Research Article

Properties of Toeplitz Operators on Some Holomorphic Banach Function Spaces

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We characterize complex measures $\mu$ on the unit ball of $\mathbb{C}^n$, for which the general Toeplitz operator $T_\mu^\alpha$ is bounded or compact on the analytic Besov spaces $B_p(B_n)$, also on the minimal Möbius invariant Banach spaces $B_1(B_n)$ in the unit ball $B_n$.

1. Introduction

Let $B_n$ be the unit ball of the $n$-dimensional complex Euclidean space $\mathbb{C}^n$. We denote the class of all holomorphic functions on the unit ball $B_n$ by $H(B_n)$. The ball centered at $z$ with radius $r$ will be denoted by $B(z, r)$. For $\alpha > -1$, let $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu$, where $d\nu$ is the normalized Lebesgue volume measure on $B_n$ and $c_\alpha = \Gamma(n+\alpha+1)/n!\Gamma(\alpha+1)$ (where $\Gamma$ denotes the Gamma function) so that $\nu_\alpha(B_n) = 1$.

For any $z = (z_1, z_2, \ldots, z_n), w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$, the inner product is defined by $\langle z, w \rangle = \sum_{k=1}^{n} z_k \bar{w}_k$. For $f \in H(B_n)$, we write

$$\nabla f(z) = \left( \frac{\partial f(z)}{\partial z_1}, \frac{\partial f(z)}{\partial z_2}, \ldots, \frac{\partial f(z)}{\partial z_n} \right),$$

$$\Re f(z) = \langle \nabla f, z \rangle = \sum_{j=1}^{n} z_j \frac{\partial f(z)}{\partial z_j}. \quad (1.1)$$
For $f \in \mathcal{A}(\mathbb{B})$ and $z \in \mathbb{B}$, set

$$Q_f(z) = \sup_{w \in \mathbb{C}^n \setminus \{0\}} \frac{|\langle \nabla f(z), \overline{w} \rangle|}{\sqrt{H_z(w, w)}},$$

where $H_z(w, w)$ is the Bergman metric on $\mathbb{B}$, that is,

$$H_z(w, w) = \left( \frac{n + 1}{2} \right) \frac{(1 - |z|^2)|w|^2 + |\langle w, z \rangle|^2}{(1 - |z|^2)^2}.$$  

(1.2)

For $1 < p < \infty$, the Besov spaces $B_p(\mathbb{B})$ consists of all functions $f \in \mathcal{A}(\mathbb{B})$ for which (see [1])

$$\|f\|_{B_p(\mathbb{B})}^p := \int_{\mathbb{B}} Q_f^p(z) \, d\nu(z) < \infty.$$

(1.3)

From [1], we know that for $n \geq 2$, the Besov space is nontrivial if and only if $p > 2n$.

The analytic Besov space is the minimal Möbius invariant Banach space $B_1(\mathbb{B})$ (see [2]) defined by

$$\|f\|_{B_1(\mathbb{B})} := \sum_{|m| = n+1} \sup_{z \in \mathbb{B}} \int_{\mathbb{B}} |\frac{\partial^m f(z)}{\partial z^m}| \, d\nu(z) < \infty.$$  

(1.4)

For $a \geq 0$, a function $f \in \mathcal{A}(\mathbb{B})$ is said to belong to the $a$-Bloch spaces $B^a(\mathbb{B})$ if (see [3])

$$b_a = \sup_{z \in \mathbb{B}} |\nabla f(z)| \left(1 - |z|^2 \right)^a < \infty.$$  

(1.5)

(1.6)

The little Bloch space $B^a_0(\mathbb{B})$ consists of all $f \in B^a(\mathbb{B})$ such that

$$\lim_{|z| \to 1} |\nabla f(z)| \left(1 - |z|^2 \right)^a = 0.$$  

(1.7)

With the norm $\|f\|_{B^a(\mathbb{B})} = |f(0)| + b_a$, we know that $B^a(\mathbb{B})$ becomes a Banach space. For $a = 1$, the spaces $B^1$ and $B^1_0$ become the Bloch and the little Bloch space (see, e.g., [2]).

For every point $a \in \mathbb{B}$, the Möbius transformation $\varphi_a : \mathbb{B} \rightarrow \mathbb{B}$ is defined by

$$\varphi_a(z) = \frac{a - P_a(z) - S_a Q_a(z)}{1 - \langle z, a \rangle}, \quad z \in \mathbb{B}.$$  

(1.8)
where $S_a = \sqrt{1 - |a|^2}$, $P_a(z) = a(z, a)/|a|^2$, $P_0 = 0$ and $Q_a = I - P_a$ (see, e.g., [2] or [4]). The map $\varphi_a$ has the following properties that $\varphi_a(0) = a$, $\varphi_a(a) = 0$, $\varphi_a = \varphi_a^{-1}$ and

\[
1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{1 - |a|^2}{1 - (a, w)}(1 - (a, w)),
\]

(1.9)

where $z$ and $w$ are arbitrary points in $\mathbb{B}_n$. In particular,

\[
1 - |\varphi_a(z)|^2 = \frac{1 - |a|^2}{1 - (a, w)}(1 - |w|^2),
\]

(1.10)

The following result can be found in [3].

**Proposition 1.1.** Let $f \in \mathcal{A}(\mathbb{B}_n)$, $2n < p < \infty$. Then $f \in B_p(\mathbb{B}_n)$ if and only if

\[
\int_{\mathbb{B}_n} \left( \frac{|f(w) - f(z)|}{|1 - \langle z, w \rangle|} \right)^p (1 - |z|^2)^{p/2} (1 - |w|^2)^{p/2} d\mu(w) d\nu(z) < \infty.
\]

(1.11)

For $\alpha > -1$ and $0 < p < \infty$, the weighted Bergman spaces $A^p_{\alpha}(\mathbb{B}_n)$ consists of all functions $f \in \mathcal{A}(\mathbb{B}_n)$ for which

\[
\|f\|_{A^p_{\alpha}} := \int_{\mathbb{B}_n} |f(z)|^p d\nu_{\alpha}(z) < \infty.
\]

(1.12)

It is clear that $A^p_{\alpha} = L^p(\mathbb{B}_n, d\nu_{\alpha}) \cap \mathcal{A}(\mathbb{B}_n)$ and $A^p_{\alpha}$ is a linear subspace of $L^p(\mathbb{B}_n, d\nu_{\alpha})$. When $\alpha = 0$, we simply write $A^p_{\alpha}(\mathbb{B}_n)$ for $A^p_{\alpha}(\mathbb{B}_n)$. In the special case when $p = 2$, $A^2_{\alpha}(\mathbb{B}_n)$ is a Hilbert space. It is well known that for $\alpha > -1$ the Bergman kernel of $A^2_{\alpha}(\mathbb{B}_n)$ is given by

\[
K^\alpha(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad z, w \in \mathbb{B}_n.
\]

(1.13)

For $\alpha > -1$, a complex measure $\mu$ such that

\[
\left| \int_{\mathbb{B}_n} \left( 1 - |w|^2 \right)^{\alpha} d\mu(w) \right| = \left| \int_{\mathbb{B}_n} d\mu_{\alpha}(w) \right| < \infty
\]

(1.14)

define a Toeplitz operator as follows:

\[
T_\mu f(z) = c_\alpha \int_{\mathbb{B}_n} \frac{\left( 1 - |w|^2 \right)^{\alpha} f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\mu(w) = \int_{\mathbb{B}_n} \frac{f(w) d\mu_{\alpha}(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}},
\]

(1.15)

where $z \in \mathbb{B}_n$ and $f \in L^1(\mathbb{B}_n, (1 - |z|^2)^\alpha d\mu)$. 

For $\alpha, \beta > -1$, define the function $P_{\alpha, \beta}(f)(z)$, for $z \in \mathbb{B}_n$ by:

$$P_{\alpha, \beta}(f)(z) = c_\alpha \int_{\mathbb{B}_n} f(w) \left(1 - |w|^2\right)^\alpha \frac{d\nu(w)}{1 - \langle z, w \rangle}.$$  \hfill (1.16)

In case $\beta = \alpha$ we write $P_\alpha$ instead of $P_{\alpha, \alpha}$ and we have that $P_\alpha(\mu)(z) = T_\mu(1)(z)$, where 1 stands for the constant function. For $\beta > \alpha$ the function $P_{\alpha, \beta}(\mu)$ is equivalent to the $(\beta - \alpha)$ fractional derivative of $P_\alpha(\mu)$. The Bergman projection $P_\alpha$ is the orthogonal form $L^2(\mathbb{B}_n, d\nu_\alpha)$ onto $A^2_\alpha(\mathbb{B}_n)$ defined by:

$$P_\alpha f(z) = c_\alpha \int_{\mathbb{B}_n} K^{\alpha}(z, w) f(w) d\nu_\alpha(w).$$  \hfill (1.17)

The Bergman projection $P_\alpha$ naturally extends to an integral operator on $L^1(\mathbb{B}_n, d\nu_\alpha)$.

Toepplitz operators have been studied extensively on the Bergman spaces by many authors. For references, see [5, 6]. Boundedness and compactness of general Toeplitz operators $T_\mu$ on the $a$-Bloch $B^a(\mathbb{D})$ spaces have been investigated in [7] on the unit disk $\mathbb{D}$ for $0 < a < \infty$. Also in [8], the authors extend the Toeplitz operator $T_\mu^a$ to $B^a(\mathbb{B}_n)$ in the unit ball of $\mathbb{C}n$ and completely characterize the positive Borel measure $\mu$ such that $T_\mu^a$ is bounded or compact on $B^a(\mathbb{B}_n)$ with $1 \leq a < 2$. Recently, in [9], general Toeplitz operators $T_\mu^a$ on the analytic Besov $B_p(\mathbb{B}_n)$ spaces with $1 < p < \infty$ have been investigated. Under a prerequisite condition, the authors characterized complex measure $\mu$ on the unit disk $\mathbb{D}$ to which $T_\mu^a$ is bounded or compact on Besov space $B_p(\mathbb{B}_n)$. For more details on several studies of different classes of Toeplitz operators we refer to [6, 10–16] and others.

In the present paper, we will extend the general Toeplitz operators $T_\mu^a$ to $B_p(\mathbb{B}_n)$ in the unit ball of $\mathbb{C}n$ and completely characterize the positive Borel measure $\mu$ such that $T_\mu^a$ is bounded or compact on the $B_p(\mathbb{B}_n)$ spaces with $2n < p < \infty$. The extension requires some different techniques from those used in [9].

Let $\beta(\cdot, \cdot)$ be the Bergman metric on $\mathbb{B}_n$. Denote the Bergman metric ball at $a$ by $B(a, r) = \{z \in \mathbb{B}_n : \beta(a, z) < r\}$, where $a \in \mathbb{B}_n$ and $r > 0$.

**Lemma 1.2** (see [2, Theorem 2.23]). For fixed $r > 0$, there is a sequence $\{w^{(j)}\} \in \mathbb{B}_n$ such that

(i) $\bigcup_{j=1}^{\infty} B(w^{(j)}, r) = \mathbb{B}_n$;

(ii) there is a positive integer $N$ such that each $z \in \mathbb{B}_n$ is contained in at most $N$ of the sets $B(w^{(j)}, 2r)$.

A positive Borel measure $\mu$ on the unit ball $\mathbb{B}_n$ is said to be a Carleson measure for $B_p(\mathbb{B}_n)$ if there exists $C > 0$ such that

$$\int_{\mathbb{B}_n} |f(z)|^p d\mu(z) \leq C\|f\|_{B_p(\mathbb{B}_n)}^p, \quad \forall f \in B_p(\mathbb{B}_n).$$  \hfill (1.18)
The following characterization of Carleson measures can be found in [2] or in [5]. A positive Borel measure $\mu$ on the unit ball $B_n$ is said to be a Carleson measure for the Bergman space $A^p_n(B_n)$ if

$$\int_{B_n} |f(z)|^p d\nu(z) \leq C\|f\|^p_{A^p_n(B_n)}, \quad \forall f \in A^p_n(B_n).$$

(1.19)

It is well known that a positive Borel measure $\mu$ is a $(A^p(B_n), p)$-Carleson measure if and only if

$$\sup_{w^{(j)} \in B_n} \frac{\mu(B(w^{(j)}, r))}{\nu(B(w^{(j)}, r))} < \infty,$$

(1.20)

where $\{w^{(j)}\}$ is the sequence in Lemma 1.2. If $\mu$ satisfies that

$$\lim_{j \to \infty} \frac{\mu(B(w^{(j)}, r))}{\nu(B(w^{(j)}, r))} = 0,$$

(1.21)

then $\mu$ is called vanishing Carleson measure for $A^p(B_n)$.

These two are special cases of a more general notion of Carleson measures on normed spaces of analytic functions.

In general, let $\mu$ be a positive measure on $B_n$ and $X$ a M"{o}bius invariant space. For $0 < p < 1$; then $\mu$ is an $(X, p)$-Carleson measure if there is a constant $C > 0$ so that (see [2])

$$\int_{B_n} |f(z)|^p d\mu(z) \leq C\|f\|^p_X, \quad \forall f \in X.$$

(1.22)

Also, define

$$\|\nu\|_{X, p} = \sup_{f \in X, ||f||_X \leq 1} \int_{B_n} |f(z)|^p d\mu(z).$$

(1.23)

We say that $\mu$ is vanishing $(X, p)$-Carleson measure if for any sequence $\{f_n\} \in X$ with $\|f_n\|_X \leq 1$ and such that $f_n \to 0$ uniformly on compact subset of $B_n$, we have that

$$\lim_{n \to \infty} \int_{B_n} |f_n(z)|^p d\mu(z) = 0.$$

(1.24)

Throughout the paper, we will say that the expressions $A$ and $B$ are equivalent, and write $A \approx B$, whenever there exist positive constants $C_1$ and $C_2$ such that $C_1 A \leq B \leq C_2 A$. As usual, the letter $C$ will denote a positive constant, possibly different on each occurrence.
2. Bounded Toeplitz Operators on $B_p(\mathbb{B}_n)$ Spaces

We are going to work with Toeplitz operators acting on Besov spaces $B_p(\mathbb{B}_n)$ in the unit ball of $\mathbb{C}^n$.

We start with the following lemma.

**Lemma 2.1.** Let $0 < p < \infty$, $-1 < \alpha$, $t < \infty$. If

$$P_{0,\alpha}f(z) = \int_{\mathbb{B}_n} \frac{f(z)}{(1 - \langle z, w \rangle)^{n+\alpha+1}} d\nu(z), \quad (2.1)$$

then $P_{0,\alpha}$ is a bounded operator from $L^p(\mathbb{B}_n, d\nu_t)$ into $A^p_{t+p\alpha}(\mathbb{B}_n)$ if and only if $-p\alpha < t + 1 < p$.

**Proof.** Let

$$Tf(z) = \left(1 - |w|^2\right)^\alpha P_{0,\alpha}f(z)$$

$$= \left(1 - |w|^2\right)^\alpha \int_{\mathbb{B}_n} \frac{f(z)}{(1 - \langle z, w \rangle)^{n+\alpha+1}} d\nu(z). \quad (2.2)$$

By Theorem 2.10 in [2], we know that $T$ is bounded on $L^p(\mathbb{B}_n, d\nu_t)$ if and only if $-p\alpha < t + 1 < p$. However, it is obvious that $P_{0,\alpha}$ is bounded from $L^p(\mathbb{B}_n, d\nu_t)$ into $A^p_{t+p\alpha}(\mathbb{B}_n)$ if and only if $T$ is bounded on $L^p(\mathbb{B}_n, d\nu_t)$.

**Theorem 2.2.** Let $2n < p < \infty$, $\alpha > -1$ and let $\mu$ be a positive Borel measure on $\mathbb{B}_n$. If $\mu$ is a $(A^p(\mathbb{B}_n), p)$-Carleson measure, then the Toeplitz operator $T^\mu_\alpha$ is bounded on $B_p(\mathbb{B}_n)$ spaces if and only if $P_{\alpha}(\mu)(w)$ is a $(B_p(\mathbb{B}_n), p)$-Carleson measure.

**Proof.** Let $2n < p/q < \infty$ where $1/p + 1/q = 1$ and let $\alpha > -1$. We know that the dual spaces of $B_p(\mathbb{B}_n)$ are $B_q(\mathbb{B}_n)$ under the paring

$$\langle f, g \rangle = f(0)g(0) + \int_{\mathbb{B}_n} \Re f(z) \Re g(z) d\nu(z), \quad f \in B_p(\mathbb{B}_n), \quad g \in B_q(\mathbb{B}_n). \quad (2.3)$$

To prove the boundedness of $T^\mu_\alpha$, it suffices to show that

$$\left| \langle T^\mu_\alpha(f), g \rangle \right| \leq C \|f\|_{B_p(\mathbb{B}_n)} \|g\|_{B_q(\mathbb{B}_n)}, \quad (2.4)$$

for all $f \in B_p(\mathbb{B}_n)$ and $g \in B_q(\mathbb{B}_n)$, where $C$ is a positive constant that does not depend on $f$ or $g$.

Now we define $G(w)$ by the following:

$$G(w) = wP_{0,\alpha+1} \Re g(w) = w \int_{\mathbb{B}_n} \frac{\Re g(z)}{(1 - \langle z, w \rangle)^{n+\alpha+2}} d\nu(z). \quad (2.5)$$
Then
\[
\left\langle T^a \mu, f, g \right\rangle = T^a \mu f(0)g(0) + \int_{B_n} T^a \mu (\Re f) (z) \Re g(z) \, d\nu(z)
\]
\[
= T^a \mu f(0)g(0) + c_a \int_{B_n} \left( \int_{B_n} \left( 1 - |w|^2 \right)^a \Re f(w) \left( 1 - (z, w) \right)^{n+a+1} \, d\mu(w) \right) \Re g(z) \, d\nu(z). \tag{2.6}
\]

Since
\[
\frac{f(w) - f(0)}{w} = f(w) - P_a \left( f \right) (w) = f(w) - c_a \int_{B_n} \frac{f(z) \left( 1 - |z|^2 \right)^a}{(1 - (z, w))^{n+a+1}} \, d\nu(z)
\]
\[
= c_a \int_{B_n} \frac{f(w) - f(z)}{(1 - (z, w))^{n+a+1}} \, d\nu_a(z), \tag{2.7}
\]
we have
\[
T^a \mu f(0) = \int_{B_n} f(w) \, d\mu_a(w)
\]
\[
= f(0) \int_{B_n} d\mu_a(w) + c_a^2 \int_{B_n} \left( \frac{f(w) - f(z)}{1 - (z, w)} \right) \, d\nu_a(z) d\mu_a(w). \tag{2.8}
\]

This implies
\[
|T^a \mu f(0)| \leq C |f(0)| + c_a^2 \int_{B_n} \frac{|f(w) - f(z)|}{1 - (z, w)} \, d\nu_a(z) d\mu_a(w). \tag{2.9}
\]

By Proposition 1.1, we have
\[
\int_{B_n} \frac{|f(w) - f(z)|}{1 - (z, w)} \, d\nu_a(z) d\mu_a(w)
\]
\[
= \left( \int_{B_n} \left( 1 - |z|^2 \right)^{p_{a-1/2}} \left( \frac{|f(w) - f(z)|}{1 - (z, w)} \right)^p \left( 1 - |w|^2 \right)^{n/2} \right)^{1/p}
\]
\[
\times \left( \frac{1 - |w|^2}{1 - (z, w)} \right)^{p(n+a)} \, d\mu(w) \, d\nu(z) \tag{2.10}
\]
\[
\leq C \|f\|_{B^p(B_n)} \int_{B_n} \left( 1 - |z|^2 \right)^{a-(1/2)} \left( \frac{1 - |w|^2}{1 - (z, w)} \right)^{a/2} \, d\mu(w) d\nu(z).
\]
Since \( \mu \) is a \((A^p(\mathbb{E}_n), p)\)-Carleson measure, taking \( \alpha - 1/2 > -1 \), then as in \cite{8} (see also Proposition 1.4.10 of \cite{4}), we get

\[
(1 - |z|^2)^{-1/2} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{\alpha - 1/2}}{|1 - \langle z, w \rangle|^\beta} \, d\mu(w) \leq C. \tag{2.11}
\]

Then,

\[
\left| T_\mu^\alpha f(0) \right| \leq C |f(0)| + C \| f \|_{B_p(\mathbb{B}_n)} \int_{\mathbb{B}_n} d\nu(z) \leq C \| f \|_{B_p(\mathbb{B}_n)}. \tag{2.12}
\]

Therefore

\[
\left| T_\mu^\alpha f(0) g(0) \right| \leq C \| f \|_{B_p(\mathbb{B}_n)} \| g \|_{B_q(\mathbb{B}_n)}. \tag{2.13}
\]

By Fubini’s Theorem we have

\[
\left\langle T_\mu^\alpha f, g \right\rangle = \int_{\mathbb{B}_n} T_\mu^\alpha (\Re f)(z) \overline{\Re g(z)} \, d\nu(z)
\]

\[
= c_\alpha \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \left( \frac{1 - |w|^2}{|1 - \langle z, w \rangle|^\beta} \Re f(w) \right) \overline{\Re g(z)} \, d\nu(z) \right) \, d\mu(w) \tag{2.14}
\]

\[
= c_\alpha \int_{\mathbb{B}_n} f(w) \left( 1 - |w|^2 \right)^\alpha \left( \overline{\mathbb{W}} \int_{\mathbb{B}_n} \frac{\overline{\Re g(z)} \, d\nu(z)}{|1 - \langle z, w \rangle|^\beta} \right) \, d\mu(w)
\]

\[
= c_\alpha \int_{\mathbb{B}_n} f(w) \overline{G(w)} \left( 1 - |w|^2 \right)^\alpha \, d\mu(w).
\]

Using the operator \( P_\alpha \), divide the integral

\[
\int_{\mathbb{B}_n} f(w) \overline{G(w)} \left( 1 - |w|^2 \right)^\alpha \, d\mu(w) = \int_{\mathbb{B}_n} \overline{f(w)} G(w) \left( 1 - |w|^2 \right)^\alpha \, d\mu(w), \tag{2.15}
\]

we have

\[
\left\langle T_\mu^\alpha f, g \right\rangle = c_\alpha \int_{\mathbb{B}_n} \left[ (I - P_\alpha) \left( f \overline{G} \right) \right](w) \left( 1 - |w|^2 \right)^\alpha \, d\mu(w)
\]

\[
+ c_\alpha \int_{\mathbb{B}_n} P_\alpha \left( f \overline{G} \right)(w) \left( 1 - |w|^2 \right)^\alpha \, d\mu(w) \tag{2.16}
\]

\[
= I_1 + I_2,
\]
where $I$ is the identity operator, and

$$
(I - P_\alpha) \left( f \overline{G} \right)(w) = f(w) \overline{G}(w) - c_\alpha \int_{B_n} \frac{f(z) G(z) \left( 1 - |z|^2 \right)^\alpha}{(1 - \langle z, w \rangle)^{n+\alpha+1}} \, d\nu(z) 
$$

$$
= c_\alpha \int_{B_n} \frac{(f(w) - f(z)) G(z) \left( 1 - |z|^2 \right)^\alpha}{(1 - \langle z, w \rangle)^{n+\alpha+1}} \, d\nu(z).
$$

(2.17)

By Proposition 1.1, we have

$$
|I_1| = c_\alpha \left| \int_{B_n} \left( (I - P_\alpha) \left( f \overline{G} \right) \right)(w) \left( 1 - |w|^2 \right)^\alpha \, d\mu(w) \right|
$$

$$
= c_\alpha^2 \left| \int_{B_n} \frac{(f(w) - f(z)) G(z) \left( 1 - |z|^2 \right)^\alpha \left( 1 - |w|^2 \right)^\alpha}{(1 - \langle z, w \rangle)^{n+\alpha+1}} \, d\nu(z) \, d\mu(w) \right|
$$

$$
= c_\alpha^2 \left| \int_{B_n} |G(z)| \left( 1 - |z|^2 \right)^\alpha \int_{B_n} \frac{|f(w) - f(z)| \left( 1 - |w|^2 \right)^\alpha}{|1 - \langle z, w \rangle|^{n+\alpha+1}} \, d\mu(w) \, d\nu(z) \right|
$$

$$
= c_\alpha^2 \left( \int_{B_n} |G(z)|^p \left( 1 - |z|^2 \right)^{p \alpha - p/2} \right)
$$

$$
\times \int_{B_n} \frac{|f(w) - f(z)|^p \left( 1 - |z|^2 \right)^{p \alpha - p/2} \left( 1 - |w|^2 \right)^{p \alpha - p/2}}{|1 - \langle z, w \rangle|^p} \, d\mu(w) \, d\nu(z)
$$

$$
\leq C \| f \|_{L^p(B_n)} \int_{B_n} |G(z)| \left( 1 - |z|^2 \right)^{\alpha - 1/2} \int_{B_n} \frac{\left( 1 - |w|^2 \right)^{\alpha - 1/2}}{|1 - \langle z, w \rangle|^{n+\alpha}} \, d\mu(w) \, d\nu(z).
$$

By Lemma 2.1, the operator $P_{0,\alpha}$ is bounded from $L^p(B_n, d\nu)$ into $A_{\alpha \cdot 1}(B_n)$ whenever $-p\alpha < t + 1 < p$. Since $g \in B_q(B_n)$ if and only if $\Re g \in A_{q-2}(B_n)$, and we have from above, $P_{0,\alpha}$ maps $A_{q-2}(B_n)$ boundedly into $A_q^{\alpha}(B_n)$ whenever $-q(\alpha + 1) < (q - 2) + 1 < q$, whenever $-p\alpha < t + 1 < p$. Since $g \in B_q(B_n)$ if and only if $\Re g \in A_{q-2}(B_n)$, and we have from above, $P_{0,\alpha}$ maps $A_{q-2}(B_n)$ boundedly into $A_q^{\alpha}(B_n)$ whenever $-q(\alpha + 1) < (q - 2) + 1 < q$. 
or \( q > 1/(\alpha + 2) \), which is always true if \( \alpha > -1 \). Thus \( G(w) \in A^q_{\gamma_\alpha}(\mathbb{B}_n) \). It can easily seen that \( G \in A^1_\alpha(\mathbb{B}_n) \) and that \( \|G\|_{A^1_\alpha(\mathbb{B}_n)} \leq C \) for all \( \alpha > -1 \). Thus

\[
|I_1| \leq C \|f\|_{B_p(\mathbb{B}_n)} \int_{\mathbb{B}_n} |G(z)| \left( 1 - |z|^2 \right)^{\alpha} \left( 1 - w_0^2 \right)^{\alpha-1} \frac{1}{\left| 1 - (z, w) \right|^{\pi\alpha+1}} d\mu(z) d\nu(z)
\]

\[
\leq C \|f\|_{B_p(\mathbb{B}_n)} \int_{\mathbb{B}_n} \|G\|_{A^1_\alpha} \left( 1 - |z|^2 \right)^{-1/2} \left( 1 - w_0^2 \right)^{\alpha-1} \frac{1}{\left| 1 - (z, w) \right|^{\pi\alpha+1}} d\mu(z) d\nu(z)
\]

\[
\leq C \|f\|_{B_p(\mathbb{B}_n)} \int_{\mathbb{B}_n} \|g\|_{B_p(\mathbb{B}_n)} \left( 1 - |z|^2 \right)^{-1/2} \left( 1 - w_0^2 \right)^{\alpha-1} \frac{1}{\left| 1 - (z, w) \right|^{\pi\alpha+1}} d\mu(z) d\nu(z).
\]

(2.19)

By (2.11), we get

\[
|I_1| \leq C \|f\|_{B_p(\mathbb{B}_n)} \|g\|_{B_p(\mathbb{B}_n)}.
\]

(2.20)

Next consider \( I_2 \), we have

\[
|I_2| = c_\alpha \int_{\mathbb{B}_n} P_\alpha \left( f \overline{G} \right)(z) d\mu(z)
\]

\[
= c_\alpha^2 \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \frac{f(w) G(w) \left( 1 - |w|^2 \right)^{\alpha}}{\left| 1 - (z, w) \right|^{\pi\alpha+1}} d\nu(w) \right) d\mu(z)
\]

\[
= c_\alpha \int_{\mathbb{B}_n} \|G\|_{A^1_\alpha} \left( 1 - |z|^2 \right)^{\alpha} \int_{\mathbb{B}_n} \left( 1 - |w|^2 \right)^{\alpha} \frac{d\mu(z)}{\left| 1 - (z, w) \right|^{\pi\alpha+1}} d\nu(w)
\]

\[
\leq C \int_{\mathbb{B}_n} \|G\|_{A^1_\alpha} \|f\|_{A^1_\alpha} \int_{\mathbb{B}_n} \frac{d\mu(z)}{\left| 1 - (z, w) \right|^{\pi\alpha+1}} d\nu(w)
\]

\[
\leq C \int_{\mathbb{B}_n} \|g\|_{B_p(\mathbb{B}_n)} \|f\|_{A^1_\alpha} \int_{\mathbb{B}_n} \frac{d\mu(z)}{\left| 1 - (z, w) \right|^{\pi\alpha+1}} d\nu(w).
\]

(2.21)

Therefore, \( T^\alpha_\mu \) is bounded on \( B_p(\mathbb{B}_n) \) if and only if

\[
\int_{\mathbb{B}_n} \|G\|_{A^1_\alpha} \left( 1 - |z|^2 \right)^{\alpha} \frac{d\mu(z)}{\left| 1 - (z, w) \right|^{\pi\alpha+1}} d\nu(w)
\]

\[
\int_{\mathbb{B}_n} \|G\|_{A^1_\alpha} \left( 1 - |z|^2 \right)^{\alpha} \frac{d\mu(z)}{\left| 1 - (z, w) \right|^{\pi\alpha+1}} d\nu(w)
\]

(2.22)

if and only if the measure \( P_\alpha(\mu)(w) \) is a \( (B_p(\mathbb{B}_n), p) \)-Carleson measure.

Now, we will characterize boundedness of Toeplitz operators on the minimal M"obius invariant Banach spaces of holomorphic functions \( B_1(\mathbb{B}_n) \) in the unit ball of \( \mathbb{C}^n \).
Theorem 2.3. Let \( \mu \) be a positive Borel measure on \( \mathbb{B}_n \). If \( \mu \) is a \((A^p(\mathbb{B}_n), p)\)-Carleson measure, then the Toeplitz operator \( T_\mu^a \) is bounded on \( B_1(\mathbb{B}_n) \) spaces if and only if

\[
\sum_{|m| = r+1} \left| \frac{\partial^m}{\partial w^m} P_\alpha(\mu)(w) \right| d\nu(w)
\]

is a \((B_1(\mathbb{B}_n), 1)\)-Carleson measure.

Proof. We will use the fact that the dual spaces of \( B_1(\mathbb{B}_n) \) are the Bloch space \( B(\mathbb{B}_n) \) under the paring

\[
\langle f, g \rangle = \int_{\mathbb{B}_n} \Re f(z)\Re g(z) d\nu(z), \quad f \in B_1(\mathbb{B}_n), \quad g \in B(\mathbb{B}_n).
\]

Similarly, as in the proof of Theorem 2.2, by duality, we have that \( T_\mu^a \) is bounded on \( B_1(\mathbb{B}_n) \) spaces if and only if

\[
\left| \left\langle T_\mu^a(f), g \right\rangle \right| = c_\alpha \left| \int_{\mathbb{B}_n} f(w)\overline{G(w)} \left(1 - |w|^2\right)^{\alpha} d\mu(w) \right| \leq C \|f\|_{B_1(\mathbb{B}_n)} \|g\|_{B(\mathbb{B}_n)},
\]

for all \( f \in B_1(\mathbb{B}_n) \) and \( g \in B(\mathbb{B}_n) \), where

\[
G(w) = wP_0_{\alpha+1}\Re g(w) = w \int_{\mathbb{B}_n} \frac{\Re g(z)}{1 - \langle z, w \rangle^{n+\alpha+2}} d\nu(z).
\]

Using the fact that

\[
\left| \int_{\mathbb{B}_n} \frac{\Re g(z)}{1 - \langle z, w \rangle^{n+\alpha+2}} d\nu(z) \right| \approx \left| \int_{\mathbb{B}_n} \frac{\left(1 - |z|^2\right)\Re g(z)}{(1 - \langle z, w \rangle)^{n+\alpha+3}} d\nu(z) \right|
\]

for \( g \in B(\mathbb{B}_n) \), we have that \( |G(w)|(1 - |w|^2)^{\alpha+1} < \infty \), which means that \( G \in B^{\alpha+2}(\mathbb{B}_n) \). Now using the operator \( P_{\alpha+1} \), we have

\[
\left\langle T_\mu^a(f), g \right\rangle = c_\alpha \left\langle (I - P_{\alpha+1})\left(f\overline{G}\right), (1 - |w|^2)^{\alpha} d\mu(w) \right\rangle
\]

\[+ c_\alpha \int_{\mathbb{B}_n} P_{\alpha+1}(f\overline{G})(w) \left(1 - |w|^2\right)^{\alpha} d\mu(w)
\]

\[= I_1 + I_2,
\]

\[
(I - P_{\alpha+1})(f\overline{G})(w) = c_{\alpha+2} \int_{\mathbb{B}_n} \frac{(f(w) - f(z))\overline{G(z)} \left(1 - |z|^2\right)^{\alpha+1}}{(1 - \langle z, w \rangle)^{n+\alpha+2}} d\nu(z).
\]
Thus, \[ |I_1| = c_\alpha \left| \int_{B_n} (I - P_{\alpha + 1}) \left( fG \right)(w) \left( 1 - |w|^2 \right)^\alpha d\mu(w) \right| \]
\[ = c_\alpha c_{\alpha + 1} \left| \int_{B_n} \frac{(f(w) - f(z)) \overline{G(z)} \left( 1 - |z|^2 \right)^{\alpha + 2} \left( 1 - |w|^2 \right)^\alpha}{|1 - \langle z, w \rangle|^{\alpha + 2}} d\nu(z) d\mu(w) \right| \]
\[ = c_\alpha c_{\alpha + 1} \int_{B_n} |G(z)| \left( 1 - |z|^2 \right)^{\alpha + 1} \int_{B_n} \frac{|f(w) - f(z)| \left( 1 - |w|^2 \right)^\alpha}{|1 - \langle z, w \rangle|^{\alpha + 2}} d\mu(w) d\nu(z) \]
\[ \leq C \int_{B_n} \| f \|_{B_p(B_n)} \| G(z) \| \left( 1 - |z|^2 \right)^{\alpha + 1} \int_{B_n} \frac{\left( 1 - |w|^2 \right)^{\alpha - 1/2}}{|1 - \langle z, w \rangle|^{\alpha + 2}} d\mu(w) d\nu(z) \]
\[ \leq C \int_{B_n} \| f \|_{B_p(B_n)} \| G \|_{A_{\gamma, \alpha}(B_n)} \left( 1 - |z|^2 \right)^{-1/2} \int_{B_n} \frac{\left( 1 - |w|^2 \right)^{\alpha - 1/2}}{|1 - \langle z, w \rangle|^{\alpha + 2}} d\mu(w) d\nu(z) \]
\[ \leq C \| f \|_{B_p(B_n)} \| G \|_{A_{\gamma, \alpha}(B_n)}. \] (2.29)

Next consider \( I_2 \), notice first that
\[ P_\alpha(\mu)(w) = c_\alpha \int_{B_n} \frac{d\mu_\alpha(z)}{|1 - \langle z, w \rangle|^{\alpha + 1}}; \]
\[ \sum_{|m| = \alpha + 1} \left| \frac{\partial^m}{\partial w^m} P_\alpha(\mu)(w) \right| \approx \int_{B_n} \frac{(\bar{z})^m d\mu_\alpha(z)}{|1 - \langle z, w \rangle|^{\alpha + 2}}. \] (2.30)

Thus,
\[ |I_2| = c_\alpha \left| \int_{B_n} (\bar{z})^m P_{\alpha + 1} \left( fG \right)(z) d\mu_\alpha(z) \right| \]
\[ = c_\alpha c_{\alpha + 1} \left| \int_{B_n} \bar{z}^m \int_{B_n} \frac{f(w)G(w) \left( 1 - |w|^2 \right)^{\alpha + 1}}{|1 - \langle z, w \rangle|^{\alpha + 2}} d\nu(w) d\mu_\alpha(z) \right| \]
\[ = c_\alpha \int_{B_n} |f(w)| \| G(w) \| \left( 1 - |w|^2 \right)^{\alpha + 1} \left( c_{\alpha + 1} \int_{B_n} \frac{(\bar{z})^m \left( 1 - |z|^2 \right)^\alpha d\mu(z)}{|1 - \langle z, w \rangle|^{\alpha + 2}} \right) d\nu(w) \]
\[ = c_\alpha \int_{B_n} |f(w)| \| G(w) \| \left( 1 - |w|^2 \right)^{\alpha + 1} \sum_{|m| = \alpha + 1} \left| \frac{\partial^m}{\partial w^m} P_\alpha(\mu)(w) \right| d\nu(w). \] (2.31)
It is known that \((A^1(\mathbb{B}_n))^* = B^{β+1}(\mathbb{B}_n)\) under the paring

\[
\langle F, H \rangle_β = c_β \int_{\mathbb{B}_n} F(w)\overline{H(w)}\left(1 - |w|^2\right)^β d\mu(w), \quad F \in A^1(\mathbb{B}_n), \ H \in B^{β+1}(\mathbb{B}_n).
\] (2.32)

Since \(G \in B^{α+2}(\mathbb{B}_n)\), \(g \in B(\mathbb{B}_n)\) for by the above duality we get

\[
\sup_{\|g\|_{B(\mathbb{B}_n)} \leq 1} |I_2| \approx C \sup_{\|g\|_{B(\mathbb{B}_n)} \leq 1} \int_{\mathbb{B}_n} \|G\|_{A^1(\mathbb{B}_n)} \|f(w)\| \left| \sum_{|m| = n+1} \frac{\partial^m}{\partial w^m} P_α(\mu)(w) \right| d\nu(w)
\]

\[
\leq C \sup_{\|g\|_{B(\mathbb{B}_n)} \leq 1} \int_{\mathbb{B}_n} \|f(w)\| \left| \sum_{|m| = n+1} \frac{\partial^m}{\partial w^m} P_α(\mu)(w) \right| d\nu(w).
\] (2.33)

Therefore, \(T^α_μ\) is bounded on \(B_p(\mathbb{B}_n)\) if and only if

\[
\int_{\mathbb{B}_n} \|f(w)\| \left| \sum_{|m| = n+1} \frac{\partial^m}{\partial w^m} P_α(\mu)(w) \right| d\nu(w) \leq C \|f\|_{B_p(\mathbb{B}_n)}
\] (2.34)

if and only if the measure \(\sum_{|m| = n+1} |(\partial^m / \partial w^m) P_α(\mu)(w)| d\nu(w)\) is a \((B_p(\mathbb{B}_n), p)\)-Carleson measure.

\section{Compact Toeplitz Operators on \(B_p(\mathbb{B}_n)\) Spaces}

In this section we will characterize compact Toeplitz operators on \(B_p(\mathbb{B}_n)\) spaces in the unit ball of \(\mathbb{C}^n\). We need the following lemma.

\textbf{Lemma 3.1.} Let \(0 < p < \infty, -1 < \alpha\) and \(T^α_μ\) be bounded linear operator from \(B_p(\mathbb{B}_n)\) into \(B_p(\mathbb{B}_n)\) in the unit ball. Then \(T^α_μ\) is compact on \(B_p(\mathbb{B}_n)\) spaces if and only if \(\|T^α_μf_j\|_{B_p(\mathbb{B}_n)} \to 0\) as \(j \to \infty\) whenever \(\{f_j\}\) is a bounded sequence in \(B_p(\mathbb{B}_n)\) that converges to 0 uniformly on \(\mathbb{B}_n\).

\textit{Proof.} This lemma can be proved by Montel’s Theorem.

\textbf{Theorem 3.2.} Let \(2n < p < \infty, \alpha > -1\) and let \(μ\) be a positive Borel measure on \(\mathbb{B}_n\). If \(μ\) is a vanishing \((A^p(\mathbb{B}_n), p)\)-Carleson measure, then the Toeplitz operator \(T^α_μ\) is compact on \(B_p(\mathbb{B}_n)\) spaces if and only if \(P_α(μ)(w)\) is a vanishing \((B_p(\mathbb{B}_n), p)\)-Carleson measure.

\textit{Proof.} Let \(2n < p, q < \infty\) where \(1/p + 1/q = 1\) and let \(\{f_j\}\) be a sequence in \(B_p(\mathbb{B}_n)\) satisfying \(\|f_j\|_{B_p(\mathbb{B}_n)} \leq 1\) and such that \(f_j\) converges to 0 uniformly as \(j \to \infty\) on compact subsets of \(\mathbb{B}_n\), and let \(g \in B_q(\mathbb{B}_n)\). By duality, we have that \(T^α_μ\) is compact on \(B_p(\mathbb{B}_n)\) if and only if

\[
\lim_{j \to \infty} \sup_{\|g\|_{B_q(\mathbb{B}_n)} \leq 1} \left| \langle T^α_μ(f_j), g \rangle \right| = 0.
\] (3.1)
As in the proof of Theorem 2.2,

\[
\langle T_\mu^\alpha(f_j), g \rangle = T_\mu^\alpha(f_j(0)\overline{g}(0)) + \int_{\mathbb{B}_n} T_\mu^\alpha(\Re f_j)(z)\overline{\Re g(z)}d\nu(z)
\]
\[
= T_\mu^\alpha(f_j(0)\overline{g}(0)) + c_\alpha \int_{\mathbb{B}_n} f_j(w)\overline{G(w)}d\mu_\alpha(w),
\]

where

\[
G(w) = wP_{0,\alpha+1}\Re g(w) = w\int_{\mathbb{B}_n} \frac{\Re g(z)}{(1 - \langle z, w \rangle)^{\alpha+r+2}}d\nu(z).
\]

Also as in the proof of Theorem 2.2,

\[
\left| T_\mu^\alpha(f) \right| \leq C\left\| f \right\|_{B_p(\mathbb{B}_n)}.
\]

Since \(|\int_{\mathbb{B}_n} d\mu_\alpha(w)| < \infty\) and \(\mu\) is a vanishing \((A^p(\mathbb{B}_n), p)\)-Carleson measure, and \(f_j\) converges to 0 uniformly as \(j \to \infty\) on compact subsets of \(\mathbb{B}_n\), we get that

\[
T_\mu^\alpha(f) \to 0 \text{ as } j \to \infty.
\]

Thus \(T_\mu^\alpha\) is compact on \(B_p(\mathbb{B}_n)\) if and only if

\[
\lim_{j \to \infty} \sup_{\|x\|_{\mathbb{B}_n} \leq 1} \left\| \int_{\mathbb{B}_n} f_j(w)\overline{G(w)}d\mu_\alpha(w) \right\| = 0.
\]

Using the operator \(P_\alpha\), we have that

\[
\int_{\mathbb{B}_n} f_j(w)\overline{G(w)}d\mu_\alpha(w) = c_\alpha \int_{\mathbb{B}_n} (I - P_\alpha)\left( f_j\overline{G} \right)(z)d\mu_\alpha(z) + c_\alpha \int_{\mathbb{B}_n} P_\alpha\left( f_j\overline{G} \right)(z)d\mu_\alpha(z).
\]

\[
= J_1 + J_2.
\]
For $0 < r < 1$ and $rB_n = \{z \in \mathbb{C}^n, |z| \leq r\}$, we have

$$
|J_1| = c_{\alpha}^2 \int_{B_n} \left| (I - P_{\alpha}) \left( f_j(\overline{G}) \right) (w) \left( 1 - |w|^2 \right)^\alpha \right| d\mu(w)
$$

$$
= c_{\alpha}^2 \int_{B_n} \frac{(f_j(w) - f_j(z)) \overline{G}(z) \left( 1 - |z|^2 \right)^\alpha \left( 1 - |w|^2 \right)^\alpha}{(1 - \langle z, w \rangle)^{n+\alpha+1}} d\nu(z) d\mu(w)
$$

$$
= c_{\alpha}^2 \left( \int_{B_n \setminus rB_n} + \int_{rB_n} \right) |G(z)| \left( 1 - |z|^2 \right)^\alpha \int_{B_n} \frac{|f_j(w) - f_j(z)| \left( 1 - |w|^2 \right)^\alpha}{|1 - \langle z, w \rangle|^{n+\alpha+1}} d\mu(w) d\nu(z)
$$

$$
= L_1 + L_2.
$$

For a fixed $\varepsilon > 0$, since $\mu$ is a vanishing $(A^p(B_n), p)$-Carleson measure, let $r$ sufficiently close to 1 so that

$$
\left( 1 - |z|^2 \right)^{-1/2} \int_{B_n \setminus rB_n} \frac{(1 - |w|^2)^{\alpha-1/2}}{|1 - \langle z, w \rangle|^{n+\alpha}} d\mu(w) d\nu < \varepsilon.
$$

(3.9)

Similarly, as in the proof of Theorem 2.2, by Proposition 1.1,

$$
L_1 = c_{\alpha}^2 \left( \int_{B_n \setminus rB_n} + \int_{rB_n} \right) |G(z)| \left( 1 - |z|^2 \right)^\alpha \int_{B_n} \frac{|f_j(w) - f_j(z)| \left( 1 - |w|^2 \right)^\alpha}{|1 - \langle z, w \rangle|^{n+\alpha+1}} d\mu(w) d\nu(z)
$$

$$
\leq C \int_{B_n \setminus rB_n} \|f_j\|_{B_p(B_n)} |G(z)| \left( 1 - |z|^2 \right)^{\alpha-1/2} \int_{B_n} \frac{(1 - |w|^2)^{\alpha-1/2}}{|1 - \langle z, w \rangle|^{n+\alpha}} d\mu(w) d\nu(z)
$$

$$
\leq C\varepsilon \|f_j\|_{B_p(B_n)} \|G\|_{\mathcal{A}_\alpha(B_n)} \leq C\varepsilon \|f_j\|_{B_p(B_n)} \|g\|_{B_p(B_n)} \leq \varepsilon.
$$

(3.10)

Since $f_j \to 0$ as $j \to \infty$ on compact subsets of $B_n$, we can choose $j$ big enough so that

$$
|G(z)| \left( 1 - |z|^2 \right)^\alpha < \varepsilon.
$$

(3.11)

Therefore,

$$
L_2 = c_{\alpha}^2 \int_{rB_n} |G(z)| \left( 1 - |z|^2 \right)^\alpha \int_{B_n} \frac{|f_j(w) - f_j(z)| \left( 1 - |w|^2 \right)^\alpha}{|1 - \langle z, w \rangle|^{n+\alpha+1}} d\mu(w) d\nu(z)
$$

$$
\leq C \int_{rB_n} \|f_j\|_{B_p(B_n)} |G(z)| \left( 1 - |z|^2 \right)^{\alpha-1/2} \int_{B_n} \frac{(1 - |w|^2)^{\alpha-(1/2)}}{|1 - \langle z, w \rangle|^{n+\alpha}} d\mu(w) d\nu(z)
$$

$$
\leq C\varepsilon \|G\|_{\mathcal{A}_\alpha(B_n)} \leq C\varepsilon \|g\|_{B_p(B_n)}.
$$

(3.12)
Hence $|J_1| < C\varepsilon$, where $C$ does not depend on $g(z)$, and so

$$\lim_{j \to \infty} \sup_{\|g\|_{B_p(B_n)} \leq 1} |J_1| = 0. \quad (3.13)$$

Thus, $T_\mu^\alpha$ is compact on $B_p(B_n)$ if and only if

$$\lim_{j \to \infty} \sup_{\|g\|_{B_p(B_n)} \leq 1} |J_2| = 0. \quad (3.14)$$

Again, as in the proof of Theorem 2.2, we have

$$|J_2| = c_\alpha \left| \int_{B_n} P_\alpha \left( f_j \mathcal{G} \right)(z) d\mu_\alpha(z) \right|$$

$$= c_\alpha^2 \left| \int_{B_n} \int_{B_n} \frac{f_j(w) \mathcal{G}(w) \left(1 - |w|^2\right)^\alpha}{(1 - \langle z, w \rangle)^{n+\alpha+1}} d\nu(w) d\mu_\alpha(z) \right|$$

$$= c_\alpha \int_{B_n} |f_j(w)| |\mathcal{G}(w)| \left(1 - |w|^2\right)^\alpha \int_{B_n} \left(1 - |z|^2\right)^\alpha d\mu(z)$$

$$\leq C \int_{B_n} \|G\|_{A_1(B_n)} \left| f_j(w) \right| P_\alpha(\mu)(w) d\nu(w)$$

$$\leq C \int_{B_n} \|g\|_{B_1(B_n)} \left| f_j(w) \right| P_\alpha(\mu)(w) d\nu(w).$$

Therefore, $T_\mu^\alpha$ is compact on $B_p(B_n)$ if and only if

$$\lim_{j \to \infty} \int_{B_n} |f_j(w)| P_\alpha(\mu)(w) d\nu(w) = 0, \quad (3.16)$$

which is equivalent to say that $P_\alpha(\mu)(w)$ is a vanishing $(B_p(B_n), p)$-Carleson measure. \qed

**Theorem 3.3.** Let $\mu$ be a positive Borel measure on $\mathbb{B}_n$. If $\mu$ is a $(A^p(\mathbb{B}_n), p)$-Carleson measure, then the Toeplitz operator $T_\mu^\alpha$ is compact on $B_1(B_n)$ spaces if and only if

$$\sum_{|m| = n+1} \left| \frac{\partial^m P_\alpha(\mu)}{\partial w^m}(w) \right| d\nu(w) \quad (3.17)$$

is a vanishing $(B_1(\mathbb{B}_n), 1)$-Carleson measure.
Proof. Let \( \{ f_j \} \) be a sequence in \( B_p(\mathbb{B}_n) \) satisfying \( \| f_j \|_{B_p(\mathbb{B}_n)} \leq 1 \) and such that \( f_j \) converges to 0 uniformly as \( j \to \infty \) on compact subsets of \( \mathbb{B}_n \), and let \( g \in \mathcal{B}(\mathbb{B}_n) \). By duality, we have that \( T^a_\mu \) is compact on \( B_1(\mathbb{B}_n) \) if and only if
\[
\lim_{j \to \infty} \sup_{\| s \|_{B_p(\mathbb{B}_n)} \leq 1} \left| \langle T^a_\mu(f_j), g \rangle \right| = 0.
\] (3.18)

Thus, \( T^a_\mu \) is compact on \( B_1(\mathbb{B}_n) \) if and only if
\[
\lim_{j \to \infty} \sup_{\| s \|_{B_p(\mathbb{B}_n)} \leq 1} \left| \int_{\mathbb{B}_n} f_j(w) \overline{G(w)} \, d\mu_a(w) \right| = 0.
\] (3.19)

Using the operator \( P_\alpha \), we have that
\[
\int_{\mathbb{B}_n} f_j(w) \overline{G(w)} \, d\mu_a(w) = c_d \int_{\mathbb{B}_n} (I - P_\alpha) \left( f_j \overline{G} \right)(z) \, d\mu_a(z) + c_d \int_{\mathbb{B}_n} P_\alpha \left( f_j \overline{G} \right)(z) \, d\mu_a(z).
\] (3.20)

As in the proof of Theorem 2.3, we have
\[
|J_1| \leq C \int_{\mathbb{B}_n} \| f_j \|_{B_p(\mathbb{B}_n)} \| G \|_{A^1(\mathbb{B})} \left( 1 - |z|^2 \right)^{-1/2} \int_{\mathbb{B}_n} \frac{|1 - |w|^2|^{a-1/2}}{|1 - \langle z, w \rangle|^{n+a+1}} \, d\mu(w) \, dv(z).
\] (3.21)

Notice that \( \| f_j \|_{B_p(\mathbb{B}_n)} \) implies that \( \| f_j \|_{B_1(\mathbb{B}_n)} \leq C \). Since \( f_j \) converges to 0 uniformly as \( j \to \infty \) on compact subsets of \( \mathbb{B}_n \), and \( \mu \) is a \( (A^p(\mathbb{B}_n), p) \)-Carleson measure, we get that \( G \in \mathcal{B}^{a+2}(\mathbb{B}_n) \) and \( \| G \|_{\mathcal{B}^{a+2}(\mathbb{B}_n)} \leq C \| G \|_{\mathcal{B}(\mathbb{B}_n)} \). Thus
\[
|J_1| \leq C \int_{\mathbb{B}_n} \| f_j \|_{B_1(\mathbb{B}_n)} \| g \|_{\mathcal{B}(\mathbb{B}_n)} \| \mu \|_{A^1(\mathbb{B})} \left( 1 - |z|^2 \right)^{-1/2} \int_{\mathbb{B}_n} \frac{|1 - |w|^2|^{a-1/2}}{|1 - \langle z, w \rangle|^{n+a+1}} \, d\mu(w) \, dv(z).
\] (3.22)

Therefore, \( T^a_\mu \) is compact on \( B_1(\mathbb{B}_n) \) if and only if
\[
\lim_{j \to \infty} \sup_{\| s \|_{B_p(\mathbb{B}_n)} \leq 1} |J_2| = 0.
\] (3.23)

We have shown in the proof of Theorem 2.3
\[
\sup_{\| s \|_{B_p(\mathbb{B}_n)} \leq 1} |J_2| \leq C \sup_{\| s \|_{B_p(\mathbb{B}_n)} \leq 1} \int_{\mathbb{B}_n} |f_j(w)| \Re P_a(\mu)(w) \, dv(w).
\] (3.24)
Therefore, $T^\sigma_\mu$ is compact on $B_1(\mathbb{B}_n)$ if and only if
\[
\lim_{j \to \infty} \int_{\mathbb{B}_n} |f_j(w)| \sum_{|m| = n+1} \frac{\partial^m}{\partial w^m} P_\alpha(\mu)(w) \left| dw(w) \right| = 0, \tag{3.25}
\]
which is equivalent to saying that the measure $\sum_{|m| = n+1} \left| \frac{\partial^m}{\partial w^m} P_\alpha(\mu)(w) \right| dw(w)$ is a vanishing $(B_1(\mathbb{B}_n), 1)$-Carleson measure.

References

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