Research Article

Dichotomies with No Invariant Unstable Manifolds for Autonomous Equations

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We analyze the existence of (no past) exponential dichotomies for a well-posed autonomous differential equation (that generates a C_0 -semigroup $\{T(t)\}_{t \geq 0}$). The novelty of our approach consists in the fact that we do not assume the $T(t)$ -invariance of the unstable manifolds. Roughly speaking, we prove that if the solution of the corresponding inhomogeneous difference equation belongs to any sequence space (on which the right shift is an isometry) for every inhomogeneity from the same class of sequence spaces, then the continuous-time solutions of the autonomous homogeneous differential equation will exhibit a (no past) exponential dichotomic behavior. This approach has many advantages among which we emphasize on the facts that the aforementioned condition is very general (since the class of sequence spaces that we use includes almost all the known sequence spaces, as the classical ℓ^p spaces, sequence Orlicz spaces, etc.) and that from discrete-time conditions we get information about the continuous-time behavior of the solutions.

1. Introduction

The exponential dichotomy is one of the most basic concepts arising in the theory of dynamical systems. For linear differential equations, the notion was introduced by Perron in [1], who was concerned with the problem of conditional stability of a system $x' = A(t)x$ and its connection with the existence of bounded solutions of the equation $x' = A(t)x + f(t, x)$, where the state space X is a finite-dimensional Banach space and the operator-valued function $A(\cdot)$ is bounded and continuous in the strong operator topology. Relevant results concerning the extension of Perron's problem in the more general framework of infinite-dimensional Banach spaces were obtained by Daleckij and Krein [2], Bellman [3], Massera and Schäffer [4], and more recently by van Neerven [5], and van Minh et al. [6].

For the case of discrete-time systems, analogous results were firstly obtained by Li in [7]. In his paper, we remark the same central concern as in Perron's work, but in other terms. In fact it was proposed that the inhomogeneous equation is responsible in some sense for the asymptotic behavior of solutions of the homogeneous equation. In this spirit, there were established connections between the condition that the inhomogeneous equation has some bounded solution for every bounded "second member" on the one hand and a certain form of conditional stability of the solutions of the homogeneous equation on the other. This idea was later extensively developed for discrete-time systems in the infinite-dimensional case by Coffman and Schäffer [8] and Henry [9]. More recently, we have the papers of Ben-Artzi and Gohberg [10], Pinto [11], and LaSalle [12]. Applications of this "discrete-time theory" to stability theory of linear continuous-time systems in infinite-dimensional spaces have been presented by Przyłuski and Rolewicz in [13].

The dichotomy of the autonomous equation (A) , $x'(t) = Ax(t)$ (assuming its well-posedness, i.e., A generates a C_0 -semigroup $\{T(t)\}_{t \geq 0}$) consists in the existence of a bounded projection P such that the solutions that start in $X_1 = \text{Im } P$ decay to zero and the solutions that start in $X_2 = \text{Im}(I - P)$ are unbounded. In the hypothesis that X_2 is $T(t)$ -invariant and finite-dimensional, the existence of a dichotomy for (A) implies that solutions starting in X_2 exist in backward time (or equivalently, $\{T(t)\}$ extends to a C_0 -group on X_2). This observation is not necessarily true in the infinite-dimensional setting, but required in many researches (see, e.g., [6, 14, 15]). The novelty of our approach consists in the fact that we do not assume *a priori* that the operators $T(t)|_{X_2}$ are invertible (we do not even assume that X_2 is $T(t)$ -invariant), and subsequently the unstable subspace is allowed to be infinite-dimensional. Roughly speaking, we prove that if the solution of the corresponding inhomogeneous difference equation belongs to any sequence space (on which the right shift is an isometry) for every inhomogeneity from the same class of sequence spaces, then the continuous-time solutions of the autonomous homogeneous differential equation will exhibit a (no past) exponential dichotomic behavior. This approach has many advantages among which we emphasize on the facts that the previous condition is very general (since the class of sequence spaces that we use includes almost all the known sequence spaces, as the classical ℓ^p spaces, sequence Orlicz spaces, etc.), and since we use a discrete-time technique, we are not forced to require any continuity or measurability hypotheses on the trajectories of the one-parameter semigroup generated by the differential system. Also, it is worth to mention that from discrete-time conditions we get information about the continuous-time behavior of the solutions.

2. Sequence Schäffer Spaces

Let $\mathbb{N}$ be the set of all nonnegative integers, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\mathbb{R}$ the set of all real numbers, and we will denote by $[t]$ the greatest integer less than or equal with t . The linear space of all real-valued sequences $s : \mathbb{N} \rightarrow \mathbb{R}$ is denoted by $\mathcal{S}$. Let $(X, \|\cdot\|)$ be a real or complex Banach space and consider $\mathcal{S}(X)$ the linear space of all sequences $f : \mathbb{N} \rightarrow X$. For an X -valued sequence $f : \mathbb{N} \rightarrow X$, we can associate the sequence $\|f\| : \mathbb{N} \rightarrow \mathbb{R}$ defining $\|f\|(n) = \|f(n)\|$ for all $n \in \mathbb{N}$. We also consider two linear operators $R, L : \mathcal{S}(X) \rightarrow \mathcal{S}(X)$ defined by

$$Rf(n) = \begin{cases} f(n-1), & n \in \mathbb{N}^*, \\ 0, & n = 0, \end{cases} \quad Lf(n) = f(n+1), \quad (2.1)$$

known as the *right shift operator*, respectively, the *left shift operator*. A simple verification gives us $LRf = f$ and $RLf(n) = f(n)$ for $n \in \mathbb{N}^*$, $RLf(0) = 0$, for all $f \in \mathcal{S}(X)$. If $A \subset \mathbb{N}$, the characteristic function of A will be denoted by χ_A and for the simplicity of notation put $\delta_k = \chi_{\{k\}}$ for each $k \in \mathbb{N}$.

Definition 2.1. A Banach space $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is said to be a *sequence Schäffer space* if $\mathcal{E} \subset \mathcal{S}$ and the following conditions hold:

$$(s_1) \quad \delta_0 \in \mathcal{E},$$

$$(s_2) \quad \text{if } f \in \mathcal{E}, \text{ then } Rf \in \mathcal{E} \text{ and } \|Rf\|_{\mathcal{E}} = \|f\|_{\mathcal{E}},$$

$$(s_3) \quad \text{if } f \in \mathcal{S} \text{ and } g \in \mathcal{E} \text{ such that } |f| \leq |g|, \text{ then } f \in \mathcal{E} \text{ and } \|f\|_{\mathcal{E}} \leq \|g\|_{\mathcal{E}}.$$

Remark 2.2. By (s_1) and (s_2) we have that any sequence with finite support is contained in any sequence Schäffer space, hence $\chi_{\{0,1,\dots,n\}} \in \mathcal{E}$ for any sequence Schäffer space $\mathcal{E}$ and $n \in \mathbb{N}$. The third property is called the *ideal property* and will play a central role in our investigations.

Example 2.3. Common instances of sequence Schäffer spaces are the spaces of p -summable sequences, namely, for $p \in [1, \infty)$,

$$\begin{aligned} \ell^p &= \left\{ f : \mathbb{N} \rightarrow \mathbb{R} : \sum_{k=0}^{\infty} |f(k)|^p < \infty \right\}, \quad \text{with the norm } \|f\|_p = \left(\sum_{k=0}^{\infty} |f(k)|^p \right)^{1/p}, \\ \ell^{\infty} &= \left\{ f : \mathbb{N} \rightarrow \mathbb{R} : \sup_{n \in \mathbb{N}} |f(n)| < \infty \right\}, \quad \text{with the norm } \|f\|_{\infty} = \sup_{n \in \mathbb{N}} |f(n)|. \end{aligned} \quad (2.2)$$

The subspace of ℓ^{∞} , $\ell_0^{\infty} = \{f \in \ell^{\infty} : \lim_{n \rightarrow \infty} f(n) = 0\}$ (often denoted by c_0) with the induced norm is another example of sequence Schäffer space.

It is easy to check that $(c, \|\cdot\|_{\infty})$ (the space of all convergent sequences) is not a sequence Schäffer space.

The spaces ℓ^1 , ℓ^{∞} , and ℓ_0^{∞} occupy particularly important positions in the class of sequence Schäffer spaces. For $\mathcal{E}$ a sequence Schäffer space, we will define the sequences $\alpha_{\mathcal{E}}, \beta_{\mathcal{E}} \in \mathcal{S}$ by

$$\begin{aligned} \alpha_{\mathcal{E}}(n) &= \inf \left\{ L > 0 : \sum_{k=0}^n |f(k)| \leq L \|f\|_{\mathcal{E}}, \forall f \in \mathcal{E} \right\}, \\ \beta_{\mathcal{E}}(n) &= \|\chi_{\{0,1,\dots,n\}}\|_{\mathcal{E}}, \end{aligned} \quad (2.3)$$

which are both nondecreasing and $\beta_{\mathcal{E}}(n) > 0$, for all $n \in \mathbb{N}$.

Example 2.4. Other remarkable examples of sequence Schäffer spaces are the *sequence Orlicz spaces*. Let $\varphi : \mathbb{R} \rightarrow [0, \infty]$ be a left continuous, nondecreasing function and not identically

0 or ∞ on $(0, \infty)$. The Young function attached to φ is defined by $\Phi(t) = \int_0^t \varphi(s) ds$, $t \geq 0$. Consider

$$\begin{aligned} \ell^\Phi &= \left\{ f \in \mathcal{S} : \text{exists } c > 0 \text{ such that } \sum_{k=0}^n \Phi(c|f(k)|) < \infty \right\} \quad \text{with the norm} \\ \|f\|_\Phi &= \inf \left\{ c > 0 : \sum_{k=0}^n \Phi(c^{-1}|f(k)|) \leq 1 \right\} \quad (\text{the Luxemburg norm}). \end{aligned} \quad (2.4)$$

For the Banach space $(\ell^\Phi, \|\cdot\|_\Phi)$ the conditions (s_1) , (s_2) , and (s_3) are verified; hence $(\ell^\Phi, \|\cdot\|_\Phi)$ is a sequence Schffer space.

For $1 \leq p < \infty$, taking $\varphi(t) = pt^{p-1}$ we have that $(\ell^\Phi, \|\cdot\|_\Phi) \equiv (\ell^p, \|\cdot\|_p)$. Even ℓ^∞ is a sequence Orlicz space, obtained from $\varphi(t) = 0$ for $t \in [0, 1]$ and $\varphi(t) = \infty$ for $t > 1$ [16, Section 3].

Remark 2.5. Let $p \in [1, \infty)$. If $(\ell^\Phi, \|\cdot\|_\Phi) = (\ell^p, \|\cdot\|_p)$, then $\lim_{t \rightarrow 0} (\Phi(t)/t^p) = 1$.

Indeed, if $(\ell^\Phi, \|\cdot\|_\Phi) = (\ell^p, \|\cdot\|_p)$, then $\|\chi_{\{0,1,\dots,n\}}\|_\Phi = \|\chi_{\{0,1,\dots,n\}}\|_p$, for all $n \in \mathbb{N}$, which is equivalent with

$$\Phi^{-1}\left(\frac{1}{n+1}\right) = \left(\frac{1}{n+1}\right)^{1/p}, \quad \forall n \in \mathbb{N}. \quad (2.5)$$

Let $x \in (0, 1]$ and $m = [1/x] \in \mathbb{N}^*$. Using the fact that Φ^{-1} is nondecreasing, we have that

$$\left(\frac{1}{m+1}\right)^{1/p} = \Phi^{-1}\left(\frac{1}{m+1}\right) \leq \Phi^{-1}(x) \leq \Phi^{-1}\left(\frac{1}{m}\right) = \left(\frac{1}{m}\right)^{1/p}, \quad (2.6)$$

which implies that

$$\left[\frac{1}{([1/x] + 1)x}\right]^{1/p} \leq \frac{\Phi^{-1}(x)}{x^{1/p}} \leq \left[\frac{1}{x[1/x]}\right]^{1/p}, \quad (2.7)$$

for all $x \in (0, 1]$. Hence $\lim_{x \rightarrow 0} \Phi^{-1}(x)x^{-(1/p)} = 1$ and

$$\lim_{u \rightarrow 0} \frac{\Phi(u)}{u^p} = \lim_{u \rightarrow 0} \frac{1}{\left[\Phi^{-1}(\Phi(u))/(\Phi(u))^{1/p}\right]^p} = 1. \quad (2.8)$$

Example 2.6. Consider $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$:

$$\varphi(t) = \sum_{m=1}^{\infty} \frac{\sqrt[m]{t}}{m^2}, \quad \Phi(t) = \int_0^t \varphi(s) ds = \sum_{m=1}^{\infty} \frac{t^{1+1/m}}{m(m+1)}. \quad (2.9)$$

We claim that $\ell^\Phi \neq \ell^p$, no matter how we choose $p \in [1, \infty)$.

Indeed, $\lim_{t \rightarrow 0} (\Phi(t)/t) = 0$ and $\lim_{t \rightarrow 0} (\Phi(t)/t^p) = \infty$, for all $p \in (1, \infty)$, and using the aforementioned remark, our claim follows easily. Also, $\ell^\Phi \neq \ell^\infty$ since $\chi_{\mathbb{N}} \in \ell^\infty \setminus \ell^\Phi$.

Remark 2.7. By simple computations we obtain that

$$\alpha_{\ell^p}(n) = (n+1)^{1-1/p}, \quad \beta_{\ell^p}(n) = (n+1)^{1/p} \quad (2.10)$$

for $1 \leq p \leq \infty$ (with the convention $1/\infty = 0$) and for the Orlicz sequence spaces,

$$\alpha_{\ell^\Phi}(n) = (n+1)\Phi^{-1}\left(\frac{1}{n+1}\right), \quad \beta_{\ell^\Phi}(n) = \frac{1}{\Phi^{-1}(1/(n+1))}. \quad (2.11)$$

For two Banach spaces $(B_1, \|\cdot\|_1)$ and $(B_2, \|\cdot\|_2)$, we say that B_1 is continuously embedded in B_2 ($B_1 \hookrightarrow B_2$) if $B_1 \subset B_2$ and there exists $c > 0$ such that $\|f\|_2 \leq c\|f\|_1$ for all $f \in B_1$. For the following three propositions, proofs can be retrieved from [8, Section 3].

Proposition 2.8. *If $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a sequence Schaffer space, then $\ell^1 \hookrightarrow \mathcal{E} \hookrightarrow \ell^\infty$ with $\beta_{\mathcal{E}}(0)\|f\|_\infty \leq \|f\|_{\mathcal{E}}$ for all $f \in \mathcal{E}$ and $\|f\|_{\mathcal{E}} \leq \beta_{\mathcal{E}}(0)\|f\|_1$ for all $f \in \ell^1$.*

Proposition 2.9. *If $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ and $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ are sequence Schaffer spaces, then $\mathcal{E} \hookrightarrow \mathcal{F}$ if and only if $\mathcal{E} \subset \mathcal{F}$.*

Proposition 2.10. *Let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ be a sequence Schaffer space. The following characterizations hold:*

- (i) $\alpha_{\mathcal{E}}$ is bounded if and only if $\mathcal{E} = \ell^1$ and $\|\cdot\|_1 \sim \|\cdot\|_{\mathcal{E}}$;
- (ii) $\beta_{\mathcal{E}}$ is bounded if and only if $\ell_0^\infty \subset \mathcal{E}$.

For $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ being a sequence Schaffer space and X being a Banach space, we consider $\mathcal{E}(X) = \{f \in \mathcal{S}(X) : \|f\| \in \mathcal{E}\}$ and $\|f\|_{\mathcal{E}(X)} = \|\|f\|\|_{\mathcal{E}}$. To prove that $(\mathcal{E}(X), \|\cdot\|_{\mathcal{E}(X)})$ is a Banach space see, for example, [17, Remark 2.1] or [8, Lemma 3.8]. The following properties of this space are simple verifications.

Proposition 2.11. *The space $(\mathcal{E}(X), \|\cdot\|_{\mathcal{E}(X)})$ is a Banach space with the following properties:*

- (i) if $f \in \mathcal{S}(X)$ has finite support, then $f \in \mathcal{E}(X)$;
- (ii) if $f \in \mathcal{E}(X)$, then $Rf \in \mathcal{E}(X)$ and $\|Rf\|_{\mathcal{E}(X)} = \|f\|_{\mathcal{E}(X)}$;
- (iii) if $f \in \mathcal{S}(X)$ and $g \in \mathcal{E}(X)$ such that $\|f\| \leq \|g\|$, then $f \in \mathcal{E}(X)$ and $\|f\|_{\mathcal{E}(X)} \leq \|g\|_{\mathcal{E}(X)}$.

To prevent any further confusion, let us fix the notation $\mathcal{B}(X)$ for the class of bounded linear operators acting on X .

3. Semigroups of Bounded Linear Operators, Exponential Dichotomy, and Admissibility

Let X be a Banach space, $x_0 \in X$, $A : D(A) \subset X \rightarrow X$ a linear operator, and consider the (abstract) Cauchy problem $(A; x_0)$:

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad t \geq 0, \\ x(0) &= x_0. \end{aligned} \tag{3.1}$$

The function $u : [0, \infty) \rightarrow D(A)$ is said to be a classical solution for the previous Cauchy problem if it is continuously differentiable on $[0, \infty)$ and satisfies $(A; x_0)$. A theorem of E. Hille states that if the resolvent set of A is nonempty, then for every $x \in D(A)$ the Cauchy problem $(A; x)$ has a unique classical solution if and only if A is the infinitesimal generator of a C_0 -semigroup. For the proof of this result we guide the reader to [18, Theorem 6.7, page 150] or [19, Theorem 1.3, page 102].

Definition 3.1. A family of bounded linear operators $\{T(t)\}_{t \geq 0}$ acting on X is called a *semigroup* if

(i) $T(0) = I$ (where I is the identity operator on X),

(ii) $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$.

If (ii) holds for any $t, s \in \mathbb{R}$, then $\{T(t)\}_{t \geq 0}$ is called a *group*. If in addition of (i) and (ii), there exist $M, \omega > 0$ such that

(iii) $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$,

then $\{T(t)\}_{t \geq 0}$ is said to be *exponentially bounded*.

If a semigroup $\{T(t)\}_{t \geq 0}$ has all its orbits $T(\cdot)x$ right-continuous in the origin, then it is in the class of C_0 -semigroups (or it is said to be *strongly continuous*). It is well known that a strongly continuous semigroup is exponentially bounded (for details we refer the reader to [19, page 4]).

If Y is a closed subspace of X and $\{T(t)\}_{t \geq 0}$ is a (C_0) -semigroup such that Y is $T(t)$ -invariant for any $t \geq 0$, then the restrictions $T(t)|_Y$ form a (C_0) -semigroup called the subspace (C_0) -semigroup on Y ([18, page 43]). A (C_0) -semigroup can be extended to a (C_0) -group if and only if there exists $t_0 > 0$ such that $T(t_0)$ is invertible (see [18, page 80] or [19, Section 1.6]).

The linear operator defined by

$$\begin{aligned} D(A) &= \left\{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}, \\ Ax &= \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \quad \text{for } x \in D(A) \end{aligned} \tag{3.2}$$

is the *infinitesimal generator* of the semigroup $\{T(t)\}_{t \geq 0}$. If $\{T(t)\}_{t \geq 0}$ is a C_0 -semigroup, then A is closed and densely defined and $\{T(t)\}_{t \geq 0}$ is exponentially bounded (for details, see [19, Section 1.2]). It is clear that a classical solution for $(A; x)$ with A being the infinitesimal

generator of $\{T(t)\}_{t \geq 0}$ exists only if $x \in D(A)$ in which case $T(\cdot)x$ is the unique classical solution; otherwise it is said to be the *mild solution* of $(A; x)$.

A linear and bounded operator T acting on a complex Banach space is said to be *hyperbolic* if $\sigma(T) \cap \Gamma = \emptyset$ (where $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ denotes the unit circle in the complex plane and $\sigma(T)$ is the spectrum of T). The spectral Riesz projection P for a hyperbolic operator T is given by

$$P = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - T)^{-1} d\lambda. \quad (3.3)$$

The projection corresponds to the part of the spectrum of T contained in the open unit disk $D(0, 1)$. We note that the projection P commutes with T . Since

$$\sigma(T|_{\text{Im } P}) = \sigma(T) \cap D(0, 1), \quad \sigma(T|_{\text{Im}(I-P)}) = \sigma(T) \cap (\mathbb{C} \setminus D(0, 1)), \quad (3.4)$$

we obtain the spectral radius $r(T|_{\text{Im } P}) < 1$ and also the operator $T|_{\text{Im}(I-P)}$ is invertible with $r(T^{-1}|_{\text{Im}(I-P)}) < 1$. Hence, if T is hyperbolic, then there exist the constants $N, \nu > 0$ such that, for all integers $n \geq 0$,

$$\begin{aligned} \|T^n x\| &\leq N e^{-\nu n} \|x\|, \quad \forall x \in \text{Im } P, \\ \|T^n x\| &\geq N^{-1} e^{\nu n} \|x\|, \quad \forall x \in \text{Im}(I - P) \end{aligned} \quad (3.5)$$

(for the previous exposure we consulted [14, page 28]).

Definition 3.2. A semigroup $\{T(t)\}_{t \geq 0}$ is *hyperbolic* if there exists $t_0 > 0$ such that $T(t_0)$ is an hyperbolic operator.

Definition 3.3. The semigroup $\{T(t)\}_{t \geq 0}$ has an *exponential dichotomy* (or that it is exponentially dichotomic) if there exist a projection P (i.e., $P \in \mathcal{B}(X)$ and $P^2 = P$) and the constants $N, \nu > 0$ such that

- (i) $PT(t) = T(t)P$, for all $t \geq 0$,
- (ii) $T(t)|_{\text{Ker } P} : \text{Ker } P \rightarrow \text{Ker } P$ is an isomorphism, for each $t \geq 0$,
- (iii) $\|T(t)x\| \leq N e^{-\nu t} \|x\|$, for all $t \geq 0$ and $x \in \text{Im } P$,
- (iv) $\|T(t)^{-1}|_{\text{Ker } P} x\| \leq N e^{-\nu t} \|x\|$, for all $t \geq 0$ and $x \in \text{Ker } P$.

The first condition in the previous definition expresses equivalently that $\text{Im } P$ and $\text{Ker } P$ are both $T(t)$ -invariant; the essence of (ii) is that $\{T(t)|_{\text{Ker } P}\}_{t \geq 0}$ can be extended to a group. The next result establishes the relation between hyperbolicity and exponential dichotomy for C_0 -semigroups on complex Banach spaces. For the proof we refer the reader to [14, Lemma 2.15], [20, Theorem 1.1], or alternatively [15].

Proposition 3.4. *For a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ acting on a complex Banach space the following statements are equivalent.*

- (i) $\{T(t)\}_{t \geq 0}$ is hyperbolic.
- (ii) $\{T(t)\}_{t \geq 0}$ has an exponential dichotomy.

Moreover, if (i) holds, then $X = X_1 \oplus X_2$ where $X_1 = \text{Im } P$, $X_2 = \text{Im}(I - P)$, and

$$P = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - T(t_0))^{-1} d\lambda \quad (3.6)$$

is the spectral Riesz projection for $T(t_0)$ that corresponds to $\sigma(T(t_0)) \cap \Gamma = \emptyset$. Also, if $\{T(t)\}_{t \geq 0}$ has an exponential dichotomy, then $\sigma(T(t)) \cap \Gamma = \emptyset$, for every $t \geq 0$.

If A generates an exponentially dichotomic C_0 -semigroup, then the differential equation $\dot{x} = Ax$ has the property that the solutions $x(\cdot)$ starting from X_1 (resp., from X_2) decay exponentially for $t > 0$ (resp., for $t < 0$) uniformly with respect to the initial data. As it can be seen, the exponential dichotomy concept generalizes strongly the exponential stability concept but it has a serious drawback. It forces the solution that starts from X_2 to exist for negative time, or in counterpart it forces the semigroup to be invertible on X_2 . We will drop off this requirement here and extend the notion of hyperbolicity by replacing the exponential decay in negative time for the solutions starting in X_2 with an exponential blow-up in positive time. We will call the “exponential decay on X_1 and exponential blow-up on X_2 ” (both on positive time) behavior as no past exponential dichotomy.

Definition 3.5. The semigroup $\{T(t)\}_{t \geq 0}$ has a *no past exponential dichotomy* if there exist a projection P and the constants $N_1, N_2, \nu > 0$ such that

- (i) $PT(t)P = T(t)P$, for all $t \geq 0$,
- (ii) $\|T(t)x\| \leq N_1 e^{-\nu t} \|x\|$, for all $t \geq 0$, $x \in \text{Im } P$,
- (iii) $\|T(t)x\| \geq N_2 e^{\nu t} \|x\|$, for all $t \geq 0$, $x \in \text{Ker } P$.

Definition 3.6. The semigroup $\{T(t)\}_{t \geq 0}$ has an *ordinary dichotomy* if there exist a projection P and the constants $N_1, N_2 > 0$ such that

- (i) $PT(t)P = T(t)P$, for all $t \geq 0$,
- (ii) $\|T(t)x\| \leq N_1 \|x\|$, for all $t \geq 0$, $x \in \text{Im } P$,
- (iii) $\|T(t)x\| \geq N_2 \|x\|$, for all $t \geq 0$, $x \in \text{Ker } P$.

Remark 3.7. Note that if P is one-to-one, then $\text{Im } P = X$ and thus the concept of (no past) exponential dichotomy overlaps the concept of *exponential stability*. Recall that $\{T(t)\}_{t \geq 0}$ is said to be exponentially stable if one of the following equivalent statements is true:

- (i) there exist $N, \nu > 0$ such that $\|T(t)\| \leq N e^{-\nu t}$ for all $t \geq 0$;
- (ii) there exist $t_0 > 0$ such that $\|T(t_0)\| < 1$.

It is obvious that the existence of an exponential dichotomy implies the existence of a no past exponential dichotomy, but the converse is not valid as the following example points out.

Example 3.8. The following C_0 -semigroup $\{S(t)\}_{t \geq 0}$ has a no past exponential dichotomy but is not exponentially dichotomic. Let A be a $p \times p$ matrix with real entries whose spectrum is contained in the open left-half plane and consider the right shift semigroup on $L^1(\mathbb{R}_+, \mathbb{R})$, given by

$$T(t)f(s) = \begin{cases} 0, & 0 \leq s < t, \\ f(s-t), & s \geq t. \end{cases} \quad (3.7)$$

Then $S(t) = e^{tA} \oplus e^t T(t)$, $S(t)(x, f) = (e^{tA}x, e^t T(t)f)$ has a no past exponential dichotomy on $X = \mathbb{R}^p \oplus L^1(\mathbb{R}_+, \mathbb{R})$ with $X_1 = \{(x, 0) : x \in \mathbb{R}^p\}$ and $X_2 = \{(0, f) : f \in L^1(\mathbb{R}_+, \mathbb{R})\}$. However, the restriction of $S(t)$ on X_2 is not onto, and thus $\{S(t)\}_{t \geq 0}$ is not exponentially dichotomic.

Lemma 3.9. Let $\{T(t)\}_{t \geq 0}$ be an exponentially bounded semigroup. If there exist Y , a vector subspace of X , $n_0 \in \mathbb{N}^*$, $H > 0$, and $\eta > 1$ such that

- (i) $\|T(n)x\| \geq H\|x\|$, for all $n \in \{0, 1, \dots, n_0\}$ and $x \in Y$,
- (ii) $\|T(n + n_0)x\| \geq \eta\|T(n)x\|$, for all $n \in \mathbb{N}$ and $x \in Y$,

then there exist $N, \nu > 0$ which satisfy $\|T(t)x\| \geq Ne^{\nu t}\|x\|$, for all $x \in Y$ and $t \geq 0$.

Proof. Let $x \in Y$, and $n \in \mathbb{N}^*$. Then, there exist $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n_0 - 1\}$ such that $n = kn_0 + r$. Therefore, $\|T(n)x\| = \|T(kn_0 + r)x\| \geq \eta^k \|T(r)x\| \geq H\eta^k \|x\|$. We take $\nu := (1/n_0) \ln \eta > 0$ to obtain

$$\|T(n)x\| \geq He^{\nu kn_0} \|x\| = He^{-\nu r} e^{\nu(kn_0+r)} \|x\| > He^{-\nu n_0} e^{\nu n} \|x\|, \quad \forall n \in \mathbb{N}. \quad (3.8)$$

For $t \geq 0$, take $n = [t] + 1$ to have $Me^{\omega} \|T(t)x\| \geq \|T(n)x\| \geq He^{-\nu n_0} e^{\nu n} \|x\|$. Put $N := HM^{-1}e^{-(\nu n_0 + \omega)} > 0$ in order to write $\|T(t)x\| \geq Ne^{\nu t}\|x\|$. $\square$

Lemma 3.10. Let $\{T(t)\}_{t \geq 0}$ be an exponentially bounded semigroup. If there exist Y a vector subspace of X , $n_0 \in \mathbb{N}^*$, $H > 0$, and $\eta \in (0, 1)$ such that

- (i) $\|T(n)x\| \leq H\|x\|$, for all $n \in \{0, 1, \dots, n_0\}$ and $x \in Y$ and
- (ii) $\|T(n + n_0)x\| \leq \eta\|T(n)x\|$, for all $n \in \mathbb{N}$ and $x \in Y$,

then there exist $N, \nu > 0$ which satisfy $\|T(t)x\| \leq Ne^{-\nu t}\|x\|$, for all $x \in Y$ and $t \geq 0$.

Proof. It is analogous with the proof of Lemma 3.9. $\square$

Consider the autonomous inhomogeneous abstract Cauchy problem $(A, f; x_0)$:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(t), \quad t \geq 0, \\ x(0) &= x_0. \end{aligned} \quad (3.9)$$

If A generates the C_0 -semigroup $\{T(t)\}_{t \geq 0}$, $x \in X$ and $f \in L^1_{\text{loc}}(\mathbb{R}_+, X)$, then the function $u : \mathbb{R}_+ \rightarrow X$ given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds \quad (3.10)$$

is said to be the *mild solution* of the Cauchy problem $(A, f; x)$. For sufficient conditions assuring that the mild solution is also a classical solution (i.e., a continuously differentiable function that verifies the initial value problem $(A, f; x)$), we refer the reader to [19, Section 4.2].

The “test functions method” (or “Perron’s method”) was often used until now (see e.g., [1–4]) to study properties of asymptotic behavior such as exponential dichotomy. According to Massera and Schäffer [4] by “test functions method” it is meant the relation between certain “test functions” f and “nice solutions” of the inhomogeneous equations $(A, f): x' = Ax + f$. The crudest expression of this method is the notion of *admissibility* of a pair of classes of functions both in $L^1_{\text{loc}}(\mathbb{R}_+, X)$ (Massera and Schäffer named these classes of functions as the class of “test functions” and the class of “nice solutions” and defined the pair to be admissible if for every “test function” f , the equation (A, f) has a “nice solution” (see [4, Chapter 5, page 124]).

In this spirit, we set the expression of the mild solution of the equation (A, f) in discrete-time to give the following definition of admissibility in terms of “test sequences” and “nice discrete-time mild solutions”. In this way, we do not need any assumption of continuity or measurability and we still obtain continuous-time asymptotic properties for the autonomous differential equation $(A): x' = Ax$.

Definition 3.11. Let $\mathcal{X}, \mathcal{F}$ be sequence Schäffer spaces. The pair $(\mathcal{X}, \mathcal{F})$ is said to be *admissible* to $\{T(t)\}_{t \geq 0}$ if for each $f \in \mathcal{X}(X)$, there exists $x \in X$ such that $p_f(\cdot; x) \in \mathcal{F}(X)$, where

$$p_f(n; x) = T(n)x + \sum_{k=0}^n T(n-k)f(k), \quad (3.11)$$

for each $n \in \mathbb{N}$.

4. Main Results

In this section, for $\{T(t)\}_{t \geq 0}$ being a semigroup on the Banach space X (with M, ω assuring the exponential boundedness if it is the case for $\{T(t)\}_{t \geq 0}$ to be exponentially bounded) and $\mathcal{X}$ and $\mathcal{F}$ being two sequence Schäffer spaces, we denote

$$\begin{aligned} X_{1, \mathcal{F}} &= \{x \in X : (T(n)x)_{n \in \mathbb{N}} \in \mathcal{F}(X)\}, \\ X_1 &= \left\{x \in X : \lim_{t \rightarrow \infty} T(t)x = 0\right\}, \end{aligned} \quad (4.1)$$

which are obviously vector subspaces of X .

Hypothesis 1. The vector subspace $X_{1, \mathcal{F}}$ is closed and admits a closed complement; that is, there exists $X_{2, \mathcal{F}}$, a closed vector subspace, such that

$$X = X_{1, \mathcal{F}} \oplus X_{2, \mathcal{F}}. \quad (4.2)$$

We denote by P_1 the projection onto $X_{1,\mathcal{F}}$ along $X_{2,\mathcal{F}}$ and set $P_2 = I - P_1$ (we will prove that in the case of a no past exponential dichotomy for $\{T(t)\}_{t \geq 0}$, $X_{1,\mathcal{F}}$ coincides always with X_1 and thus the $\mathcal{F}$ -independent notation for projectors is consistent).

Remark 4.1. We have that $T(t)X_{1,\mathcal{F}} \subset X_{1,\mathcal{F}}$ and $T(t)x \neq 0$, for each $x \in X_{2,\mathcal{F}} \setminus \{0\}$ and $t \geq 0$.

Proof. If $x \in X_{1,\mathcal{F}}$ and $t \geq 0$, then it is to see that $\|T(n)T(t)x\| \leq \|T(t)\|\|T(n)x\|$ for all $n \in \mathbb{N}$, and since $(\|T(n)x\|)_{n \in \mathbb{N}} \in \mathcal{F}$, it follows that $T(t)x \in X_{1,\mathcal{F}}$.

For the second part, assume for a contradiction that there exist $t \geq 0$ and $x \in X_{2,\mathcal{F}} \setminus \{0\}$ such that $T(t)x = 0$. Then, $T(n)x = T(n-t)T(t)x = 0$ for every $n \in \mathbb{N}$, $n \geq t$ and thus $(T(n)x)_{n \in \mathbb{N}} \in \mathcal{F}(X)$. It follows that $x \in X_{1,\mathcal{F}}$, which is not possible since $x \in X_{2,\mathcal{F}} \setminus \{0\}$. Thus, $T(t)|_{X_{2,\mathcal{F}}}$ is one-to-one, for all $t \geq 0$. $\square$

Proposition 4.2. *If the pair $(\mathcal{X}, \mathcal{F})$ is admissible to the semigroup $\{T(t)\}_{t \geq 0}$, then for each $f \in \mathcal{X}(X)$, there exists unique $x_f \in X_{2,\mathcal{F}}$ such that $p_f(\cdot; x_f) \in \mathcal{F}(X)$.*

Proof. Let $f \in \mathcal{X}(X)$ and $x \in X$ from Definition 3.11. For $y = x - P_1x = P_2x$ we have that $y \in X_{2,\mathcal{F}}$ and $p_f(n; y) = p_f(n; x) - T(n)P_1x$. Since $p_f(\cdot; x) \in \mathcal{F}(X)$ and $(T(n)P_1x) \in \mathcal{F}(X)$, it follows that $p_f(\cdot; y) \in \mathcal{F}(X)$.

To prove the uniqueness of y , suppose that there exists $z \in X_{2,\mathcal{F}}$ with the property $p_f(\cdot; z) \in \mathcal{F}(X)$. Since $p_f(n; y) - p_f(n; z) = T(n)(y - z)$, we have that $y - z \in X_{1,\mathcal{F}} \cap X_{2,\mathcal{F}}$ and therefore $z = y$.

The unique vector $y \in X_{2,\mathcal{F}}$ will be denoted by x_f . $\square$

Proposition 4.3. *If the pair $(\mathcal{X}, \mathcal{F})$ is admissible to the semigroup $\{T(t)\}_{t \geq 0}$, there exists $K > 0$ such that*

$$\|x_f\| \leq K\|f\|_{\mathcal{X}(X)}, \quad \|p_f(\cdot; x_f)\|_{\mathcal{F}(X)} \leq K\|f\|_{\mathcal{X}(X)}, \quad (4.3)$$

for all $f \in \mathcal{X}(X)$.

Proof. We define the operator

$$\mathcal{U} : \mathcal{X}(X) \longrightarrow X_{2,\mathcal{F}} \oplus \mathcal{F}(X), \quad \mathcal{U}f = (x_f, p_f(\cdot; x_f)). \quad (4.4)$$

It is obvious that $\mathcal{U}$ is a linear operator. Now, we will show that it is also closed.

Let $(f_n)_{n \in \mathbb{N}} \subset \mathcal{X}(X)$ such that $\|f_n - f\|_{\mathcal{X}(X)} \xrightarrow{n \rightarrow \infty} 0$ and

$$\|\mathcal{U}f_n - (y, g)\|_{X_{2,\mathcal{F}} \oplus \mathcal{F}(X)} \xrightarrow{n \rightarrow \infty} 0, \quad (4.5)$$

where $f \in \mathcal{X}(X)$, $y \in X_{2,\mathcal{F}}$, and $g \in \mathcal{F}(X)$. For each $n \in \mathbb{N}$, we take $x_n = x_{f_n} \in X_{2,\mathcal{F}}$ and $u_n = p_{f_n}(\cdot; x_n) \in \mathcal{F}(X)$. We have that $\|x_n - y\| \xrightarrow{n \rightarrow \infty} 0$ and $\|u_n - g\|_{\mathcal{F}(X)} \xrightarrow{n \rightarrow \infty} 0$.

Since

$$\|u_n(m) - g(m)\| \leq \frac{1}{\beta_F(0)} \|u_n - g\|_{\mathcal{F}(X)}, \quad (4.6)$$

we have that $\lim_{n \rightarrow \infty} u_n(m) = g(k)$, for all $m \in \mathbb{N}$. On the other hand,

$$\begin{aligned} \|u_n(m) - p_f(m; y)\| &= \left\| T(m)(x_n - y) + \sum_{j=0}^m T(m-j)(f_n(j) - f(j)) \right\| \\ &\leq \|T(m)\| \|x_n - y\| + \sum_{j=0}^m \|T(m-j)\| \|f_n(j) - f(j)\|, \end{aligned} \quad (4.7)$$

which implies $\lim_{n \rightarrow \infty} u_n(m) = p_f(m; y)$, for all $m \in \mathbb{N}$. It follows that $p_f(\cdot; y) = g \in \mathcal{F}(X)$ and by Proposition 4.2 we have $y = x_f$. Therefore $\mathcal{U}f = (y; g)$.

Hence, $\mathcal{U}$ is a closed linear operator, and by the Closed-Graph Theorem it is also bounded which means that there exists $K > 0$ such that

$$\|x_f\| + \|p_f(\cdot; x_f)\|_{\mathcal{F}(X)} = \|\mathcal{U}f\|_{X_2 \oplus \mathcal{F}(X)} \leq K \|f\|_{\mathcal{E}(X)}, \quad (4.8)$$

and the proof is complete. $\square$

A simple and useful evaluation that results from Proposition 4.3 and Proposition 2.8 is given in the following remark.

Remark 4.4. If $(\mathcal{E}, \mathcal{F})$ is admissible to the semigroup $\{T(t)\}_{t \geq 0}$, then $(\mathcal{E}, \ell^\infty)$ is admissible to $\{T(t)\}_{t \geq 0}$ and $\|p_f(n; x_f)\| \leq (K/\beta_{\mathcal{F}}(0))\|f\|_{\mathcal{E}(X)}$, for all $f \in \mathcal{E}(X)$ and $n \in \mathbb{N}$.

With this intermediate result we are able to prove that the admissibility of $(\mathcal{E}, \mathcal{F})$ to an exponentially bounded semigroup $\{T(t)\}_{t \geq 0}$ is a sufficient condition for a no past exponential dichotomy of $\{T(t)\}_{t \geq 0}$. The restriction over such a pair $(\mathcal{E}, \mathcal{F})$ is that $\mathcal{E}$ and $\mathcal{F}$ are not simultaneously the bounds of the chain of sequence Schäffer spaces (in the sense of Proposition 2.8).

Theorem 4.5. Let $\mathcal{E}$ and $\mathcal{F}$ be sequence Schäffer spaces such that $\ell^1 \neq \mathcal{E}$ or $\ell_0^\infty \not\subset \mathcal{F}$ and let $\{T(t)\}_{t \geq 0}$ be an exponentially bounded semigroup. If $(\mathcal{E}, \mathcal{F})$ is admissible to $\{T(t)\}_{t \geq 0}$, then it has a no past exponential dichotomy and $X_{1, \mathcal{F}} = X_1$.

Proof (Part I). The exponential decay of $\{T(t)\}_{t \geq 0}$ on $X_{1, \mathcal{F}}$.

Let $x \in X_{1, \mathcal{F}}$ and consider the sequence

$$f : \mathbb{N} \longrightarrow X, \quad f(n) = \delta_0(n)x, \quad (4.9)$$

which is in $\mathcal{E}(X)$ with $\|f\|_{\mathcal{E}(X)} = \beta_{\mathcal{E}}(0)\|x\|$. Observe that $p_f(n; 0) = T(n)x$, for all $n \in \mathbb{N}$. Then, $p_f(\cdot; 0) \in \mathcal{F}(X)$ and from $\|p_f(n; 0)\| \leq K/(\beta_{\mathcal{F}}(0))\|f\|_{\mathcal{E}(X)}$ we obtain $\|T(n)x\| \leq K(\beta_{\mathcal{E}}(0)/\beta_{\mathcal{F}}(0))\|x\|$, for all $n \in \mathbb{N}$. For $t \geq 0$, taking $n = [t]$ we have

$$\|T(t)x\| \leq Me^{\omega(t-n)}\|T(n)x\| \leq Me^{\omega} K \frac{\beta_{\mathcal{E}}(0)}{\beta_{\mathcal{F}}(0)} \|x\|. \quad (4.10)$$

Since the constant $C_1 := Me^\omega K(\beta_\mathcal{E}(0)/\beta_F(0))$ does not depend on x , we can write down

$$\|T(t)x\| \leq C_1\|x\|, \quad \forall t \geq 0, x \in X_{1,\mathcal{F}}. \quad (4.11)$$

For $n, m \in \mathbb{N}$ and $x \in X_{1,\mathcal{F}}$, we evaluate

$$\begin{aligned} \|T(n)x\| \sum_{j=0}^n \delta_j(m) &= \sum_{j=0}^n \|T(n-j)T(j)x\| \delta_j(m) \leq C_1 \sum_{j=0}^n \|T(j)x\| \delta_j(m) \\ &\leq C_1 \sum_{j=0}^{\infty} \|T(j)x\| \delta_j(m) = C_1 \|p_f(m; 0)\|, \end{aligned} \quad (4.12)$$

which implies $\|T(n)x\| \sum_{j=0}^n \delta_j \leq C_1 \|p_f(\cdot; 0)\|$ and therefore

$$\|T(n)x\| \|\chi_{\{0,1,\dots,n\}}\|_{\mathcal{F}} \leq C_1 \|p_f(\cdot; 0)\|_{\mathcal{F}(X)}. \quad (4.13)$$

By Proposition 4.3 we get that

$$\|T(n)x\| \leq \frac{C_1 K \beta_\mathcal{E}(0)}{\beta_\mathcal{F}(n)} \|x\|, \quad \forall n \in \mathbb{N}, x \in X_{1,\mathcal{F}}. \quad (4.14)$$

If $\ell_0^\infty \not\subset \mathcal{F}$, then $\beta_\mathcal{F}$ is not bounded and therefore there exists $n_0 \in \mathbb{N}^*$ such that $\eta := C_1 K \beta_\mathcal{E}(0)/\beta_\mathcal{F}(n_0) < 1$ and $\|T(n_0)x\| \leq \eta \|x\|$, for all $x \in X_{1,\mathcal{F}}$.

If $\ell_0^\infty \subset \mathcal{F}$, then $\mathcal{E} \neq \ell^1$. From Proposition 2.8, it follows that there exists $h \in \mathcal{E} \setminus \ell^1$. Consider

$$\gamma : \mathbb{N} \longrightarrow \mathbb{R}, \quad \gamma(n) = \sum_{j=0}^n |h(j)|, \quad (4.15)$$

which is nondecreasing and $\lim_{n \rightarrow \infty} \gamma(n) = \infty$. For $n \in \mathbb{N}$ and $x \in X_{1,\mathcal{F}}$, the sequence

$$g : \mathbb{N} \longrightarrow X, \quad g(m) = \chi_{\{0,1,\dots,n\}}(m) |h(m)| T(m)x \quad (4.16)$$

has finite support, and thus $g \in \mathcal{E}(X)$. Observing that $\|g(m)\| \leq C_1 \|x\| |h(m)|$ for every $m \in \mathbb{N}$, we are led to the evaluation $\|g\|_{\mathcal{E}(X)} \leq L \|h\|_{\mathcal{E}} \|x\|$. Also,

$$\begin{aligned} p_g(m; 0) &= \sum_{j=0}^m T(m-j)g(j) = \sum_{j=0}^m T(m-j)T(j)x |h(j)| \chi_{\{0,1,\dots,n\}}(j) \\ &= \left(\sum_{j=0}^m \chi_{\{0,1,\dots,n\}}(j) |h(j)| \right) T(m)x, \end{aligned} \quad (4.17)$$

and thus we have $\|p_g(m; 0)\| \leq \gamma(n)\|T(m)x\|$, for all $m \in \mathbb{N}$. Since $x \in X_{1,\mathcal{F}}$, $(T(m)x)_{m \in \mathbb{N}} \in \mathcal{F}(X)$, we have that $p_g(\cdot; 0) \in \mathcal{F}(X)$. Taking $m = n$ in $\|p_g(m; 0)\| \leq (K/\beta_{\mathcal{F}}(0))\|g\|_{\mathcal{E}(X)}$, we can deduce that

$$\gamma(n)\|T(n)x\| \leq \frac{KC_1}{\beta_{\mathcal{F}}(0)}\|h\|_{\mathcal{E}}\|x\|. \quad (4.18)$$

It follows that there exists $n_0 \in \mathbb{N}^*$ such that $\eta := (KC_1\|h\|_{\mathcal{E}}/\beta_{\mathcal{F}}(0)\gamma(n_0)) < 1$ and $\|T(n_0)x\| \leq \eta\|x\|$, for all $x \in X_{1,\mathcal{F}}$.

In both cases, we obtained the existence of some $n_0 \in \mathbb{N}^*$ and some constant $\eta \in (0, 1)$ such that $\|T(n_0)x\| \leq \eta\|x\|$ for all $x \in X_{1,\mathcal{F}}$. Therefore, the semigroup $\{T(t)_{|X_{1,\mathcal{F}}}\}_{t \geq 0}$ is exponentially stable (see Remark 3.7). Subsequently, there exist $N_1, \nu_1 > 0$ such that

$$\|T(t)x\| \leq N_1 e^{-\nu_1 t} \|x\|, \quad \forall t \geq 0, x \in X_{1,\mathcal{F}}. \quad (4.19)$$

□

Proof (Part II). The exponential blow-up of $\{T(t)\}_{t \geq 0}$ on $X_{2,\mathcal{F}}$.

Let $n \in \mathbb{N}$, $n_0 \in \mathbb{N}^*$ and $x \in X_{2,\mathcal{F}} \setminus \{0\}$, and consider the sequence

$$f : \mathbb{N} \longrightarrow X, \quad f(m) = \delta_{n+n_0}(m) \frac{T(m)x}{\|T(n+n_0)x\|}. \quad (4.20)$$

We have that $f \in \mathcal{E}(X)$ with $\|f\|_{\mathcal{E}(X)} = \beta_{\mathcal{E}}(0)$. On the one hand,

$$\begin{aligned} - \sum_{j=m+1}^{\infty} \delta_{n+n_0}(j) \frac{T(m)x}{\|T(n+n_0)x\|} &= - \sum_{j=0}^{\infty} \delta_{n+n_0}(j) \frac{T(m)x}{\|T(n+n_0)x\|} + \sum_{j=0}^m \delta_{n+n_0}(k) \frac{T(m-j)T(j)x}{\|T(n+n_0)x\|} \\ &= T(m) \left(\frac{-x}{\|T(n+n_0)x\|} \right) + \sum_{j=0}^m T(m-j)f(j) = p_f(m; y), \end{aligned} \quad (4.21)$$

for all $m \in \mathbb{N}$ (where $y = -x/\|T(n+n_0)x\| \in X_{2,\mathcal{F}}$), while on the other hand

$$- \sum_{j=m+1}^{\infty} \delta_{n+n_0}(j) \frac{T(m)x}{\|T(n+n_0)x\|} = \begin{cases} - \frac{T(m)x}{\|T(n+n_0)x\|}, & m < n+n_0, \\ 0, & m \geq n+n_0, \end{cases} \quad (4.22)$$

which implies that $p_f(\cdot; y) \in \mathcal{F}(X)$. Then, $\|p_f(m; y)\| \leq K/(\beta_{\mathcal{F}}(0))\|f\|_{\mathcal{E}(X)}$ and

$$\|T(m)x\| \leq K \frac{\beta_{\mathcal{E}}(0)}{\beta_{\mathcal{F}}(0)} \|T(n+n_0)x\|, \quad \forall 0 \leq m < n+n_0. \quad (4.23)$$

Taking now $m = n = 0$ we can write down

$$\|T(n_0)x\| \geq K^{-1} \frac{\beta_{\mathcal{F}}(0)}{\beta_{\mathcal{E}}(0)} \|x\|, \quad \forall n_0 \in \mathbb{N}^*, x \in X_{2,\mathcal{F}}. \quad (4.24)$$

For $t \geq 0$, we put $n_0 = [t] + 1$ to evaluate $Me^{\omega} \|T(t)x\| \geq \|T(n_0)x\| \geq K^{-1}(\beta_{\mathcal{F}}(0)/\beta_{\mathcal{E}}(0))\|x\|$. Denoting $C_2 := (KMe^{\omega})^{-1}\beta_{\mathcal{F}}(0)/\beta_{\mathcal{E}}(0)$ we have that

$$\|T(t)x\| \geq C_2\|x\|, \quad \forall t \geq 0, \quad x \in X_{2,\mathcal{F}}. \quad (4.25)$$

Also, from (4.23), taking $C' := 1 + K(\beta_{\mathcal{E}}(0)/\beta_{\mathcal{F}}(0))$ we have that

$$C'\|T(h)x\| \geq \|T(n)x\|, \quad \forall h \geq n, \quad x \in X_{2,\mathcal{F}}. \quad (4.26)$$

Since n, n_0 , and x are randomly taken (see the beginning of *Part II*), we have that

$$\begin{aligned} \frac{\|T(n)x\|}{\|T(n+n_0)x\|} \sum_{j=n}^{n+n_0-1} \delta_j(m) &\leq C' \sum_{j=n}^{n+n_0-1} \frac{\|T(j)x\|}{\|T(n+n_0)x\|} \delta_j(m) \\ &= C' \sum_{j=n}^{n+n_0-1} \|p_f(j; y)\| \delta_j(m) \\ &\leq C' \|p_f(m; y)\|, \quad \forall m \in \mathbb{N}, \end{aligned} \quad (4.27)$$

and therefore $\|T(n)x\|/\|T(n+n_0)x\| \chi_{\{n, n+1, \dots, n+n_0-1\}} \leq C' \|p_f(\cdot; y)\|$. It follows that

$$\frac{\|T(n)x\|}{\|T(n+n_0)x\|} \|R^n \chi_{\{0,1,\dots,n_0-1\}}\|_{\mathcal{F}} \leq C' \|p_f(\cdot; y)\|_{\mathcal{F}}, \quad (4.28)$$

or equivalently,

$$\frac{\|T(n)x\|}{\|T(n+n_0)x\|} \beta_{\mathcal{F}}(n_0-1) \leq C' \|p_f(\cdot; y)\|_{\mathcal{F}(X)}. \quad (4.29)$$

Using the fact that $\|p_f(\cdot; y)\|_{\mathcal{F}(X)} \leq K\|f\|_{\mathcal{E}(X)}$, we deduce that

$$\|T(n+n_0)x\| \geq \frac{\beta_{\mathcal{F}}(n_0-1)}{C'K\beta_{\mathcal{E}}(0)} \|T(n)x\|, \quad \forall n \in \mathbb{N}, \quad n_0 \in \mathbb{N}^*, \quad x \in X_{2,\mathcal{F}}. \quad (4.30)$$

If $\ell_0^\infty \not\subset \mathcal{F}$, then $\beta_{\mathcal{F}}$ is not bounded and therefore there exists $n_0 \in \mathbb{N}^*$ such that $\eta := (\beta_{\mathcal{F}}(n_0-1))/(C'K\beta_{\mathcal{E}}(0)) > 1$ and $\|T(n+n_0)x\| \geq \eta\|T(n)x\|$ for all $n \in \mathbb{N}$ and $x \in X_{2,\mathcal{F}}$.

If $\ell_0^\infty \subset \mathcal{F}$, then $\mathcal{E} \neq \ell^1$ and therefore there exists $h \in \mathcal{E} \setminus \ell^1$. Consider γ as in (4.15), and for $n, n_0 \in \mathbb{N}$ and $x \in X_{2,\mathcal{F}} \setminus \{0\}$ we define

$$g : \mathbb{N} \longrightarrow X, \quad g(m) = \sum_{j=0}^{n_0} \delta_{n+j}(m) |h(j)| \frac{T(n+j)x}{\|T(n+n_0)x\|}. \quad (4.31)$$

Since $\|g(m)\| \leq C' \chi_{\{n, n+1, \dots, n+n_0\}}(m) |(R^n h)(m)| \leq C' |(R^n h)(m)|$ for each $m \in \mathbb{N}$, we have that $g \in \mathcal{X}(X)$ with $\|g\|_{\mathcal{X}(X)} \leq C' \|R^n h\|_{\mathcal{X}} = C' \|h\|_{\mathcal{X}}$. Note that $g(m) = \chi_{\{n, n+1, \dots, n+n_0\}}(m) |R^n h(m)| T(m)x / \|T(n+n_0)x\|$, for all $m \in \mathbb{N}$. On the one hand,

$$\begin{aligned}
& - \sum_{j=m+1}^{\infty} \chi_{\{n, \dots, n+n_0\}}(j) |R^n h(j)| \frac{T(m)x}{\|T(n+n_0)x\|} \\
& = - \sum_{j=0}^{\infty} \chi_{\{n, \dots, n+n_0\}}(j) |R^n h(j)| \frac{T(m)x}{\|T(n+n_0)x\|} \\
& \quad + \sum_{j=0}^m T(m-j) \chi_{\{n, \dots, n+n_0\}}(j) |R^n h(j)| \frac{T(j)x}{\|T(n+n_0)x\|} \\
& = T(m) \left(\gamma(n_0) \frac{-x}{\|T(n+n_0)x\|} \right) + \sum_{j=0}^m T(m-j) g(j) \\
& = p_g(m; z),
\end{aligned} \tag{4.32}$$

for all $m \in \mathbb{N}$ (where $z := \gamma(n_0)(-x/\|T(n+n_0)x\|) \in X_{2, \mathcal{F}}$), while on the other hand

$$\begin{aligned}
& - \sum_{j=m+1}^{\infty} \chi_{\{n, \dots, n+n_0\}}(j) |R^n h(j)| \frac{T(m)x}{\|T(n+n_0)x\|} \\
& = \begin{cases} -\gamma(n_0) \frac{T(m)x}{\|T(n+n_0)x\|}, & m < n, \\ -(\gamma(n_0) - \gamma(m-n)) \frac{T(m)x}{\|T(n+n_0)x\|}, & n \leq m < n+n_0, \\ 0, & m \geq n+n_0. \end{cases}
\end{aligned} \tag{4.33}$$

From (4.32) and (4.33), it follows that $p_g(\cdot; z)$ has finite support, and thus it belongs to $\mathcal{F}(X)$. Therefore, $\|p_g(n; z)\| \leq (K/\beta_{\mathcal{F}}(0)) \|g\|_{\mathcal{X}(X)}$ and using that $\|g\|_{\mathcal{X}(X)} \leq C' \|h\|_{\mathcal{X}}$ we obtain that

$$(\gamma(n_0) - |h(0)|) \frac{\|T(n)x\|}{\|T(n+n_0)x\|} \leq \frac{KC'}{\beta_{\mathcal{F}}(0)} \|h\|_{\mathcal{X}}. \tag{4.34}$$

Thus, $\|T(n+n_0)x\| \geq (\beta_{\mathcal{F}}(0)/KC') \|h\|_{\mathcal{X}} (\gamma(n_0) - |h(0)|) \|T(n)x\|$, for all $n, n_0 \in \mathbb{N}$ and $x \in X_{2, \mathcal{F}}$. Then, there exist $n_0 \in \mathbb{N}^*$ and $\eta > 1$ such that

$$\|T(n+n_0)x\| \geq \eta \|T(n)x\|, \quad \forall n \in \mathbb{N}, x \in X_{2, \mathcal{F}}. \tag{4.35}$$

In both cases, we obtained the existence of some $n_0 \in \mathbb{N}^*$ and some constant $\eta > 1$ satisfying the condition (ii) from Lemma 3.9, while the first condition is assured by (4.25). Subsequently, there exist $N_2, \nu_2 > 0$ such that

$$\|T(t)x\| \geq N_2 e^{\nu_2 t} \|x\|, \quad \forall t \geq 0, x \in X_{2, \mathcal{F}}. \tag{4.36}$$

If we take $\nu := \min\{\nu_1, \nu_2\} > 0$, all the conditions guaranteeing the existence of a no past exponential dichotomy are met. $\square$

Proof (Part III). We prove that $X_{1,\mathcal{F}} = X_1$, no matter how we choose the sequence Schäffer space $\mathcal{F}$. If $x \in X_{1,\mathcal{F}}$, then $\|T(t)x\| \leq N_1 e^{-\nu_1 t} \|x\|$, for all $t \geq 0$, which implies $x \in X_1$.

Conversely, let $x \in X_1$, $u \in X_{1,\mathcal{F}}$, and $v \in X_{2,\mathcal{F}}$ such that $x = u + v$. For every $n \in \mathbb{N}$, we have that

$$\begin{aligned} \|T(n)x\| &= \|T(n)v - (-T(n)u)\| \geq \|T(n)v\| - \|T(n)u\| \\ &\geq N_2 e^{\nu_2 n} \|v\| - N_1 e^{-\nu_1 n} \|u\|. \end{aligned} \quad (4.37)$$

If we suppose that $v \neq 0$, then $\|T(n)x\| \xrightarrow{n \rightarrow \infty} \infty$ contradicting the fact that $x \in X_1$. It follows immediately that $v = 0$ and $x = u \in X_{1,\mathcal{F}}$. $\square$

Remark 4.6. As we pointed out in the introduction, there is an extensive literature on the connection between *admissibility* and *hyperbolicity* (or equivalently, exponential dichotomy). Latest there is known the equivalence between the admissibility of the pair (ℓ^p, ℓ^q) $1 \leq p \leq q \leq \infty$ and $(p, q) \neq (1, \infty)$ and the hyperbolicity of a C_0 -semigroup $\{T(t)\}_{t \geq 0}$, when we assume *a priori* that the kernel of the splitting projection is $T(t)$ -invariant and $T(t)|_{\text{Ker } P}$ are invertible. For details we refer the reader to [21]. We try to extend this line of results in two directions. First, we do not assume *a priori* that $T(t)|_{\text{Ker } P}$ is invertible (we do not even assume that $\text{Ker } P$ is $T(t)$ -invariant) and still we succeed to prove that the admissibility of any pair of sequence Schäffer spaces implies the existence of a *no past exponential dichotomy*. Secondly, it is worth to note that the class of sequence Schäffer spaces is extremely reachable (see, e.g., Examples 2.3 and 2.4) and this fact allows the reader to choose the “test sequences” in various ways and in the same time it does not force the “output” or “nice discrete-time mild solutions” (i.e., the solution of the inhomogeneous difference equation problem) to stay in $\ell^q(X)$, as before. Moreover, this approach can provide interesting *input spaces* (i.e., the spaces consisting in “test sequences”) which are different from the classical ℓ^p spaces (we refer the reader to Example 2.6). Also, it is worth to note that if there exists a pair of vector-valued sequence Schäffer spaces $(\mathcal{E}, \mathcal{F})$, which is admissible to $\{T(t)\}_{t \geq 0}$, and with the property that $\ell^1 \neq \mathcal{E}$ or $\ell_0^\infty \not\subset \mathcal{F}$, then the subspace $X_{1,\mathcal{F}}$ (which induces the no past exponential dichotomy) is actually the regular stable subspace X_1 . If we would impose in addition that the complement of X_1 (denoted by X_2) is also $T(t)$ -invariant, then the aforementioned admissibility condition would imply that $\{T(t)\}_{t \geq 0}$ extends automatically to a C_0 -group on X_2 , and thus we would get hyperbolicity for $\{T(t)\}_{t \geq 0}$ (see Theorem 4.12 below). Therefore, we can conclude that “admissibility” converts to “no past exponential dichotomy” “admissibility” and “ $T(t)$ -invariance of X_2 ” converts to “hyperbolicity”.

The next example shows that the condition “ $E \neq \ell^1$ or $\ell_0^\infty \not\subset \mathcal{F}$ ” in the statement of Theorem 4.5 is essential.

Example 4.7. Let $X = \mathbb{R}$ and consider the semigroup $\{T(t)\}_{t \geq 0}$ by $T(t) = I_{\mathbb{R}}$ for all $t \geq 0$. If $f \in \ell^1$, then there exists $x = -\sum_{k=0}^\infty f(k) \in \mathbb{R}$ (the series being absolutely convergent) such that $p_f(\cdot; x) \in \ell_0^\infty(X)$. Therefore, the pair (ℓ^1, ℓ_0^∞) is admissible to $\{T(t)\}_{t \geq 0}$, but one can easily check that $\{T(t)\}_{t \geq 0}$ does not possess a no past exponential dichotomy.

The above theorem shows that the space $X_{1,\mathcal{F}}$ induces a no past exponential dichotomy for $\{T(t)\}_{t \geq 0}$ —an exponentially bounded semigroup—and that $X_{1,\mathcal{F}}$ is actually X_1 . Concerning the hypothesis over $\mathcal{E}$ and $\mathcal{F}$ from the Theorem 4.5 we make one last remark.

Remark 4.8. In [8] there was introduced the order relation between pairs of Banach spaces: (B_1, D_1) is said to be *stronger than* (B, D) (or the later pair is *weaker than* the former) if $B \hookrightarrow B_1$ and $D_1 \hookrightarrow D$. In the case of sequence Schäffer spaces the two inclusions with continuous injection are equivalent with the corresponding algebraic inclusion (Proposition 2.9); so the pair (ℓ^1, ℓ_0^∞) is stronger than $(\mathcal{E}, \mathcal{F})$ if $\mathcal{E} \subset \ell^1$ and $\ell_0^\infty \subset \mathcal{F}$, or equivalently (considering Proposition 2.8), $\mathcal{E} = \ell^1$ and $\ell_0^\infty \subset \mathcal{F}$. Therefore, the hypothesis “ $\mathcal{E} \neq \ell^1$ or $\ell_0^\infty \not\subset \mathcal{F}$ ” can be replaced by, “the pair (ℓ^1, ℓ_0^∞) is not stronger than the pair $(\mathcal{E}, \mathcal{F})$ ”.

Dropping off the restriction “ $\mathcal{E} \neq \ell^1$ or $\ell_0^\infty \not\subset \mathcal{F}$ ”, the proof of Theorem 4.5 still provides useful information. More accurately, the admissibility of any pair of sequence Schäffer spaces implies the existence of an ordinary dichotomy for the semigroup $\{T(t)|_{X_{1,\mathcal{F}}}\}_{t \geq 0}$ (in the sense of Definition 3.6).

Corollary 4.9. *Let $(\mathcal{E}, \mathcal{F})$ be a pair sequence Schäffer space and, $\{T(t)\}_{t \geq 0}$ be an exponentially bounded semigroup. If the pair $(\mathcal{E}, \mathcal{F})$ is admissible to $\{T(t)\}_{t \geq 0}$, then $\{T(t)\}_{t \geq 0}$ has an ordinary dichotomy; that is, there exist two constants $C_1, C_2 > 0$ such that*

$$\begin{aligned} \|T(t)x\| &\leq C_1 \|x\|, \quad \forall t \geq 0, x \in X_{1,\mathcal{F}}, \\ \|T(t)x\| &\geq C_2 \|x\|, \quad \forall t \geq 0, x \in X_{2,\mathcal{F}}. \end{aligned} \tag{4.38}$$

Proof. Note that in the proof of Theorem 4.5, we do not use the hypothesis “ $\mathcal{E} \neq \ell^1$ or $\ell_0^\infty \not\subset \mathcal{F}$ ” to prove the relations (4.11) and (4.25). $\square$

Corollary 4.10. *Let $p, q \in [1, \infty]$ such that $(p, q) \neq (1, \infty)$ and let $\{T(t)\}_{t \geq 0}$ be an exponentially bounded semigroup. If the pair (ℓ^p, ℓ^q) is admissible to $\{T(t)\}_{t \geq 0}$, then $\{T(t)\}_{t \geq 0}$ has a no past exponential dichotomy.*

Proof. We have the equivalence (due to Proposition 2.10 and the fact that $\beta_{\ell^q}(n) = (n+1)^{1/q}$ for each $n \in \mathbb{N}$): $\ell_0^\infty \subset \ell^q$ if and only if $q = \infty$. Thus, given $(p, q) \neq (1, \infty)$ the hypothesis of Theorem 4.5 is satisfied and the conclusion is immediate. $\square$

Corollary 4.11. *Let ℓ^Φ be a sequence Orlicz space and let $\{T(t)\}_{t \geq 0}$ be an exponentially bounded semigroup. If the pair (ℓ^Φ, ℓ^Φ) is admissible to $\{T(t)\}_{t \geq 0}$, then $\{T(t)\}_{t \geq 0}$ has a no past exponential dichotomy.*

Proof. It follows from Theorem 4.5 and Example 2.4.

With Theorem 4.12 we prove that if we impose the $T(t)$ -invariance of X_2 , we can deduce the invertibility of $T(t)|_{X_2}$, thus obtaining the exponential dichotomy (or equivalently, hyperbolicity) for $\{T(t)\}_{t \geq 0}$. Note that the first condition of the following theorem does not require that X_2 is invariant to all operators $T(t)$. However, we can prove that $\{T(t)|_{X_2}\}_{t \geq 0}$ is a semigroup (exponentially bounded if $\{T(t)\}_{t \geq 0}$ is exponentially bounded); hence the invertibility of all its operators follows from the invertibility of just one of them. $\square$

Theorem 4.12. Let $\mathcal{E}$ and $\mathcal{F}$ be two sequence Schäffer spaces such that $\ell^1 \neq \mathcal{E}$ or $\ell_0^\infty \not\subset \mathcal{F}$ and let $\{T(t)\}_{t \geq 0}$ be an exponentially bounded semigroup. If

- (i) there exists $t_0 > 0$ such that $P_2 T(t) P_2 = T(t) P_2$ for all $t \in [0, t_0]$,
- (ii) $(\mathcal{E}, \mathcal{F})$ is admissible to T ,

then $T(t) : X_2 \rightarrow X_2$ is invertible for each $t \geq 0$. Therefore, $\{T(t)\}_{t \geq 0}$ has an exponential dichotomy.

Proof. Let $t > t_0$ and consider $n \in \mathbb{N}$ and $\delta \in [0, t_0]$ such that $t = nt_0 + \delta$. If $x \in X_2$, we have that $T(t)x = T(\delta)T(t_0)^n x \in X_2$. Therefore, the condition (i) assures that the operator $T(t) : X_2 \rightarrow X_2$ is well defined (with respect to the range) for each $t \geq 0$ and from Theorem 4.5 we have that $\{T(t)\}_{t \geq 0}$ has a no past exponential dichotomy.

Let $t \geq 0$, $n_0 = [t] + 1$, $y \in X_2$ and consider the sequence

$$f : \mathbb{N} \rightarrow X, \quad f(n) = -\delta_{n_0}(n)T(n_0 - t)y, \quad (4.39)$$

which belongs to $\mathcal{E}(X)$ with $\|f\|_{\mathcal{E}(X)} = \beta_{\mathcal{E}}(0)\|T(n_0 - t)y\|$. Then, according to Proposition 4.2, there exists a unique $x \in X_2$ such that $p_f(\cdot; x) \in \mathcal{F}(X)$. Since

$$\begin{aligned} p_f(n; x) &= T(n)x - \sum_{k=0}^n \delta_{n_0}(k)T(n-k)T(n_0 - t)y \\ &= T(n)x - T(n - n_0)T(n_0 - t)y = T(n - t)T(t)x - T(n - t)y \\ &= T(n - t)(T(t)x - y), \end{aligned} \quad (4.40)$$

for all $n \in \mathbb{N}$, $n \geq n_0$, we have that $T(t)x, y \in X_2$ and

$$\|p_f(n; x)\| = \|T(n - t)(T(t)x - y)\| \geq N_2 e^{\gamma_2(n-t)} \|T(t)x - y\|, \quad \forall n \geq n_0. \quad (4.41)$$

If we assume that $T(t)x - y \neq 0$, then $\lim_{n \rightarrow \infty} \|p_f(n; x)\| = \infty$ that contradicts the fact that $p_f(\cdot; x) \in \ell^\infty(X)$. It follows that $T(t)x = y$. We proved that $T(t)|_{X_2}$ is onto, and since the one-to-one property was already proved in the Remark 4.1, we get the invertibility.

Since only the property of invertibility of operators on X_2 restricts the no past exponential dichotomy to be an exponential dichotomy, we completed the proof.

In what follows we try to answer concerns regarding the converse of what was obtained with Theorem 4.12. It will be clear (see Example 4.17) that for a semigroup $\{T(t)\}_{t \geq 0}$ that has exponential dichotomy, not every pair $(\mathcal{E}, \mathcal{F})$ of sequence Schäffer spaces is admissible to $\{T(t)\}_{t \geq 0}$. $\square$

Remark 4.13. If $\mathcal{E}$ and $\mathcal{F}$ are two sequence Schäffer spaces such that $L\mathcal{E} \subset \mathcal{F}$, then $\mathcal{E} \subset \mathcal{F}$ and $\|Lf\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}}$ for all $f \in \mathcal{E}$.

Proof. Let $f \in \mathcal{E}$. Then, $Rf \in \mathcal{E}$ and therefore $f = L(Rf) \in \mathcal{F}$ and $Lf \in \mathcal{F}$. Since, $|RLf(m)| \leq |f(m)|$ for each $m \in \mathbb{N}$, we deduce that

$$\|Lf\|_{\mathcal{F}} = \|RLf\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}}. \quad (4.42) \quad \square$$

Theorem 4.14. *Let $\mathcal{E}$ and $\mathcal{F}$ be two sequence Schffer spaces such that $L(\mathcal{E} \cup \mathcal{F}) \subset \mathcal{F}$. If the semigroup $\{T(t)\}_{t \geq 0}$ has an exponential dichotomy, then the pair $(\mathcal{E}, \mathcal{F})$ is admissible to $\{T(t)\}_{t \geq 0}$.*

Proof. We have that there exist $N, \nu > 0$ such that

$$\|T(t)P_1x\| \leq Ne^{-\nu t}\|P_1x\|, \quad \left\|T(t)|_{\text{Im } P_2}^{-1}P_2x\right\| \leq Ne^{-\nu t}\|P_2x\|, \quad (4.43)$$

for all $t \geq 0$ and $x \in X$.

Let $f \in \mathcal{E}(X)$ and consider the sequence

$$\varphi : \mathbb{N} \longrightarrow X, \quad \varphi(n) = \sum_{k=0}^n T(n-k)P_1f(k) - \sum_{k=n+1}^{\infty} T(k-n)^{-1}P_2f(k), \quad (4.44)$$

which is well defined, since $f \in \ell^\infty(X)$ and

$$\begin{aligned} \sum_{k=n+1}^{\infty} \|T(k-n)^{-1}P_2f(k)\| &\leq N\|P_2\| \sum_{k=n+1}^{\infty} e^{-\nu(k-n)}\|f(k)\| \\ &\leq \frac{N\|P_2\| \|f\|_\infty}{1 - e^{-\nu}}, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (4.45)$$

For any $n \in \mathbb{N}$, we have the following evaluation:

$$\begin{aligned} \|\varphi(n)\| &\leq N\|P_1\| \sum_{k=0}^n e^{-\nu(n-k)}\|f(k)\| + N\|P_2\| \sum_{k=n+1}^{\infty} e^{-\nu(k-n)}\|f(k)\|, \\ &\leq N(\|P_1\| + \|P_2\|) \left(\sum_{k=0}^n e^{-\nu k} \|R^k f\|(n) + \sum_{k=1}^{\infty} e^{-\nu k} \|L^k f\|(n) \right). \end{aligned} \quad (4.46)$$

From the hypothesis and Remark 4.13 we have that $f, R^k f, L^k f \in \mathcal{F}(X)$ with $\|L^k f\|_{\mathcal{F}(X)} \leq \|f\|_{\mathcal{F}(X)}$ for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned} \sum_{k=1}^{\infty} \|e^{-\nu k} \|L^k f\|\|_{\mathcal{F}} &\leq \sum_{k=1}^{\infty} e^{-\nu k} \|f\|_{\mathcal{F}(X)} = \frac{e^{-\nu}}{1 - e^{-\nu}} \|f\|_{\mathcal{F}(X)}, \\ \sum_{k=0}^{\infty} \|e^{-\nu k} \|R^k f\|\|_{\mathcal{F}} &= \sum_{k=0}^{\infty} e^{-\nu k} \|f\|_{\mathcal{F}(X)} = \frac{1}{1 - e^{-\nu}} \|f\|_{\mathcal{F}(X)}, \end{aligned} \quad (4.47)$$

and thus $g := \sum_{k=0}^n e^{-\nu k} \|R^k f\| + \sum_{k=1}^{\infty} e^{-\nu k} \|L^k f\|$ exists as an element in $\mathcal{F}$.

From (4.46), denoting $C := N(\|P_1\| + \|P_2\|)$ we have that $\|\varphi(n)\| \leq Cg(n)$ for all $n \in \mathbb{N}$, and therefore, $\varphi \in \mathcal{F}(X)$. Note that for $k \geq n$, since $T(k)x = T(k-n)T(n)x$, we have that $T(k-n)^{-1}x = T(n)T(k)^{-1}x$ for all $x \in X_2$.

Then,

$$\begin{aligned}
 \varphi(n) &= \sum_{k=0}^n T(n-k)P_1f(k) - \sum_{k=0}^{\infty} T(n)T(k)^{-1}P_2f(k) \\
 &\quad + \sum_{k=0}^n T(n-k)T(k)T(k)^{-1}P_2f(k) \\
 &= T(n) \left(-\sum_{k=0}^{\infty} T(k)^{-1}P_2f(k) \right) + \sum_{k=0}^n T(n-k)(P_1f(k) + P_2f(k)) \\
 &= T(n)x + \sum_{k=0}^n T(n-k)f(k) = p_f(n; x), \quad \forall n \in \mathbb{N},
 \end{aligned} \tag{4.48}$$

where $x = -\sum_{k=0}^{\infty} T(k)^{-1}P_2f(k) \in X_2$ (the series being absolutely convergent). It follows that there exists $x \in X$ such that $p_f(\cdot; x) = \varphi \in \mathcal{F}(X)$. $\square$

Remark 4.15. Let us examine more closely the condition $L(\mathcal{E} \cup \mathcal{F}) \subset \mathcal{F}$ and the proof of the above theorem. The condition is in fact equivalent with $L\mathcal{F} \subset \mathcal{F}$ and $\mathcal{E} \subset \mathcal{F}$ (see Remark 4.13). If we keep the condition $\mathcal{E} \subset \mathcal{F}$ and prove (in some other setting) that $\varphi \in \mathcal{E}$, the argument presented previously still works. Such a new setting is given by $L\mathcal{E} \subset \mathcal{E}$ (the space $\mathcal{E}$ is invariant under the left shift), since the series defining the element g will be absolutely convergent in $\mathcal{E}$.

In the following result we put all the pieces together to provide a necessary and sufficient condition for the exponential dichotomy of an exponentially bounded semigroup.

Corollary 4.16. *Let $\{T(t)\}_{t \geq 0}$ be an exponentially bounded semigroup and let $\mathcal{E}$ and $\mathcal{F}$ be two sequence Schaffer spaces with the following properties:*

- (i) *there exists $t_0 > 0$ such that $P_2T(t)P_2 = T(t)P_2$ for all $t \in [0, t_0]$;*
- (ii) $\mathcal{E} \subset \mathcal{F}$;
- (iii) $L\mathcal{E} \subset \mathcal{E}$ or $L\mathcal{F} \subset \mathcal{F}$;
- (iv) $\ell^1 \neq \mathcal{E}$ or $\ell_0^\infty \not\subset \mathcal{F}$.

Then, $\{T(t)\}_{t \geq 0}$ has an exponential dichotomy if and only if the pair $(\mathcal{E}, \mathcal{F})$ is admissible to $\{T(t)\}_{t \geq 0}$.

Proof. The *necessity* follows from Theorem 4.14 and Remark 4.15, while the *sufficiency* is proved by Theorems 4.5 and 4.12. $\square$

Example 4.17. Consider the semigroup acting on $(\mathbb{R}, |\cdot|)$ defined by $T(t)x = e^{-t}x$. Clearly, $\{T(t)\}_{t \geq 0}$ defines a C_0 -semigroup with $\|T(t)x\| = e^{-t}|x|$ for all $t \geq 0$ and $x \in X$. Therefore, $\{T(t)\}_{t \geq 0}$ is exponentially stable (thus it is exponentially dichotomic).

Consider the sequence $f : \mathbb{N} \rightarrow \mathbb{R}$ $f(n) = 1/(n+1)$. We have that $f \in \ell^2 \setminus \ell^1$, and for any $x \in \mathbb{R}$,

$$p_f(n; x) = T(n)x + \sum_{k=0}^n T(n-k)f(k) = e^{-n}x + \sum_{k=0}^n e^{-(n-k)} \frac{1}{k+1}. \tag{4.49}$$

It follows that $|p_f(n; x)| \geq 1/(n+1) - e^{-n}|x|$, for all $n \in \mathbb{N}$. Then,

$$\sum_{n=0}^{\infty} |p_f(n; x)| \geq \sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{|x|}{1-e^{-1}}. \quad (4.50)$$

We obtained that $p_f(\cdot; x) \notin \ell^1$ for any $x \in X$. Thus, the pair (ℓ^2, ℓ^1) is not admissible to $\{T(t)\}_{t \geq 0}$.

This example shows that the condition $\mathcal{E} \subset \mathcal{F}$ in Theorem 4.14 cannot be dropped. If the semigroup is exponentially stable, then $X_2 = \{0\}$, and therefore, the invariance condition of either $\mathcal{E}$ or $\mathcal{F}$ under the left shift is not necessary.

Corollary 4.18. *Let $\mathcal{E}$ and $\mathcal{F}$ be two sequence Schäffer spaces such that $\mathcal{E} \subset \mathcal{F}$. If $\{T(t)\}_{t \geq 0}$ is an exponentially stable semigroup, then the pair $(\mathcal{E}, \mathcal{F})$ is admissible to $\{T(t)\}_{t \geq 0}$.*

Proof. Exponential stability is an exponential dichotomy with $P_1 = I$. With this remark, observe that the expression of φ in (4.44) reduces to

$$\varphi(n) = \sum_{k=0}^n T(n-k)f(k), \quad (4.51)$$

and, respectively, the evaluation from (4.46) becomes

$$\|\varphi(n)\| \leq N \sum_{k=0}^n e^{-\nu k} \|R^k f\| \leq N \sum_{k=0}^{\infty} e^{-\nu k} \|R^k f\| \quad (4.52)$$

The sequence $g := \sum_{k=0}^{\infty} e^{-\nu k} \|R^k f\|$ exists in $\mathcal{E}$ and $\|\varphi(n)\| \leq g(n)$ for all $n \in \mathbb{N}$. Therefore, $\varphi \in \mathcal{E}$, and since $\varphi = p_f(\cdot; 0)$, $\mathcal{E} \subset \mathcal{F}$, we deduce that there exists $x \in X$ such that $p_f(\cdot; 0) \in \mathcal{F}(X)$. $\square$

Corollary 4.19. *Let $p, q \in [1, \infty]$ such that $p \leq q$. If the semigroup $\{T(t)\}_{t \geq 0}$ is exponentially dichotomic, then the pair (ℓ^p, ℓ^q) is admissible to $\{T(t)\}_{t \geq 0}$.*

Proof. The spaces ℓ^p ($1 \leq p \leq \infty$) are invariant under the left shift, and with $p \leq q$ we also have that $\ell^p \subset \ell^q$. Applying now Theorem 4.14 we obtain the previously statement. $\square$

Corollary 4.20. *Let ℓ^Φ be a sequence Orlicz space. The semigroup $\{T(t)\}_{t \geq 0}$ is exponentially dichotomic if and only if the pair (ℓ^Φ, ℓ^Φ) is admissible to $\{T(t)\}_{t \geq 0}$.*

Proof. It follows immediately from Theorem 4.14 and Corollary 4.11, since any ℓ^Φ is invariant under the left shift. $\square$

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