Phi_y\| = \left\| m^{-1} g(y - \cdot) \right\|_2. \quad (2.3)$$

Proof. Let $y \in \mathbb{R}$ be fixed, we have

$$\begin{aligned} \|\Phi_y\|^2 &= \langle V_g \Phi_y \cdot m, V_g \Phi_y \cdot m \rangle \\ &= \left\langle V_g V_g^* \left(\frac{1}{m^2} \cdot \pi^* g(y) \right) \cdot m, V_g V_g^* \left(\frac{1}{m^2} \cdot \pi^* g(y) \right) \cdot m \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \left\langle \frac{1}{m^2} \cdot \pi^* g(y) \cdot m, \frac{1}{m^2} \cdot \pi^* g(y) \cdot m \right\rangle = \left\langle \frac{1}{m^2} \cdot \pi^* g(y), \pi^* g(y) \right\rangle \\
&= \iint \frac{1}{m(x, \omega)^2} \cdot M_\omega T_x g(y) \overline{M_\omega T_x g(y)} dx d\omega \\
&= \iint \frac{1}{m(x, \omega)^2} |g(y - x)|^2 dx d\omega.
\end{aligned} \tag{2.4}$$

□

Corollary 2.2. Let $f \in M_{m_1}^2(\mathbb{R})$ and let m_2 be equivalent to m_1 with equivalence constant c . Denote the respective reproducing kernels with Φ_1 and Φ_2 .

If $\tilde{f}(y) = \langle f, \Phi_{2,y} \rangle_{M_{m_1}^2}$ then

$$c^{-2}f(y) \leq \tilde{f}(y) \leq c^2f(y). \tag{2.5}$$

Proof. Instead of using $\Phi_{1,y}$ to reconstruct the function f at point y in the modulation space defined with respect to m_1 , we use m_2 to obtain the approximation $\tilde{f}$ as follows:

$$\begin{aligned}
\tilde{f}(y) &= \langle V_g f \cdot m_1, V_g \Phi_{2,y} \cdot m_1 \rangle = \langle V_g f \cdot m_1, m_2^{-2} \pi^* g(y) \cdot m_1 \rangle \\
&= \langle V_g f, m_2^{-2} m_1^2 \pi^* g(y) \rangle.
\end{aligned} \tag{2.6}$$

Then it holds $c^{-2} \langle V_g f, \pi^* g(y) \rangle \leq \tilde{f}(y) \leq c^2 \langle V_g f, \pi^* g(y) \rangle$, which is equivalent to (2.5). □

For instance, if the equivalence constant between weights m_1 and m_2 is $c = 1.05$, then for all $f \in M_{m_1}^2$, it holds that $0.907f(y) \leq \tilde{f}(y) \leq 1.1025f(y)$.

2.1. Reproducing Kernel for the Sobolev Space

In analogy to (1.8), we calculate the RK for the modulation space $M_v^2(\mathbb{R})$, equivalent to $\mathcal{H}_v^2(\mathbb{R})$, defined with respect to weight (1.1). Since we now employ the inner product (2.1), the RK of $M_v^2(\mathbb{R})$ depends on the used analysis window g .

Let the RK function be denoted by $\Phi_y(t) = \Phi(t, y)$, then (2.2) implies that $\Phi_y = V_g^*(v^{-2} \pi^* g(y))$, and by the pointwise version of (1.14)

$$\begin{aligned}
\Phi(t, y) &= \iint \frac{1}{(1 + \omega^2)^s} e^{-2\pi i \omega y} \overline{g(y - x)} e^{2\pi i \omega t} g(t - x) dx d\omega \\
&= \int \frac{e^{2\pi i \omega(t-y)}}{(1 + \omega^2)^s} d\omega \int \overline{g(y - x)} g(t - x) dx.
\end{aligned} \tag{2.7}$$

We use (1.4) to obtain the RK formula (2.8).

Corollary 2.3. Let $s > 1/2$ and $v(x, \omega) = (1 + \omega^2)^{s/2}$. The reproducing kernel of $M_v^2(\mathbb{R})$ is

$$\Phi(t, y) = T_y \left(\mathcal{F}^{-1} \left(v^{-2} \right) (t) \right) \cdot G(t, y), \quad (2.8)$$

where $\|\Phi_y\|_{M_v^2}^2 = 2/(2s - 1)$ and

$$G(t, y) = \int g(t - x) \overline{g(y - x)} dx = \langle g, T_{t-y} g \rangle. \quad (2.9)$$

Proof. From (2.3), it follows

$$\begin{aligned} \|\Phi_y\|_{M_v^2}^2 &= \|v^{-1} g(y - \cdot)\|_2^2 \\ &= \iint_{\mathbb{R}^2} \frac{1}{(1 + |\omega|^2)^s} |g(y - x)|^2 dx d\omega \\ &= \int_{\mathbb{R}} \frac{1}{(1 + |\omega|^2)^s} d\omega \cdot \int_{\mathbb{R}} |g(y - x)|^2 dx = \frac{2}{2s - 1} \|g\|_2^2 = \frac{2}{2s - 1}. \end{aligned} \quad (2.10)$$

The last equality indicates that the condition $s > 1/2$ is necessary. $\square$

Note 1. G is decaying fast for a fast-decaying window g . For instance, if $g(t) = e^{-\pi t^2}$ we have

$$G(t, y) = e^{-(\pi/2)(t-y)^2} \int e^{-2\pi(x - ((t+y)/2))^2} dx = \frac{\sqrt{2}}{2} e^{-(\pi/2)(t-y)^2}, \quad (2.11)$$

which decays fast the further away are t and y . In the discrete setting, G is a diagonal-like matrix with fast off-diagonal decay.

Note 2. Compared to (1.8), the reproducing kernel in (2.8) has an extra multiplier G , which varies for different windows.

3. Customized Weights on the Time-Frequency Plane

The idea of variable bandwidth was first described in Slepian's talk [12]; implicitly, it was studied in [13, 14] and many others, while explicitly it was explored in [15–18]. Here we work with the VB concept we have developed in [5].

Let the function $b(x) \geq 0$ describe the *time-varying* broadness of a strip ST_b in the time-frequency plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, that is,

$$ST_b := \left\{ (x, \omega) \in \mathbb{R}^2 : |\omega| \leq b(x) \right\}. \quad (3.1)$$

We call the set ST_b a strip with *variable bandwidth* (VB strip).

The vertical distance function d_b at point $z = (x, \omega) \in \mathbb{R}^2$ is given by

$$d_b(z) := \begin{cases} 0, & \text{if } z \in \text{ST}_b, \\ |\omega| - b(x), & \text{if } z \notin \text{ST}_b. \end{cases} \quad (3.2)$$

In [5], we defined a *variable bandwidth weight* of order s on the time-frequency plane (VB weight) by

$$m_{b,s}(z) = (1 + d_b(z))^s, \quad s > 0 \quad (3.3)$$

and proved it is a moderate weight. In this paper, we work with a Sobolev-type weight

$$m_{b,s}(z) = \left(1 + d_b(z)^2\right)^{s/2}, \quad s > 0, \quad (3.4)$$

which is equivalent to (3.3). The most simple example is a weight with respect to a constant bandwidth $a > 0$ for some fixed $s > 0$.

$$m_a(z) = \begin{cases} 1, & \text{if } |\omega| \leq a, \\ \left(1 + (|\omega| - a)^2\right)^{s/2}, & \text{otherwise.} \end{cases} \quad (3.5)$$

Notice that, whenever we choose $a \equiv 0$ in the last equation, we get the standard Sobolev weight $(1 + |\omega|^2)^{s/2}$.

We choose to work with $s > 0$ in (3.4) for the following reasoning: when applying weight (3.4) on the time-frequency plane, we add graded weight to the exterior of ST_b . It can be shown that, for a well-localizing window g , if the weighted $V_g f$ is integrable for some $s > 0$, then $V_g f$ is decaying faster than a polynomial of order s outside the strip. In other words, this weight is giving us the opportunity to locally describe the STFT decay.

It is a simple exercise to prove that (3.4) is moderate; all one needs is the equivalence inequality with respect to (3.3), which is analogue to the equivalence of the two submultiplicative weights in (1.11). Thus, with a simple adaptation of Proposition 1 from [5], the following result holds.

Proposition 3.1. *Let $s > 0$, take b to be a nonnegative function on $\mathbb{R}$ and define weight (3.4) for d_b given as in (3.2). If for some $k \geq 0$ the boundary function b satisfies*

$$(\forall x, y \in \mathbb{R}) \quad |b(x) - b(y)| \leq k|x - y|, \quad (3.6)$$

then the weight m_b is moderate with respect to

$$v_s(x, \omega) = \left(1 + (|x| + |\omega|)^2\right)^{s/2}, \quad s > 0, \quad (3.7)$$

That is, it holds $m_b(x_1 + x_2, \omega_1 + \omega_2) \leq k^s v_s(x_1, \omega_1) m_b(x_2, \omega_2)$ for all $x_1, x_2, \omega_1, \omega_2 \in \mathbb{R}$.

As a consequence, m_b is moderate if b' is bounded (since b is then satisfying (3.6), known as the Lipschitz condition). This corresponds to $k = 0$ in Proposition 3.1, thus the bandwidth function b is bounded and consequently, the controlling weight depends only on the frequency variable ω .

Corollary 3.2. *If b is a bounded function, then the weight m_b , given by (3.4), is moderate with respect to $v_s^o(x, \omega) = (1 + |\omega|^2)^{s/2}$.*

Due to the uncertainty principle it does not make sense to talk of the bandwidth at a given point, nor to try to describe rapid changes of local bandwidth. Accordingly, as seen in [5], the concept of variable bandwidth must be designed with some built-in robustness, and small local changes of the parameters should not effect the resulting spaces.

Proposition 3.3. *Let $|h(x)| < c$ for all $x \in \mathbb{R}$, and let $b \geq 0$ generate a moderate weight m_b (3.4). Then $b + h$ generates a moderate weight m_{b+h} , equivalent to m_b , provided $b + h \geq 0$.*

Proof. We work here with $c = \sqrt{2}/2$: let $0 \leq h(x) < \sqrt{2}/2$ for all $x \in \mathbb{R}$. The equivalence inequality is trivially satisfied on ST_{b+h} as both weights have value 1. Let $b(x) < |\omega| \leq b(x) + h(x)$. Then $|\omega| - b(x) \leq h(x)$, $m_{b+h}(x, \omega) = 1$ and we have

$$m_{b+h}(x, \omega) = 1 < m_b(x, \omega) \leq \left(1 + h(x)^2\right)^{s/2} \leq 2^{s/2} \cdot 1 = 2^{s/2} m_{b+h}(x, \omega). \quad (3.8)$$

If $|\omega| - m(x) - h(x) > 0$, then both weights have nontrivial values. Using this estimate

$$0 \leq (|\omega| - b(x) - 2h(x))^2 + 1 - 2h^2(x), \quad (3.9)$$

which holds true whenever $h(x) < \sqrt{2}/2$, we derive that

$$1 + (|\omega| - b(x))^2 \leq 2 \left(1 + (|\omega| - b(x) - h(x))^2\right). \quad (3.10)$$

Then it holds $m_{b+h}(x, \omega) \leq m_b(x, \omega) \leq 2^{s/2} m_{b+h}(x, \omega)$. □

Equivalent weights are moderate simultaneously, thus m_{b+h} is moderate.

If we apply Proposition 3.3 a finite number for shifting the constant bandwidth in (3.5) toward 0, we have the following Corollary.

Corollary 3.4. *A VB weight m_b , defined with respect to a bounded bandwidth b , is equivalent to $(1 + |\omega|^2)^{s/2}$.*

3.1. Banach Spaces of Variable Bandwidth Functions

Using the moderate weights defined with respect to a variable bandwidth strip (3.1), we can proceed to the definition of functions with variable bandwidth, using the tools from Subsection 1.4. We have seen that VB weights provide for a certain flexibility, that is, the precise knowledge of the bandwidth is not necessary as finite changes give equivalent weights. This will provide for equivalent norms on the function spaces level.

Definition 3.5. Let f be defined on $\mathbb{R}$ and fix $s > 0$. We say that f is a function with variable bandwidth $b = b(t)$ if it belongs to a modulation space $M_m^{p,q}(\mathbb{R})$ for some $1 \leq p, q \leq \infty$ and a VB weight $m = m_{b,s}$, related to b and defined as in (3.4).

We will call the space $M_m^{p,q}(\mathbb{R})$ a *variable bandwidth space* and refer to it by VB space when parameters p, q are known and $m = m_{b,s}$ is fixed up to equivalence. As usual, if $p = q$ we write M_m^p instead of $M_m^{p,p}$. The modulation space (and VB space) norm in this paper is marked with $\|\cdot\|$ or $\|\cdot\|_{M_m^{p,q}}$, if necessary. Since weights (3.3) and (3.4) are equivalent, no matter which of the two weights we use, they define one VB space. Thus the properties of the spaces defined in Definition 3.5 are the same as described in [5].

4. Special Cases of VB Reproducing Kernels

Corollary 4.1. *The reproducing kernel for the VB-space $M_{m_b}^2(\mathbb{R})$, defined with respect to (3.4), is $\Phi_y = V_g^*(m_b^{-2} \cdot \pi^* g(y))$ and the norm of Φ_y depends on the bandwidth function b*

$$\|\Phi_y\|^2 = 2 \left\| g(y - \cdot) \sqrt{b(\cdot)} \right\|_2^2 + \frac{2}{2s-1}, \quad s > \frac{1}{2}. \quad (4.1)$$

Proof. Due to (2.3) and the symmetry properties of m_b (3.4), it holds

$$\begin{aligned} \|\Phi_y\|^2 &= \iint \frac{1}{m(x, \omega)^2} |g(y - x)|^2 dx d\omega \\ &= \int |g(y - x)|^2 \left(\int_{-b(x)}^{b(x)} 1 d\omega + 2 \int_{b(x)}^{+\infty} \frac{1}{(1 + \omega - b(x))^{2s}} d\omega \right) dx \\ &= 2 \int |g(y - x)|^2 \left(b(x) + \int_1^{+\infty} \frac{1}{\xi^{2s}} d\xi \right) dx \\ &= 2 \int |g(y - x)|^2 \left(b(x) + \frac{1}{2s-1} \right) dx \\ &= 2 \left\| g(y - \cdot) \sqrt{b(\cdot)} \right\|_2^2 + \frac{2}{2s-1} \|g\|_2^2. \end{aligned} \quad (4.2)$$

Our a priori assumption is that we work with a normalized window g that is, $\|g\|_2 = 1$, so (4.1) holds. $\square$

A similar split of integration, as seen in the proof of Corollary 2.3, is possible for a constant bandwidth weight m_a (3.5). Locally, m_a depends only on ω .

Corollary 4.2. *The RK for $M_{m_a}^2(\mathbb{R})$, where m_a is as in (3.5), is*

$$\begin{aligned} \Phi(t, y) &= G(t, y) \frac{e^{2\pi i a(t-y)} - e^{-2\pi i a(t-y)}}{2\pi i(t-y)} \\ &\quad + G(t, y) \left(e^{-2\pi i a(t-y)} T_y \mathcal{F} \left((v_s^0)_+^{-2} \right) (t) + e^{2\pi i a(t-y)} T_y \mathcal{F}^{-1} \left((v_s^0)_+^{-2} \right) (t) \right), \end{aligned} \quad (4.3)$$

where G is defined in (2.9).

Proof. By (2.2), we have $\Phi_y(t) = \int g(t-x) \overline{g(y-x)} (\int (e^{2\pi i \omega(t-y)} / m_a^2(x, \omega)) d\omega) dx$. Recall (3.5) and its symmetry property; therefore, we can use the notation $(v_s^o)_+$ to describe the weight $(v_s^o(x, \omega))_+ = (1 + \omega^2)^{s/2}$, for $x > 0$ and $(v_s^o(x, \omega))_+ = 0$ otherwise and obtain

$$\begin{aligned}
 \int \frac{e^{2\pi i \omega(t-y)}}{m_a^2(x, \omega)} d\omega &= \int_{-a}^a e^{2\pi i \omega(t-y)} d\omega + \int_a^{+\infty} e^{2\pi i \omega(t-y)} (1 + (\omega - a)^2)^{-s} d\omega \\
 &\quad + \int_{-\infty}^{-a} e^{2\pi i \omega(t-y)} (1 + (-\omega - a)^2)^{-s} d\omega \\
 &= \frac{e^{2\pi i a(t-y)} - e^{-2\pi i a(t-y)}}{2\pi i(t-y)} + e^{2\pi i a(t-y)} \int_0^\infty e^{2\pi i \xi(t-y)} (1 + \xi^2)^{-s} d\xi \\
 &\quad + e^{-2\pi i a(t-y)} \int_0^\infty e^{-2\pi i \xi(t-y)} (1 + \xi^2)^{-s} d\xi \\
 &= \frac{e^{2\pi i a(t-y)} - e^{-2\pi i a(t-y)}}{2\pi i(t-y)} + e^{-2\pi i a(t-y)} T_y \mathcal{F} \left((v_s^o)_+^{-2} \right) (t) \\
 &\quad + e^{2\pi i a(t-y)} T_y \mathcal{F}^{-1} \left((v_s^o)_+^{-2} \right) (t).
 \end{aligned} \tag{4.4}$$

□

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