Research Article

Boundedness of Commutators of High-Dimensional Hausdorff Operators

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We prove the boundedness of commutators of high-dimensional Hausdorff operators $H_{\Phi,b}$ on Lebesgue space with central BMO function or Lipschitz function. Furthermore, the boundedness on Herz spaces and Morrey-Herz spaces is also obtained.

1. Introduction

The one-dimensional Hausdorff operator is defined as

$$h_{\Phi}f(x) = \int_0^{\infty} \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) dt, \quad (1.1)$$

where Φ is a locally integrable function on $(0, \infty)$. The operator h_{Φ} has a deep root in the study of the one-dimensional Fourier analysis. Particularly, it is closely related to the summability of the classical Fourier series. The reader can see ([1–3]) to find details of background and recent development of the Hausdorff operator.

Recently, Hausdorff operator in the high dimensional space $\mathbb{R}^n$ was studied. Three extensions of the one-dimensional Hausdorff operator in $\mathbb{R}^n$ were recently introduced and studied in [4]. One of them is the operator

$$H_{\Phi}f(x) = \int_{\mathbb{R}^n} \frac{\Phi(|y|^{-1}x)}{|y|^n} f(y) dy, \quad (1.2)$$

where Φ is a radial function defined on $\mathbb{R}^+$. Replacing Φ with

$$\Phi_1(t) = t^{-n} \chi_{(1,\infty)}(t), \quad \Phi_2(t) = \chi_{(0,1)}(t), \quad (1.3)$$

$H_\Phi f$ becomes the high dimensional Hardy operator

$$Hf(x) = \frac{1}{|x|^n} \int_{|y| < |x|} f(y) dy, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad (1.4)$$

and its adjoint operator

$$H^* f(x) = \int_{|y| \geq |x|} \frac{f(y)}{|y|^n} dy, \quad (1.5)$$

respectively. It is well known that Hardy operators are important operators in Harmonic analysis and a quite number of papers have appeared in [5–9]. And in [10], the authors have obtained the boundedness of commutators of one-dimensional Hausdorff operator with one-sided dyadic CMO functions on the Lebesgue space.

These observations motivate us to study the boundedness of commutators of n -dimensional Hausdorff operators on some function spaces. Let $b(x)$ be a real measurable, locally integrable function; we define the commutators of Hausdorff operators as follows:

$$\begin{aligned} H_{\Phi,b} f(x) &= b(x) H_\Phi f(x) - H_\Phi(bf)(x) = \int_{\mathbb{R}^n} \frac{\Phi(|y|^{-1}x)}{|y|^n} (b(x) - b(y)) f(y) dy \\ &= \int_{|y| < |x|} \frac{\Phi(|y|^{-1}x)}{|y|^n} (b(x) - b(y)) f(y) dy \\ &\quad + \int_{|y| \geq |x|} \frac{\Phi(|y|^{-1}x)}{|y|^n} (b(x) - b(y)) f(y) dy \\ &:= H_{\Phi,b}^1 f(x) + H_{\Phi,b}^2 f(x). \end{aligned} \quad (1.6)$$

Definition 1.1 (see [11]). Let $1 < q < \infty$. A function $f \in L_{\text{loc}}^q(\mathbb{R}^n)$ is said to belong to the space $\text{CMO}^q(\mathbb{R}^n)$ if

$$\|f\|_{\text{CMO}^q(\mathbb{R}^n)} = \sup_{r>0} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_B|^q dx \right)^{1/q} < \infty, \quad (1.7)$$

where $f_B = (1/|B(0,r)|) \int_{B(0,r)} f(x) dx$. Obviously, we have $\text{BMO}(\mathbb{R}^n) \subset \text{CMO}^q(\mathbb{R}^n)$ for all $1 \leq q < \infty$, and $\text{CMO}^q(\mathbb{R}^n) \subset \text{CMO}^p(\mathbb{R}^n)$, $1 \leq p < q < \infty$.

Definition 1.2 (see [12]). Let $0 < \gamma < 1$. The Besov-Lipschitz space $\dot{\Lambda}_\gamma(\mathbb{R}^n)$ is defined by

$$\|b\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} = \sup_{x, h \in \mathbb{R}^n} \frac{|b(x+h) - b(x)|}{|h|^\gamma} < \infty. \quad (1.8)$$

Herz-type spaces are important function spaces in harmonic analysis. It should be pointed out that Lu and Yang make tremendous contributions on this spaces. Their book (joint with Hu) [13] is the unique research book in this topic. Below, we briefly recall the definition of the Herz-type spaces. We denote by $B(x, r)$ the ball centered at x with radius r . C is a constant which may vary from line to line. For $k \in \mathbb{Z}$, let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $\Delta_k = B_k \setminus B_{k-1}$, and χ_k ($k \in \mathbb{Z}$) denote the characteristic function of the set Δ_k .

Definition 1.3 (see [13]). Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, and $0 < q < \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha, p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha, p}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha, p}(\mathbb{R}^n)} < \infty \right\}, \quad (1.9)$$

where

$$\|f\|_{\dot{K}_q^{\alpha, p}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \right)^{1/p}. \quad (1.10)$$

Obviously, $\dot{K}_q^{0, q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ and $\dot{K}_q^{\alpha/q, q}(\mathbb{R}^n) = L^q(\mathbb{R}^n, |x|^\alpha)$, so the Herz space is the natural generalization of the Lebesgue spaces with power weight $|x|^\alpha$.

Definition 1.4 (see [13]). Let $\alpha \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q < \infty$ and $\lambda \geq 0$. The homogeneous Morrey-Herz space $M\dot{K}_{p, q}^{\alpha, \lambda}(\mathbb{R}^n)$ is defined by

$$M\dot{K}_{p, q}^{\alpha, \lambda}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}_{p, q}^{\alpha, \lambda}(\mathbb{R}^n)} < \infty \right\}, \quad (1.11)$$

where

$$\|f\|_{M\dot{K}_{p, q}^{\alpha, \lambda}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \right)^{1/p} \quad (1.12)$$

with the usual modification made when $p = \infty$.

In [14], the Morrey space $M_q^\lambda(\mathbb{R}^n)$ is defined by

$$M_q^\lambda(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n) : \sup_{\lambda > 0, x \in \mathbb{R}^n} \frac{1}{r^\lambda} \int_{|x-y| < r} |f(y)|^q dy < \infty \right\}. \quad (1.13)$$

Obviously, $M\dot{K}_{p, q}^{\alpha, 0}(\mathbb{R}^n) = \dot{K}_q^{\alpha, p}(\mathbb{R}^n)$ and $M_q^\lambda(\mathbb{R}^n) \subset M\dot{K}_{q, q}^{0, \lambda}(\mathbb{R}^n)$.

2. Main Results

Now, we state our main results.

Theorem 2.1. Let $1 < p < \infty$, $1 < r < \min(p, p')$, and $b \in \text{CMO}^{\max(p, pr/(p-r))}(\mathbb{R}^n)$:

(a) if one sets

$$A_{\Phi, r} = \left(\int_0^\infty |\Phi(t)|^{r'} t^{-1+n(r'/r)} dt \right)^{1/r'} < \infty, \quad (2.1)$$

then $H_{\Phi, b}^1$ is bounded on $L^p(\mathbb{R}^n)$;

(b) if one sets

$$B_{\Phi, r} = \left(\int_0^\infty |\Phi(t)|^{r'} t^{-1+n} dt \right)^{1/r'} < \infty, \quad (2.2)$$

then $H_{\Phi, b}^2$ is bounded on $L^p(\mathbb{R}^n)$;

(c) if Φ satisfies (2.1) and (2.2), then $H_{\Phi, b}$ is bounded on $L^p(\mathbb{R}^n)$.

Remark 2.2. When $\Phi(t) = t^{-n} \chi_{(1, \infty)}(t)$, $H_{\Phi, b} f(x) = H_{\Phi, b}^1 f(x) = H_b f(x)$. We can check that $\Phi(t)$ satisfies (2.1); therefore, the boundedness of commutator of Hardy operator is obtained. When $\Phi(t) = \chi_{(0, 1)}(t)$, $H_{\Phi, b} f(x) = H_{\Phi, b}^2 f(x) = H_b^* f(x)$. We also know that $\Phi(t)$ satisfies (2.2); therefore, we get the boundedness of commutator of the adjoint Hardy operator see [9].

Remark 2.3. Using the same method, we can get the generalized result of Theorem 2.1(b). Let $1 < p < \infty$, $1 < r < p$, $0 \leq \lambda < \min(r'/r, r'/p)$, and $b \in \text{CMO}^{\max(p, pr/(p-r))}(\mathbb{R}^n)$. If

$$B_{\Phi, r, \lambda} = \left(\int_0^\infty |\Phi(t)|^{r'} t^{-1+n\lambda} dt \right)^{1/r'} < \infty, \quad (2.3)$$

then $H_{\Phi, b}^2$ is bounded on $L^p(\mathbb{R}^n)$.

Theorem 2.4. Let $1 < p, q < \infty$, $1 < r < \min(p, q')$, $1/p - 1/q = \gamma/n$, and $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$ ($0 < \gamma < 1$). Then

- (a) if Φ satisfies (2.1), then $H_{\Phi, b}^1$ are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$;
- (b) if Φ satisfies (2.2), then $H_{\Phi, b}^2$ are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$;
- (c) if Φ satisfies (2.1) and (2.2), then $H_{\Phi, b}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

Remark 2.5. Just like in Remark 2.2, we get Lipschitz estimates for commutator of Hardy operator or the adjoint Hardy operator. See the details in [15].

Theorem 2.6. Let $0 < p < \infty, 1 < q < \infty, 1 < r < q$, and $b \in \text{CMO}^{\max(q, qr/(q-r))}(\mathbb{R}^n)$. Then

- (a) if Φ satisfies (2.1) and $\alpha < n(1/r - 1/q)$, then $H_{\Phi, b}^1$ is bounded on $\dot{K}_q^{\alpha, p}(\mathbb{R}^n)$;
- (b) if Φ satisfies (2.2) and $\alpha > -n(1/q - 1/r')$, then $H_{\Phi, b}^2$ is bounded on $\dot{K}_q^{\alpha, p}(\mathbb{R}^n)$;
- (c) if Φ satisfies (2.1), (2.2), and $-n(1/q - 1/r') < \alpha < n(1/r - 1/q)$, then $H_{\Phi, b}$ is bounded on $\dot{K}_q^{\alpha, p}(\mathbb{R}^n)$.

Remark 2.7. Let $\alpha = 0$ in Theorem 2.6, then Theorem 2.1 can be obtained.

Theorem 2.8. Let $0 < p_1 \leq p_2 < \infty$, $1 < q_1, q_2 < \infty$, $1 < r < q_1$, $1/q_1 - 1/q_2 = \gamma/n$, and $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$ ($0 < \gamma < 1$). Then

- (a) if Φ satisfies (2.1) and $\alpha < n(1/r - 1/q_1)$, then $H_{\Phi,b}^1$ is bounded from $\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ to $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$;
- (b) if Φ satisfies (2.2) and $\alpha > -n(1/q_2 - 1/r')$, then $H_{\Phi,b}^2$ are bounded from $\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ to $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$;
- (c) if Φ satisfies (2.1), (2.2), and $-n(1/q_2 - 1/r') < \alpha < n(1/r - 1/q_1)$, then $H_{\Phi,b}$ is bounded from $\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ to $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$.

Remark 2.9. Let $\alpha = 0$ in Theorem 2.8, then Theorem 2.4 can be obtained.

Theorem 2.10. Let $0 < p < \infty$, $1 < q < \infty$, $1 < r < q$, $\lambda \geq 0$, and $b \in \dot{C}MO^{\max(q,qr/(q-r))}(\mathbb{R}^n)$. Then

- (a) if Φ satisfies (2.1) and $\alpha < n(1/r - 1/q) + \lambda$, then $H_{\Phi,b}^1$ is bounded on $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$;
- (b) if Φ satisfies (2.2) and $\alpha > -n(1/q - 1/r') + \lambda$, then $H_{\Phi,b}^2$ is bounded on $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$;
- (c) if Φ satisfies (2.1), (2.2), and $-n(1/q - 1/r') + \lambda < \alpha < n(1/r - 1/q) + \lambda$, then $H_{\Phi,b}$ is bounded on $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$.

Theorem 2.11. Let $\lambda \geq 0$, $0 < p_1 \leq p_2 < \infty$, $1 < q_1, q_2 < \infty$, $1 < r < q_1$, $1/q_1 - 1/q_2 = \gamma/n$, and $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$ ($0 < \gamma < 1$).

- (a) if Φ satisfies (2.1) and $\alpha < n(1/r - 1/q_1) + \lambda$, then $H_{\Phi,b}^1$ is bounded from $M\dot{K}_{p_1,q_1}^{\alpha,\lambda}(\mathbb{R}^n)$ to $M\dot{K}_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)$;
- (b) if Φ satisfies (2.2) and $\alpha > -n(1/q_2 - 1/r') + \lambda$, then $H_{\Phi,b}^2$ is bounded from $M\dot{K}_{p_1,q_1}^{\alpha,\lambda}(\mathbb{R}^n)$ to $M\dot{K}_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)$;
- (c) if Φ satisfies (2.1), (2.2), and $-n(1/q_2 - 1/r') + \lambda < \alpha < n(1/r - 1/q_1) + \lambda$, then $H_{\Phi,b}$ is bounded from $M\dot{K}_{p_1,q_1}^{\alpha,\lambda}(\mathbb{R}^n)$ to $M\dot{K}_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)$.

3. Proof of the Main Results

We first give several lemmas.

Lemma 3.1. Let $1 < r \leq p < \infty$ and $\lambda \geq 0$, then

- (i) $\int_{\Delta_i} (|\Phi(x/|y|)|/|y|^n) |f(y)| dy \leq CA_{\Phi,r} 2^{in(1/r-1/p)} |x|^{-n/r} \|f\chi_i\|_{L^p(\mathbb{R}^n)},$
- (ii) $\int_{\Delta_i} (|\Phi(x/|y|)|/|y|^n) |f(y)| dy \leq CB_{\Phi,r,\lambda} 2^{in(\lambda/r'-1/p)} |x|^{-n\lambda/r'} \|f\chi_i\|_{L^p(\mathbb{R}^n)}.$

Proof. (i) By Hölder's inequality and polar coordinate, we obtain

$$\begin{aligned}
 \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} |f(y)| dy &\leq \left(\int_{\Delta_i} \frac{|\Phi(x/|y|)|^{r'}}{|y|^{nr'}} dy \right)^{1/r'} \left(\int_{\Delta_i} |f(y)|^r dy \right)^{1/r} \\
 &\leq \left(\int_{|x|/2^i}^{|x|/2^{i-1}} |S^{n-1}| |\Phi(t)|^{r'} t^{-1+nr'/r} |x|^{n(1-r')} dt \right)^{1/r'} \\
 &\quad \times \left(\int_{\Delta_i} |f(y)|^p dy \right)^{1/p} 2^{in(1/r-1/p)} \\
 &\leq CA_{\Phi,r} 2^{in(1/r-1/p)} |x|^{-n/r} \|f\chi_i\|_{L^p(\mathbb{R}^n)}.
 \end{aligned} \tag{3.1}$$

(ii) By Hölder's inequality and polar coordinate, we have

$$\begin{aligned}
 \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} |f(y)| dy &\leq \left(\int_{\Delta_i} \frac{|\Phi(x/|y|)|^{r'}}{|y|^{(n(\lambda+1)/r')r'}} dy \right)^{1/r'} \left(\int_{\Delta_i} \frac{|f(y)|^r}{|y|^{(n-n(\lambda+1)/r')r}} dy \right)^{1/r} \\
 &\leq C \left(\int_{2^{i-1}}^{2^i} \int_{S^{n-1}} |\Phi(|x|R^{-1})|^{r'} R^{-1-n\lambda} d\sigma(y') dR \right)^{1/r'} \\
 &\quad \times 2^{-in(1-(\lambda+1)/r')} \|f\chi_i\|_{L^r} \\
 &\leq C \left(\int_{2^{i-1}}^{2^i} \int_{S^{n-1}} |\Phi(|x|R^{-1})|^{r'} R^{-1-n\lambda} d\sigma(y') dR \right)^{1/r'} \\
 &\quad \times 2^{in(\lambda/r'-1/p)} \|f\chi_i\|_{L^p} \\
 &\leq CB_{\Phi,r,\lambda} 2^{in(\lambda/r'-1/p)} |x|^{-n\lambda/r'} \|f\chi_i\|_{L^p}.
 \end{aligned} \tag{3.2}$$

□

Lemma 3.2 (see [9]). *Let b be a $\dot{CMO}^1(\mathbb{R}^n)$ function and $i, k \in \mathbb{Z}$, then*

$$|b(t) - b_{B_i}| \leq |b(t) - b_{B_k}| + 2|i - k| \|b\|_{\dot{CMO}^1(\mathbb{R}^n)}. \tag{3.3}$$

Lemma 3.3 (see [12]). *Let $0 < \gamma < 1$ and $f \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$. Then*

$$|f(x) - f(y)| \leq |x - y|^\gamma \|f\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} \leq (|x|^\gamma + |y|^\gamma) \|f\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)}. \tag{3.4}$$

Proof of Theorem 2.6. (a) When Φ satisfies (2.1) and $\alpha < n(1/r - 1/q)$, we get

$$\begin{aligned}
 \|(H_{\Phi,b}^1 f)\chi_k\|_{L^q}^q &\leq C_q \int_{\Delta_k} \left(\sum_{i=-\infty}^k \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} |(b(x) - b_{B_k})f(y)| dy \right)^q dx \\
 &\quad + C_q \int_{\Delta_k} \left(\sum_{i=-\infty}^k \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} |(b(y) - b_{B_k})f(y)| dy \right)^q dx \\
 &:= I + II.
 \end{aligned} \tag{3.5}$$

For I , using Lemma 3.1(i), we have

$$\begin{aligned}
 I &\leq C_q A_{\Phi,r}^q \int_{B_k} |b(x) - b_{B_k}|^q \left(\sum_{i=-\infty}^k 2^{in(1/r-1/q)} |x|^{-n/r} \|f\chi_i\|_{L^q} \right)^q dx \\
 &\leq C_q A_{\Phi,r}^q \|b\|_{\text{CMO}^q(\mathbb{R}^n)}^q |B_k| \left(\sum_{i=-\infty}^k 2^{in(1/r-1/q)} 2^{-kn/r} \|f\chi_i\|_{L^q} \right)^q \\
 &= C_q A_{\Phi,r}^q \|b\|_{\text{CMO}^q(\mathbb{R}^n)}^q \left(\sum_{i=-\infty}^k 2^{(i-k)n(1/r-1/q)} \|f\chi_i\|_{L^q} \right)^q.
 \end{aligned} \tag{3.6}$$

For II , by Lemma 3.2, we have

$$\begin{aligned}
 II &\leq C_q \int_{\Delta_k} \left(\sum_{i=-\infty}^k \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} |(b(y) - b_{B_i})f(y)| dy \right)^q dx \\
 &\quad + C_q \int_{\Delta_k} \left(\sum_{i=-\infty}^k \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} (k-i) \|b\|_{\text{CMO}^1(\mathbb{R}^n)} |f(y)| dy \right)^q dx \\
 &:= II_1 + II_2.
 \end{aligned} \tag{3.7}$$

For II_1 , by Lemma 3.1(i) and Hölder's inequality, we have

$$\begin{aligned}
 II_1 &\leq C_q A_{\Phi,r}^q \int_{\Delta_k} \left(\sum_{i=-\infty}^k |x|^{-n/r} \left(\int_{\Delta_i} |(b(y) - b_{B_i})f(y)|^r dy \right)^{1/r} \right)^q dx \\
 &\leq C_q A_{\Phi,r}^q \int_{\Delta_k} \left(\sum_{i=-\infty}^k |x|^{-n/r} \left(\int_{\Delta_i} |(b(y) - b_{B_i})|^{rq/(q-r)} dy \right)^{(q-r)/qr} \left(\int_{\Delta_i} |f(y)|^q dy \right)^{1/q} \right)^q dx \\
 &\leq C_q A_{\Phi,r}^q \left(\sum_{i=-\infty}^k 2^{kn(1/q-1/r)} \|b\|_{\text{CMO}^{qr/(q-r)}(\mathbb{R}^n)} |B_i|^{(q-r)/qr} \|f\chi_i\|_{L^q} \right)^q \\
 &= C_q A_{\Phi,r}^q \|b\|_{\text{CMO}^{qr/(q-r)}(\mathbb{R}^n)}^q \left(\sum_{i=-\infty}^k 2^{(i-k)n(1/q-1/r)} \|f\chi_i\|_{L^q} \right)^q.
 \end{aligned} \tag{3.8}$$

For II_2 , by Lemma 3.1(i), we have

$$\begin{aligned}
 II_2 &\leq C_q A_{\Phi,r}^q \|b\|_{\text{CMO}^1(\mathbb{R}^n)}^q \int_{\Delta_k} \left(\sum_{i=-\infty}^k (k-i) 2^{in(1/r-1/q)} |x|^{-n/r} \|f\chi_i\|_{L^q} \right)^q dx \\
 &\leq C_q A_{\Phi,r}^q \|b\|_{\text{CMO}^1(\mathbb{R}^n)}^q 2^{kn} \left(\sum_{i=-\infty}^k (k-i) 2^{in(1/r-1/q)} 2^{-kn/r} \|f\chi_i\|_{L^q} \right)^q \\
 &= C_q A_{\Phi,r}^q \|b\|_{\text{CMO}^1(\mathbb{R}^n)}^q \left(\sum_{i=-\infty}^k (k-i) 2^{(i-k)n(1/r-1/q)} \|f\chi_i\|_{L^q} \right)^q.
 \end{aligned} \tag{3.9}$$

Following the estimates of I , II_1 , and II_2 , we can obtain that

$$\begin{aligned}
\|H_{\Phi,b}^1 f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &\leq CA_{\Phi,r} \|b\|_{\text{CMO}^q(\mathbb{R}^n)} \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{i=-\infty}^k 2^{(i-k)n(1/r-1/q)} \|f\chi_i\|_{L^q} \right)^p \right)^{1/p} \\
&\quad + CA_{\Phi,r} \|b\|_{\text{CMO}^{qr/(q-r)}(\mathbb{R}^n)} \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{i=-\infty}^k 2^{(i-k)n(1/r-1/q)} \|f\chi_i\|_{L^q} \right)^p \right)^{1/p} \\
&\quad + CA_{\Phi,r} \|b\|_{\text{CMO}^1(\mathbb{R}^n)} \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{i=-\infty}^k (k-i) 2^{(i-k)n(1/r-1/q)} \|f\chi_i\|_{L^q} \right)^p \right)^{1/p} \\
&:= S.
\end{aligned} \tag{3.10}$$

Obviously,

$$S \leq CA_{\Phi,r} \|b\|_{\text{CMO}^{\max(q,qr/(q-r))}(\mathbb{R}^n)} \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{i=-\infty}^k (k-i) 2^{(i-k)n(1/r-1/q)} \|f\chi_i\|_{L^q} \right)^p \right)^{1/p}. \tag{3.11}$$

When $0 < p \leq 1$, we get

$$\begin{aligned}
S^p &\leq C_p A_{\Phi,r}^p \|b\|_{\text{CMO}^{\max(q,qr/(q-r))}(\mathbb{R}^n)}^p \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^k 2^{i\alpha p} \|f\chi_i\|_{L^q}^p (k-i)^p 2^{(i-k)(n/r-n/q-\alpha)p} \\
&= C_p A_{\Phi,r}^p \|b\|_{\text{CMO}^{\max(q,qr/(q-r))}(\mathbb{R}^n)}^p \sum_{i=-\infty}^{\infty} 2^{i\alpha p} \|f\chi_i\|_{L^q}^p \sum_{k=i}^{\infty} (k-i)^p 2^{(i-k)(n/r-n/q-\alpha)p} \\
&= C_p A_{\Phi,r}^p \|b\|_{\text{CMO}^{\max(q,qr/(q-r))}(\mathbb{R}^n)}^p \|f\|_{\dot{K}_q^{\alpha,p}}^p.
\end{aligned} \tag{3.12}$$

For the case $p > 1$, it follows from Hölder's inequality that

$$\begin{aligned}
S^p &\leq C_p A_{\Phi,r}^p \|b\|_{\text{CMO}^{\max(q,qr/(q-r))}(\mathbb{R}^n)}^p \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^k 2^{i\alpha p} \|f\chi_i\|_{L^q}^p 2^{(i-k)(n/r-n/q-\alpha)p/2} \\
&\quad \times \left(\sum_{i=-\infty}^k (k-i)^{p'} 2^{(i-k)(n/r-n/q-\alpha)p'/2} \right)^{p/p'} \\
&\leq C_p A_{\Phi,r}^p \|b\|_{\text{CMO}^{\max(q,qr/(q-r))}(\mathbb{R}^n)}^p \sum_{i=-\infty}^{\infty} 2^{i\alpha p} \|f\chi_i\|_{L^q}^p \sum_{k=i}^{\infty} 2^{(i-k)(n/r-n/q-\alpha)p/2} \\
&= C_p A_{\Phi,r}^p \|b\|_{\text{CMO}^{\max(q,qr/(q-r))}(\mathbb{R}^n)}^p \|f\|_{\dot{K}_q^{\alpha,p}}^p.
\end{aligned} \tag{3.13}$$

When Φ satisfies (2.2) and $\alpha > -n(1/q - 1/r')$, we get

$$\begin{aligned} \|(H_{\Phi,b}^2 f)\chi_k\|_{L^q(\mathbb{R}^n)}^q &\leq C_q \int_{\Delta_k} \left(\sum_{i=k}^{\infty} \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} |(b(x) - b_{B_k})f(y)| dy \right)^q dx \\ &\quad + C_q \int_{C_k} \left(\sum_{i=k}^{\infty} \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} |(b(y) - b_{B_k})f(y)| dy \right)^q dx \\ &:= J + JJ. \end{aligned} \quad (3.14)$$

For J , using Lemma 3.1(ii), we have

$$\begin{aligned} J &\leq C_q B_{\Phi,r}^q \int_{\Delta_k} |b(x) - b_{B_k}|^q \left(\sum_{i=k}^{\infty} 2^{in(1/r'-1/q)} |x|^{-n/r'} \|f\chi_i\|_{L^q} \right)^q dx \\ &\leq C_q B_{\Phi,r}^q \|b\|_{\text{CMO}^q(\mathbb{R}^n)}^q |B_k| \left(\sum_{i=k}^{\infty} 2^{in(1/r'-1/q)} 2^{-kn/r} \|f\chi_i\|_{L^q} \right)^q \\ &= C_q B_{\Phi,r}^q \|b\|_{\text{CMO}^q(\mathbb{R}^n)}^q \left(\sum_{i=k}^{\infty} 2^{(k-i)n(1/q-1/r')} \|f\chi_i\|_{L^q} \right)^q. \end{aligned} \quad (3.15)$$

For JJ , by Lemma 3.2, we have

$$\begin{aligned} JJ &\leq C_q \int_{\Delta_k} \left(\sum_{i=k}^{\infty} \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} |(b(y) - b_{B_i})f(y)| dy \right)^q dx \\ &\quad + C_q \int_{\Delta_k} \left(\sum_{i=k}^{\infty} \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} (i-k) \|b\|_{\text{CMO}^1(\mathbb{R}^n)} |f(y)| dy \right)^q dx \\ &:= JJ_1 + JJ_2. \end{aligned} \quad (3.16)$$

For JJ_1 , by Lemma 3.1(ii) and Hölder's inequality, similarly to II_1 , we have

$$JJ_1 \leq C_q B_{\Phi,r}^q \|b\|_{\text{CMO}^{qr/(q-r)}(\mathbb{R}^n)}^q \left(\sum_{i=k}^{\infty} 2^{(k-i)n(1/q-1/r')} \|f\chi_i\|_{L^q} \right)^q. \quad (3.17)$$

For JJ_2 , also due to Lemma 3.1(ii), we have

$$\begin{aligned} JJ_2 &\leq C_q B_{\Phi,r}^q \|b\|_{\text{CMO}^1(\mathbb{R}^n)}^q \int_{\Delta_k} \left(\sum_{i=k}^{\infty} (i-k) 2^{in(1/r'-1/q)} |x|^{-n/r'} \|f\chi_i\|_{L^q} \right)^q dx \\ &\leq C_q B_{\Phi,r}^q \|b\|_{\text{CMO}^1(\mathbb{R}^n)}^q 2^{kn} \left(\sum_{i=k}^{\infty} (i-k) 2^{in(1/r'-1/q)} 2^{-kn/r'} \|f\chi_i\|_{L^q} \right)^q \\ &= C_q B_{\Phi,r}^q \|b\|_{\text{CMO}^1(\mathbb{R}^n)}^q \left(\sum_{i=k}^{\infty} (i-k) 2^{(k-i)n(1/q-1/r')} \|f\chi_i\|_{L^q} \right)^q. \end{aligned} \quad (3.18)$$

The remaining proof is similar to the proof of (a), so that of (b) can be obtained easily.

(c) When Φ satisfies (2.1), (2.2), and $-n(1/q - 1/r') < \alpha < n(1/r - 1/q)$, due to

$$H_{\Phi,b}f(x) = H_{\Phi,b}^1f(x) + H_{\Phi,b}^2f(x), \quad (3.19)$$

therefore, by (a) and (b), we can easily get (c). $\square$

Proof of Theorem 2.8. (a) When Φ satisfies (2.1) and $\alpha < n(1/r - 1/q_1)$, by Lemmas 3.3, 3.1(i), and $1/q_1 - 1/q_2 = \gamma/n$, we get

$$\begin{aligned} \left\| (H_{\Phi,b}^1f)\chi_k \right\|_{L^{q_2}}^{q_2} &\leq \|b\|_{\Lambda_\gamma}^{q_2} \int_{\Delta_k} \left(\sum_{i=-\infty}^k \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} (|x|^\gamma + |y|^\gamma) |f(y)| dy \right)^{q_2} dx \\ &\leq C \|b\|_{\Lambda_\gamma}^{q_2} \int_{\Delta_k} \left(|x|^\gamma \sum_{i=-\infty}^k \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} |f(y)| dy \right)^{q_2} dx \\ &\leq C_{q_2} A_{\Phi,r}^{q_2} \|b\|_{\Lambda_\gamma}^{q_2} 2^{kn} \left(2^{k\gamma - kn/r} \sum_{i=-\infty}^k 2^{in(1/r - 1/q_1)} \|f\chi_i\|_{L^{q_1}} \right)^{q_2} \\ &= C_{q_2} A_{\Phi,r}^{q_2} \|b\|_{\Lambda_\gamma}^{q_2} \left(\sum_{i=-\infty}^k 2^{(i-k)n(1/r - 1/q_1)} \|f\chi_i\|_{L^{q_1}} \right)^{q_2}. \end{aligned} \quad (3.20)$$

The remaining proof is similar to the proof of (a) of Theorem 2.6, so that of (a) can be obtained easily.

(b) When Φ satisfies (2.2) and $\alpha > -n(1/q_2 - 1/r')$, by Lemmas 3.3, 3.1(ii), and $(1/q_1) - (1/q_2) = \gamma/n$, we get

$$\begin{aligned} \left\| (H_{\Phi,b}^2f)\chi_k \right\|_{L^{q_2}(\mathbb{R}^n)}^{q_2} &\leq \|b\|_{\Lambda_\gamma}^{q_2} \int_{\Delta_k} \left(\sum_{i=k}^{\infty} \int_{\Delta_i} \frac{|\Phi(x/|y|)|}{|y|^n} (|x|^\gamma + |y|^\gamma) |f(y)| dy \right)^{q_2} dx \\ &\leq C_{q_2} \|b\|_{\Lambda_\gamma}^{q_2} \int_{\Delta_k} \left(\sum_{i=k}^{\infty} \int_{\Delta_i} |y|^\gamma \frac{|\Phi(x/|y|)|}{|y|^n} |f(y)| dy \right)^{q_2} dx \\ &\leq C_{q_2} B_{\Phi,r}^{q_2} \|b\|_{\Lambda_\gamma}^{q_2} \int_{\Delta_k} \left(\sum_{i=k}^{\infty} 2^{i\gamma} 2^{in(1/r' - 1/q_1)} |x|^{-n/r'} \|f\chi_i\|_{L^{q_1}} \right)^{q_2} dx \\ &\leq C_{q_2} B_{\Phi,r}^{q_2} \|b\|_{\Lambda_\gamma}^{q_2} \left(\sum_{i=k}^{\infty} 2^{(k-i)n(1/q_2 - 1/r')} \|f\chi_i\|_{L^{q_1}} \right)^{q_2}. \end{aligned} \quad (3.21)$$

The remaining proof is similar to the proof of (a) of Theorem 2.6, so that of (b) can be obtained easily.

(c) The proof is the same as the proof of (c) of Theorem 2.6. $\square$

Proof of Theorem 2.10. We only give the proof of (a) when $\lambda > 0$. The proof of (b) is similar to that for (a), and the proof of (c) is similar to that for (c) of Theorem 2.6. By the definition of Morrey-Herz spaces and the estimates for I , II_1 , and II_2 above, we have

$$\begin{aligned}
& \|H_{\Phi,b}^1 f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}} \\
& \leq CA_{\Phi,r} \|b\|_{\text{CMO}^q(\mathbb{R}^n)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{i=-\infty}^k 2^{(i-k)(n/r-n/q)} \|f\chi_i\|_{L^q} \right)^p \right)^{1/p} \\
& \quad + CA_{\Phi,r} \|b\|_{\text{CMO}^{qr/(q-r)}(\mathbb{R}^n)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{i=-\infty}^k 2^{(i-k)(n/r-n/q)} \|f\chi_i\|_{L^q} \right)^p \right)^{1/p} \\
& \quad + CA_{\Phi,r} \|b\|_{\text{CMO}^1(\mathbb{R}^n)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{i=-\infty}^k (k-i) 2^{(i-k)(n/r-n/q)} \|f\chi_i\|_{L^q} \right)^p \right)^{1/p} \\
& := E_1 + E_2 + E_3.
\end{aligned} \tag{3.22}$$

For E_1 , noting that $\alpha < n(1/r - 1/q) + \lambda$, we have

$$\begin{aligned}
E_1 & \leq CA_{\Phi,r} \|b\|_{\text{CMO}^q} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} \left(\sum_{i=-\infty}^k 2^{(i-k)(n/r-n/q-\alpha)} \left(\sum_{l=-\infty}^i 2^{lap} \|f\chi_l\|_{L^q}^p \right)^{1/p} \right)^p \right)^{1/p} \\
& \leq CA_{\Phi,r} \|b\|_{\text{CMO}^q} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{kp\lambda} \left(\sum_{i=-\infty}^k 2^{(i-k)(n/r-n/q-\alpha+\lambda)} \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}} \right)^p \right)^{1/p} \\
& \leq CA_{\Phi,r} \|b\|_{\text{CMO}^q} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{kp\lambda} \right)^{1/p} \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}} \\
& = CA_{\Phi,r} \|b\|_{\text{CMO}^q(\mathbb{R}^n)} \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)}.
\end{aligned} \tag{3.23}$$

Similarly to the proof of E_1 , we have

$$\begin{aligned}
E_2 & \leq CA_{\Phi,r} \|b\|_{\text{CMO}^{qr/(q-r)}(\mathbb{R}^n)} \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)}, \\
E_3 & \leq CA_{\Phi,r} \|b\|_{\text{CMO}^1(\mathbb{R}^n)} \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)}.
\end{aligned} \tag{3.24}$$

This finishes the proof of Theorem 2.10. $\square$

Proof of Theorem 2.11. The proof is similar to the proof of Theorem 2.10, so we omit the details. $\square$

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