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Research Article

The Exponential Attractors for the g-Navier-Stokes Equations

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We consider the exponential attractors for the two-dimensional g-Navier-Stokes equations in bounded domain Ω. We establish the existence of the exponential attractor in $L^2(\Omega)$.

1. Introduction

In this paper, we study the behavior of solutions of the g-Navier-Stokes equations in spatial dimension 2. These equations are a variation of the standard Navier-Stokes equations, and they assume the form,

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in } \Omega, \\ &\frac{1}{g}(\nabla\cdot gu) = \frac{\nabla g}{g}\cdot u + \nabla\cdot u = 0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $g = g(x_1, x_2)$ is a suitable smooth real-valued function defined on Ω and Ω is a suitable bounded domain in $\mathbb{R}^2$. Notice that if $g(x_1, x_2) = 1$, then (1.1) reduce to the standard Navier-Stokes equations.

In Roh [1] the author established the global regularity of solutions of the g-Navier-Stokes equations. One can refer to [2] for details. For the boundary conditions, we will consider the periodic boundary conditions, while same results can be got for the Dirichlet boundary conditions on the smooth bounded domain. Before we present the derivation of the g-Navier-Stokes equations, it is convenient to recall some relevant aspects of

the classical theory of the Navier-Stokes equations. For many years, the Navier-Stokes equations were investigated by many authors and the existence of the attractors for 2D Navier-Stokes equations was first proved by Ladyzhenskaya [3] and independently by Foiaş and Temam [4]. The finite-dimensional property of the global attractor for general dissipative equations was first proved by Mallet-Paret [5]. For the analysis on the Navier-Stokes equations, one can refer to [6].

In the past decades, many papers in the literature show that the long-time behavior of dissipative systems can be understood through the concept of attractors, see [7–14]. In addition, in [15] the authors introduced the so-called exponential attractors, which is an interesting intermediate object between the usual (global) attractors and an inertial manifold and satisfies some nice properties like those of inertial manifolds (e.g., finite fractal dimension, exponential attracting, stable with respect to some perturbations). Indeed it now seems clear that the interesting object to investigate is the exponential attractor, rather than the usual (global) attractor (which is recovered as a byproduct). See [16, 17], and so forth. The exponential attractor is a compact and positively invariant set having finite fractal dimension which contains the global attractor and attracts every trajectory at an exponential rate. It is also known that the exponential attractor enjoys stronger robustness than the global attractor. When the semigroup of a dynamical system depends continuously on a parameter, the global attractor is in general only upper-semicontinuous. In turn, under some reasonable assumptions, if an exponential attractor exists, it can depend continuously on the parameter. Such a continuous dependence was recently studied by Efendiev and Yagi [18]. When the underlying space is a Hilbert space, it is known by the same reference [15] quoted above that the squeezing property of semigroup implies existence of exponential attractors and provides a sharp estimate of attractor dimensions. When the underlying space is a Banach space, it is known by Efendiev et al. [19] that the compact smoothing property of semigroup implies existence of exponential attractors (Theorem 2.3). Another construction of exponential attractors in Banach spaces was proposed by Dung and Nicolaenko in [20]. We also refer to [17, 21–25] for more details.

In the paper, compared with the result obtained in [26], taking advantage of a recent result due to Efendiev et al. [19] (Theorem 2.3), we construct the exponential attractor. This paper is organized as follows. In Section 2, we first recall some basic results, and then, give an important technique tool [19], that is, Theorem 2.3. In Section 3, we study the existence of compact exponential attractor for the two-dimensional g-Navier-Stokes equations in the periodic boundary conditions Ω.

2. Preliminary Results

Let $\Omega = (0,1) \times (0,1)$ and we assume that the function $g(x) = g(x_1, x_2)$ satisfies the following properties:

(1) $g(x) \in C^{\infty}_{\text{per}}(\Omega)$

(2) there exist constants $m_0 = m_0(g)$ and $M_0 = M_0(g)$ such that, for all $x \in \Omega$, $0 < m_0 \leq g(x) \leq M_0$. Note that the constant function $g \equiv 1$ satisfies these conditions.

We denote by $L^2(\Omega, g)$ the space with the scalar product and the norm given by

$$(u,v)_g = \int_\Omega (u \cdot v) g \, dx, \qquad |u|_g^2 = (u,u)_g, \tag{2.1}$$

as well as $H^1(\Omega, g)$ with the norm

$$\|u\|_{H^1(\Omega,g)} = \left[(u,u)_g + \sum_{i=1}^{2}(D_i u, D_i u)_g\right]^{1/2}, \tag{2.2}$$

where $\partial u/\partial x_i = D_i u$.

Then for the functional setting of the problems (1.1), we use the following functional spaces:

$$\begin{aligned} H_g &= Cl_{L^2_{\mathrm{per}}(\Omega,g)}\left\{u \in C^\infty_{\mathrm{per}}(\Omega) : \nabla \cdot gu = 0, \int_\Omega u\,dx = 0\right\}, \\ V_g &= \left\{u \in H^1_{\mathrm{per}}(\Omega, g) : \nabla \cdot gu = 0, \int_\Omega u\,dx = 0\right\}, \end{aligned} \tag{2.3}$$

where H_g is endowed with the scalar product and the norm in $L^2(\Omega, g)$, and V_g is the spaces with the scalar product and the norm given by

$$((u,v))_g = \int_\Omega (\nabla u \cdot \nabla v) g\,dx, \qquad \|u\|_g = ((u,u))_g. \tag{2.4}$$

Also, we define the orthogonal projection P_g as

$$P_g : L^2_{\mathrm{per}}(\Omega, g) \longrightarrow H_g \tag{2.5}$$

and we have that $Q \subseteq H_g^\perp$, where

$$Q = Cl_{L^2_{\mathrm{per}}(\Omega,g)}\left\{\nabla\phi : \phi \in C^1\left(\overline{\Omega}, \mathbb{R}\right)\right\}. \tag{2.6}$$

Then, we define the g-Laplacian operator

$$-\Delta_g u \equiv \frac{1}{g}(\nabla \cdot g\nabla)u = -\Delta u - \frac{1}{g}(\nabla g \cdot \nabla)u \tag{2.7}$$

to have the linear operator

$$A_g u = P_g\left[-\frac{1}{g}(\nabla \cdot (g\nabla u))\right]. \tag{2.8}$$

For the linear operator A_g, the following hold (see Roh [1]):

(1) A_g is a positive, self-adjoint operator with compact inverse, where the domain of $A_g, D(A_g) = V_g \cap H^2(\Omega, g)$.

(2) There exist countable eigenvalues of A_g satisfying

$$0 < \lambda_g \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots, \tag{2.9}$$

where $\lambda_g = 4\pi^2 m/M$ and λ_1 is the smallest eigenvalue of A_g. In addition, there exists the corresponding collection of eigenfunctions $\{e_1, e_2, e_3, \ldots\}$ which forms an orthonormal basis for H_g.

Next, we denote the bilinear operator $B_g(u,v) = P_g(u \cdot \nabla)v$ and the trilinear form

$$b_g(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i(D_i v_j) w_j g \, dx = \left(P_g(u \cdot \nabla)v, w\right)_g, \tag{2.10}$$

where u, v, w lie in appropriate subspaces of $L^2(\Omega, g)$. Then, the form b_g satisfies

$$b_g(u,v,w) = -b_g(u,w,v) \quad \text{for } u,v,w \in H_g. \tag{2.11}$$

We denote a linear operator R on V_g by

$$Ru = P_g\left[\frac{1}{g}(\nabla g \cdot \nabla)u\right] \quad \text{for } u \in V_g, \tag{2.12}$$

and have R as a continuous linear operator from V_g into H_g such that

$$|(Ru,u)| \leq \frac{|\nabla g|_\infty}{m_0}\|u\|_g |u|_g \leq \frac{|\nabla g|_\infty}{m_0 \lambda_g^{1/2}}\|u\|_g \quad \text{for } u \in V_g. \tag{2.13}$$

We now rewrite (1.1) as abstract evolution equations,

$$\begin{gathered} \frac{du}{dt} + \nu A_g u + B_g u + \nu R u = P_g f, \\ u(0) = u_0. \end{gathered} \tag{2.14}$$

Let us first recall some basic matters on the dynamical system. Let E be a Banach space and let K be a subset of E, K being a metric space equipped with the distance induced from the norm of E. Let $S(t)$, $0 \leq t < \infty$ be a family of mappings from K into itself having the following properties: (i) $S(0) = I$ (the identity mapping); (ii) $S(t)S(s) = S(t+s)$, $0 \leq t$, $s < \infty$ (the semigroup property); (iii) the mapping $G : [0,\infty) \times K \to K$, $(t, u_0) \to S(t)u_0$, is continuous. Such a family is called a continuous (nonlinear) semigroup acting on K. The image of $S(\cdot)u_0$ drawn in K is called the trajectory starting from K. The whole of such

trajectories is the dynamical system $(S(t), K, E)$, where K and E are called the phase-space and the universal space, respectively.

A subset $\mathcal{A}$ of the phase-space K is the global attractor of $(S(t), K, E)$ if the following conditions are satisfied: (i) $\mathcal{A}$ is a compact subset of E; (ii) $\mathcal{A}$ is an invariant set, that is, $S(t)\mathcal{A} = \mathcal{A}$ for every $0 < t < \infty$; (iii) $\mathcal{A}$ attracts every bounded subset of K, namely, for any bounded subset $B \subset K$, it holds that $\lim_{t\to\infty} \operatorname{dist}(S(t)B, \mathcal{A}) = 0$, where $\operatorname{dist}(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_E$ denotes the Hausdorff pseudodistance between two sets A and B.

We recall the definition of an exponential attractor (see, e.g., [15, 17]).

Definition 2.1. A compact set $\mathfrak{A} \subset E$ is an exponential attractor for $S(t)$ if

(1) it has finite fractal dimension, $\dim_F \mathfrak{A} < +\infty$,

(2) it is positively invariant, $S(t)\mathfrak{A} \subseteq \mathfrak{A}$, for all $t \geq 0$,

(3) it attracts exponentially the bounded subsets of E in the following sense:

$$\forall B \subset E \text{ bounded}, \ \operatorname{dist}(S(t)B, \mathfrak{A}) \leq Q(\|B\|_E)e^{-\alpha t}, \quad t \geq 0, \tag{2.15}$$

where the positive constant α, the monotonic function Q are independent of B.

Remark 2.2. We note that the existence of an exponential attractor $\mathfrak{A}$ for the semigroup $S(t)$ automatically implies the existence of the global attractor $\mathcal{A}$ and the embedding $\mathcal{A} \subset \mathfrak{A}$. We note however that, in contrast to the global attractor, an exponential attractor is not uniquely defined.

To construct an exponential attractor, we make use of the following result due to Efendiev et al. [19].

Theorem 2.3. *Let X, Y be two Banach spaces such that Y is compactly embedded in X. Let Z be a bounded closed subset of X. Assume that a semigroup $(S(t))_{t>0}$ on X satisfies the following conditions: there exists a time $t_* > 0$, constants $L_1, L_2 > 0$, and exponents $\gamma_1, \gamma_2 > 0$ such that $S(t_*)$ maps Z into itself and*

$$\begin{gathered} \|S(t_*)u_0 - S(t_*)v_0\|_Y \leq L_1\|u_0 - v_0\|_X, \\ \|S(s)u_0 - S(t)v_0\|_X \leq L_2\left(|s-t|^{\gamma_1} + \|u_0 - v_0\|_X^{\gamma_2}\right) \end{gathered} \tag{2.16}$$

hold for any u_0, $v_0 \in Z$ and s, $t \in [0, t_*]$. Then the dynamical system $((S(t))_{t>0}, Z)$ admits an exponential attractor.

Hereafter c will denote a generic scale invariant positive constant, which is independent of the physical parameters in the equation and may be different from line to line and even in the same line.

3. Exponential Attractor of g-Navier-Stokes Equations

This section deals with the existence of the exponential attractor for the two-dimensional g-Navier-Stokes equations with periodic boundary condition.

In Roh [1], the authors have shown that the semigroup $S(t) : H_g \to H_g$ $(t \geq 0)$ associated with the systems (2.14) possesses a global attractor in H_g and V_g. The main objective of this section is to prove that the system (2.14) has exponential attractors in H_g.

To this end, we first state some of the following results of existence and uniqueness of solutions of (2.14).

Theorem 3.1. *Let $f \in V_g'$ be given. Then for every $u_0 \in H_g$ there exists a unique solution $u = u(t)$ on $[0, \infty)$ of* (2.14). *Moreover, one has*

$$u(t) \in C[0, T; H_g) \cap L^2(0, T; V_g), \quad \forall T > 0. \tag{3.1}$$

Finally, if $u_0 \in V_g$, then

$$u(t) \in C[0, T; V_g) \cap L^2(0, T; D(A_g)), \quad \forall T > 0. \tag{3.2}$$

Proof. The Proof of Theorem 3.1 is similar to Roh [1] and Kwaket al. [26] and Temam [12]. □

In a similar manner as in [13, 14], we can establish the following a priori estimate for for (2.14).

Lemma 3.2. *Let $\mathcal{B}$ be a bounded subset of H_g. The semigroup $\{S(t)\} : H_g(V_g) \to H_g(V_g)$ associated with* (2.14) *possesses absorbing sets*

$$\begin{aligned} \mathcal{B}_0 &= \left\{ u \in H_g \mid |u|_g \leq \rho_0 \right\} \quad \forall t \geq t_0(\mathcal{B}), \\ \mathcal{B}_1 &= \left\{ u \in V_g \mid \|u\|_g \leq \rho_1 \ \forall t \geq t_1(\mathcal{B}) = t_0(\mathcal{B}) + 1 \right\} \end{aligned} \tag{3.3}$$

which absorb all bounded sets of H_g. Moreover $\mathcal{B}_0$ and $\mathcal{B}_1$ absorb all bounded sets of H_g and V_g in the norms of H_g and V_g, respectively.

Let $(S(t))_{t\geq 0}$ be the semigroup associated with (2.14). Since Ω is bounded, V_g is compactly embedded in H_g. Then we consider H_g, V_g as X, Y in Theorem 2.3, respectively. The crucial point is the choice of the bounded subset Z. Let

$$\mathfrak{A} = \overline{\bigcup_{t\geq\tau} S(t)\mathcal{B}_1}, \tag{3.4}$$

where $\overline{B}$ denote the closure of B in H_g and τ is the time when $\mathcal{B}_1$ absorbs itself. We claim that $\mathfrak{A}$ has all properties required for Z. In fact, it is easy to see that $\mathfrak{A}$ is positively invariant under the semiflow $S(t)$. In order to see that $\mathfrak{A}$ has the other required properties, we begin with constructing uniform a priori estimates in time t for the solution u to (2.14).

Now we consider difference of two solutions of (2.14) starting from $\mathcal{B}_0$.

Proposition 3.3. *Let the assumptions of Theorem 2.3 hold. Then, there exists a time $t_* > 0$, constants $L_1 > 0$, and exponents $\gamma_1,\ \gamma_2 > 0$ such that $S(t_*)$ maps $\mathfrak{A}$ into itself and*

$$|S(s)u_0 - S(t)v_0|_g \le L_1\left(|s-t|^{\gamma_1} + |u_0 - v_0|_g^{\gamma_2}\right) \tag{3.5}$$

holds for any $u_0,\ v_0 \in \mathfrak{A}$ and $s, t \in [0, t_*]$.

Proof. Let $u_1^0,\ u_2^0 \in \mathfrak{A}$ and let u_1 and u_2 be two solutions to (2.14) with $u_1(0) = u_1^0,\ u_2(0) = u_2^0$, respectively.

Let $\tilde{u} = u_1 - u_2$ which satisfies

$$\frac{d\tilde{u}}{dt} + \nu A_g\tilde{u} + B_g u_1 - B_g u_2 + \nu R\tilde{u} = 0. \tag{3.6}$$

Multiplying (3.6) by $\tilde{u}$, we have

$$\left(\frac{d\tilde{u}}{dt}, \tilde{u}\right)_g + \left(\nu A_g\tilde{u}, \tilde{u}\right)_g + \left(B(u_1,u_1) - B(u_2,u_2), \tilde{u}\right)_g + \left(R\tilde{u}, \tilde{u}\right)_g = 0, \tag{3.7}$$

$$\frac{1}{2}\frac{d}{dt}|\tilde{u}|_g^2 + \nu\left\|\tilde{u}_g\right\|_g^2 + b_g(\tilde{u}, u_2, \tilde{u}) + \left(R\tilde{u}, \tilde{u}\right)_g = 0. \tag{3.8}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}|\tilde{u}|_g^2 + \nu\left\|\tilde{u}_g\right\|_g^2 \le \left|b_g(\tilde{u}, u_2, \tilde{u})\right| + \left(R\tilde{u}, \tilde{u}\right)_g. \tag{3.9}$$

Since b_g satisfies the following inequality (see Temam [12]):

$$\left|b_g(u,v,w)\right| \le c|u|_g^{1/2}\|u\|_g^{1/2}\|v\|_g|w|_g^{1/2}\|w\|_g^{1/2}, \qquad \forall u, v, w \in V_g, \tag{3.10}$$

thus,

$$\begin{aligned}\left|b_g(\tilde{u}, u_2, \tilde{u})\right| &\le c\|u_2\|_g\|\tilde{u}\|_g|\tilde{u}|_g \\ &\le \frac{\nu}{4}\|\tilde{u}\|_g^2 + c\|u_2\|_g^2|\tilde{u}|_g^2.\end{aligned} \tag{3.11}$$

Next, the Cauchy inequality,

$$\begin{aligned}\left|\left(R\tilde{u}, \tilde{u}\right)_g\right| &= \left|\left(\frac{\nu}{g}(\nabla g\cdot\nabla)\tilde{u}, \tilde{u}\right)_g\right| \\ &\le \frac{\nu}{m_0}\left|\nabla g\right|_\infty\|\tilde{u}\|_g|\tilde{u}|_g \\ &\le \frac{\nu}{4}\|\tilde{u}\|_g^2 + c|\tilde{u}|_g^2.\end{aligned} \tag{3.12}$$

Finally, putting (3.11)-(3.12) together, there exist constant $M_0 = M_0(m_0, |\nabla g|_\infty, \rho_0, \rho_1)$ such that

$$\frac{1}{2}\frac{d}{dt}|\widetilde{u}|_g^2 + \frac{1}{2}\nu\|\widetilde{u}\|_g^2 \leq M_0|\widetilde{u}|_g^2. \tag{3.13}$$

Therefore, we deduce that

$$\frac{d}{dt}|\widetilde{u}|_g^2 \leq c|\widetilde{u}|_g^2. \tag{3.14}$$

By the Gronwall inequality, the above inequality implies that

$$|\widetilde{u}|_g \leq e^{ct}\left|u_1^0 - u_2^0\right|_g. \tag{3.15}$$

Next we multiply (2.14) by u, and we have

$$\left(\frac{du}{dt}, u\right)_g + \left(\nu A_g u, u\right)_g + \left(B(u,u), u\right)_g = \left(f, u\right)_g - \left(Ru, u\right)_g. \tag{3.16}$$

Since Ω is bounded, the Poincaré inequality holds:

$$\lambda_g|u|_g^2 \leq \|u\|_g^2 \quad \forall u \in V_g. \tag{3.17}$$

There exist constant $M_1 = M_1(m_0,\ |\nabla g|_\infty, \rho_0,\ \rho_1)$ such that we deduce that

$$\frac{d}{dt}|u|_g^2 + \nu\lambda|u|_g^2 \leq 2M_1 + \frac{3}{\nu}\left|f\right|_{V_g'}^2. \tag{3.18}$$

Multiplying (2.14) by $A_g u$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_g^2 &+ \nu\left|A_g u\right|_g^2 \\ &\leq \left|\left(B_g(u,u), A_g u\right)_g\right| + \left|\left(f, A_g u\right)_g\right| + \left|\left(Ru, A_g u\right)_g\right|.\end{aligned} \tag{3.19}$$

Expanding and using Young's inequality, together with b_g satisfying inequalities [12], there exists a constant $M_2 = M_2(m_0, |\nabla g|_\infty, \rho_0, \rho_1)$ such that

$$\frac{d}{dt}\|u\|_g^2 + \nu\lambda_g\|u\|_g^2 \leq \frac{3}{\nu}\left|f\right|_g + M_2. \tag{3.20}$$

Since if $u_0 \in \mathfrak{A} \subset V_g$ then the solution u with $u(0) = u_0$ satisfies $du/dt \in L^2(0,T;H_g)$, it holds that

$$|u(s) - u(t)|_g \leq \left|\int_t^s \left|\frac{du}{dr}(r)|dr\right| \leq |s-t|^{1/2}\left\|\frac{du}{dt}\right\|_{L^2(0,T;H_g)}. \tag{3.21}$$

From (3.18), (3.20) and Lemma 3.2 we can show that there exists a constant $M > 0$ which satisfies $\|du/dt\|_{L^2(0,T;H_g)} \leq M$ and depends on T but not on u_0. Putting (3.15) and (3.21) together, therefore (3.5) turns out to be valid with exponents $\gamma_1 = 1/2$ and $\gamma_2 = 1$. □

Proposition 3.4. *Let the assumptions of Theorem 2.3 hold. Then, there exists a time $t_* > 0$ and constants $L_2 > 0$ such that $S(t_*)$ maps $\mathfrak{A}$ into itself and*

$$\|S(t_*)u_0 - S(t_*)v_0\|_g \leq L_2|u_0 - v_0|_g \tag{3.22}$$

hold for any $u_0, v_0 \in \mathfrak{A}$ and $s, t \in [0, t_*]$.

Proof. Multiplying (3.6) by $tA_g\tilde{u}$, we have

$$\begin{aligned}\left(\frac{d\tilde{u}}{dt}, tA_g\tilde{u}\right) + (\nu A_g\tilde{u}, tA_g\tilde{u}) + b_g(\tilde{u}, u_2, tA_g\tilde{u})_g \\ + b_g(u_1, \tilde{u}, tA_g\tilde{u})_g + (R\tilde{u}, tA_g\tilde{u})_g = 0.\end{aligned} \tag{3.23}$$

It follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}t\|\tilde{u}\|_g^2 + \nu t\left|A_g\tilde{u}\right|_g^2 - \frac{1}{2}\|\tilde{u}\|_g^2 \\ \leq t\left|b_g(\tilde{u}, u_2, A_g\tilde{u})_g\right| + t\left|b_g(u_1, \tilde{u}, A_g\tilde{u})_g\right| + t\left|(R\tilde{u}, A_g\tilde{u})_g\right|.\end{aligned} \tag{3.24}$$

To estimate b_g, we recall some inequalities [12]: for every $u, v \in D(A_g)$,

$$\left|B_g(u,v)\right| \leq c\begin{cases}|u|_g^{1/2}\|u\|_g^{1/2}\|v\|_g^{1/2}\left|A_g v\right|_g^{1/2}, \\ |u|_g^{1/2}\left|A_g u\right|_g^{1/2}\|v\|_g.\end{cases} \tag{3.25}$$

Expanding and using Young's inequality, together with (3.25), we have

$$\begin{aligned}\left|b_g(\tilde{u}, u_2, A_g\tilde{u})_g\right| &\leq |\tilde{u}|_g^{1/2}\left|A_g\tilde{u}\right|_g^{1/2}\|u_2\|_g\left|A_g\tilde{u}\right|_g \\ &\leq \frac{\nu}{3}\left|A_g\tilde{u}\right|_g^2 + \frac{c}{\nu}|\tilde{u}|_g^2\|u_2\|_g^4,\end{aligned} \tag{3.26}$$

$$\begin{aligned}\left|b_g(u_1, \tilde{u}, A_g\tilde{u})_g\right| &\leq |u_1|_g^{1/2}\|u_1\|_g^{1/2}\|\tilde{u}\|_g^{1/2}\left|A_g\tilde{u}\right|_g^{1/2}\left|A_g\tilde{u}\right|_g \\ &\leq \frac{\nu}{3}\left|A_g\tilde{u}\right|_g^2 + \frac{c}{\nu}|u_1|_g^2\|u_1\|_g^2\|\tilde{u}\|_g^2.\end{aligned} \tag{3.27}$$

Next, using the Cauchy inequality,

$$\begin{aligned}\left|\left(R\tilde{u}, A_g\tilde{u}\right)_g\right| &= \left|\left(\frac{1}{g}(\nabla g\cdot\nabla)\tilde{u}, A_g\tilde{u}\right)_g\right| \\ &\le \frac{|\nabla g|_\infty}{m_0}\|\tilde{u}\|_g\left|A_g\tilde{u}\right|_g \\ &\le \frac{\nu}{3}\left|A_g\tilde{u}\right|_g^2 + \frac{3}{4\nu}|\nabla g|_\infty^2\|\tilde{u}\|_g^2.\end{aligned} \tag{3.28}$$

Since (3.13), we have

$$\|\tilde{u}\|_g^2 \le c|\tilde{u}|_g^2, \tag{3.29}$$

Putting (3.26)–(3.29) together, therefore we have

$$\frac{d}{dt}t\|\tilde{u}\|_g^2 \le c\left(1+|u_1|_g^2\|u_1\|_g^2\right)t\|\tilde{u}\|_g^2 + c\left(t\|u_2\|_g^4+1\right)|\tilde{u}|_g^2. \tag{3.30}$$

By the Gronwall inequality and (3.15), the above inequality implies

$$t\|\tilde{u}\|_g^2 \le C(t)\left|u_1^0-u_2^0\right|_g^2, \tag{3.31}$$

where

$$C(t) = \int_0^t c\exp\left(s+\int_s^t\left(1+|u_1|_g^2\|u_1\|_g^2\right)dr\right)\left(s\|u_2\|_g^4+1\right)ds. \tag{3.32}$$

By taking $t = t_1(\mathcal{B}_1)$, we complete the proof. □

Now, we give our main theorem which relies on the Propositions 3.3 and 3.4 to construct an exponential attractor.

Theorem 3.5. *There exists a subset $\mathfrak{A}$ of H_g such that $S(t)$ maps $\mathfrak{A}$ into itself and the dynamical system $((S(t))_{t>0}, \mathfrak{A})$ admits an exponential attractor.*

Based on the abve results (Propositions 3.3 and 3.4) and applying Theorem 2.3, we can deduce Theorem 3.5.

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References

[1] J. Roh, "Dynamics of the g-Navier-Stokes equations," *Journal of Differential Equations*, vol. 211, no. 2, pp. 452–484, 2005.

[2] J. Roh, *g-Navier-stokes equations [thesis]*, University of Minnesota, 2001.

[3] O. A. Ladyzhenskaya, "On the dynamical system generated by the Navier-Stokes equations," *Zapiskii of Nauchnish Seminarovs LOMI*, vol. 27, pp. 91–114, 1972, English translation: *Journal of Soviet Mathematics*, vol. 3, 1975.

[4] C. Foiaş and R. Temam, "Some analytic and geometric properties of the solutions of the evolution Navier-Stokes equations," *Journal de Mathématiques Pures et Appliquées*, vol. 58, no. 3, pp. 339–368, 1979.

[5] J. Mallet-Paret, "Negatively invariant sets of compact maps and an extension of a theorem of Cartwright," *Journal of Differential Equations*, vol. 22, no. 2, pp. 331–348, 1976.

[6] P. Constantin and C. Foias, *Navier-Stokes Equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, Ill, USA, 1988.

[7] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, vol. 49 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI, USA, 2002.

[8] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, vol. 25 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 1988.

[9] A. Haraux, *Systèmes Dynamiques Dissipatifs et Applications*, vol. 17 of *Recherches en Mathématiques Appliquées*, Masson, Paris, France, 1991.

[10] S. S. Lu, H. Q. Wu, and C. K. Zhong, "Attractors for non-autonomous 2D Navier-Stokes equations with *normal* external forces," *Dynamics of Continuous, Discrete and Impulsive Systems*, vol. 8, pp. 585–597, 2005.

[11] Q. Ma, S. Wang, and C. Zhong, "Necessary and sufficient conditions for the existence of global attractors for semigroups and applications," *Indiana University Mathematics Journal*, vol. 51, no. 6, pp. 1541–1559, 2002.

[12] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, vol. 68 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2nd edition, 1997.

[13] D. Wu, "On the dimension of the pullback attractors for g-Navier-Stokes equations," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 893240, 16 pages, 2010.

[14] D. Wu and C. Zhong, "The attractors for the nonhomogeneous nonautonomous Navier-Stokes equations," *Journal of Mathematical Analysis and Applications*, vol. 321, no. 1, pp. 426–444, 2006.

[15] A. Eden, C. Foias, B. Nicolaenko, and R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, vol. 37 of *Research in Applied Mathematics*, John Wiley & Sons, New York, NY, USA, 1994.

[16] M. Efendiev, S. Zelik, and A. Miranville, "Exponential attractors and finite-dimensional reduction for non-autonomous dynamical systems," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 135, no. 4, pp. 703–730, 2005.

[17] A. Miranville and S. V. Zelik, "Attractors for dissipative partial differential equations in bounded and unbounded domains," in *Handbook of Differential Equations, Evolutionary Equations*, C. M. Dafermos and M. Pokorny, Eds., vol. 4, Elsevier, Amsterdam, The Netherlands, 2008.

[18] M. Efendiev and A. Yagi, "Continuous dependence on a parameter of exponential attractors for chemotaxis-growth system," *Journal of the Mathematical Society of Japan*, vol. 57, no. 1, pp. 167–181, 2005.

[19] M. Efendiev, A. Miranville, and S. Zelik, "Exponential attractors for a nonlinear reaction-diffusion system in $\mathbf{R}^3$," *Comptes Rendus de l'Académie des Sciences. Série I*, vol. 330, no. 8, pp. 713–718, 2000.

[20] L. Dung and B. Nicolaenko, "Exponential attractors in Banach spaces," *Journal of Dynamics and Differential Equations*, vol. 13, no. 4, pp. 791–806, 2001.

[21] A. Babin and B. Nicolaenko, "Exponential attractors of reaction-diffusion systems in an unbounded domain," *Journal of Dynamics and Differential Equations*, vol. 7, no. 4, pp. 567–590, 1995.

[22] A. Eden, C. Foias, and V. Kalantarov, "A remark on two constructions of exponential attractors for α-contractions," *Journal of Dynamics and Differential Equations*, vol. 10, no. 1, pp. 37–45, 1998.

[23] M. Efendiev, A. Miranville, and S. Zelik, "Exponential attractors for a singularly perturbed Cahn-Hilliard system," *Mathematische Nachrichten*, vol. 272, pp. 11–31, 2004.

[24] M. Efendiev, A. Miranville, and S. Zelik, "Infinite dimensional exponential attractors for a non-autonomous reaction-diffusion system," *Mathematische Nachrichten*, vol. 248/249, pp. 72–96, 2003.

[25] Y. Zhong and C. Zhong, "Exponential attractors for reaction-diffusion equations with arbitrary polynomial growth," *Nonlinear Analysis*, vol. 71, no. 3-4, pp. 751–765, 2009.

[26] M. Kwak, H. Kwean, and J. Roh, "The dimension of attractor of the 2D g-Navier-Stokes equations," *Journal of Mathematical Analysis and Applications*, vol. 315, no. 2, pp. 436–461, 2006.