Research Article

On Divergence of Fourier Series by Some Methods of Summability

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A new summability method of series is introduced and studied. The particular cases of this method are, for example, variable-order Cesaro and Riesz methods. Applications to divergence problem of Fourier series are given. An extension of Kolmogorov, Schipp, and Bočkarev’s well-known theorems on divergence of Fourier trigonometric, Walsh, and orthonormal series is established.

1. A New Summability Method of Series

Let

\[ \Lambda = \| \lambda_n(k) \|, \quad n = 0, 1, 2, \ldots, \quad k = 0, 1, 2, \ldots, n, \]  \hspace{1cm} (1.1)

be such triangular matrix which satisfies the following conditions:

\[
\begin{align*}
(1) \quad & 0 \leq \lambda_n(k+1) \leq \lambda_n(k) \leq 1, \quad 0 \leq k \leq n; \\
(2) \quad & \lambda_n(0) = 1, \quad \lambda_n(k) = 0, \quad k \geq n + 1. 
\end{align*}
\]  \hspace{1cm} (1.2)

By \( s_n \) we denote a partial sum of a series

\[ \sum_{k=0}^{\infty} u_k. \]  \hspace{1cm} (1.3)
and by $\sigma_n$ we denote a mean constructed by the $\Lambda$ matrix, that is,

$$
S_n = \sum_{k=0}^{n} u_k, \quad \sigma_n = \sum_{k=0}^{n} \lambda_n(k) u_k.
$$

\[ (1.4) \]

**Theorem 1.1.** Let matrix (1.1) satisfies an inequality

$$
\lim_{n \to \infty} \lambda_n(n) > \frac{1}{2}.
$$

\[ (1.5) \]

Then for any series (1.3) which satisfies the following condition:

$$
\lim_{n \to \infty} |S_n| = +\infty,
$$

\[ (1.6) \]

an equality

$$
\lim_{n \to \infty} |\sigma_n| = +\infty
$$

\[ (1.7) \]

holds.

Below we prove a Lemma which is used to prove Theorem 1.1.

**Lemma 1.2.** For every natural number $n$ an inequality

$$
|S_n - \sigma_n| \leq 2(1 - \lambda_n(n)) \cdot \max_{1 \leq k \leq n} |s_k|
$$

\[ (1.8) \]

holds.

**Proof of the Lemma.** Using Abel transformation and $\lambda_n(0) = 1$ we get

$$
S_n - \sigma_n = \sum_{k=0}^{n} u_k - \sum_{k=0}^{n} \lambda_n(k) u_k
$$

\[ (1.9) \]

\begin{align*}
&= \sum_{k=1}^{n} u_k - \sum_{k=1}^{n} \lambda_n(k) u_k \\
&= \sum_{k=1}^{n} (1 - \lambda_n(k)) u_k \\
&= \sum_{k=1}^{n} (\lambda_n(k+1) - \lambda_n(k)) s_k + (1 - \lambda_n(n)) s_n.
\end{align*}
Therefore,

\[ |s_n - \sigma_n| \leq \sum_{k=1}^{n-1} |\lambda_n(k + 1) - \lambda_n(k)| \cdot |s_k| + |1 - \lambda_n(n)| \cdot |s_n| \]

\[ \leq \max_{1 \leq k \leq n} |s_k| \cdot \left( \sum_{k=1}^{n-1} |\lambda_n(k + 1) - \lambda_n(k)| + |1 - \lambda_n(n)| \right). \]  

(1.10)

Thus, taking into account (1.1) we immediately get

\[ |s_n - \sigma_n| \leq \max_{1 \leq k \leq n} |s_k| \cdot \left( \sum_{k=1}^{n-1} |\lambda_n(k + 1) - \lambda_n(k)| + 1 - \lambda_n(n) \right) \]

\[ = \max_{1 \leq k \leq n} |s_k| \cdot (\lambda_n(1) - \lambda_n(n) + 1 - \lambda_n(n)) \]

\[ \leq \max_{1 \leq k \leq n} |s_k| \cdot (1 - \lambda_n(n) + 1 - \lambda_n(n)) \]

\[ = 2 \cdot (1 - \lambda_n(n)) \cdot \max_{1 \leq k \leq n} |s_k|. \]  

(1.11)

So the Lemma is proved.

Proof of Theorem 1.1. According to the condition of Theorem 1.1 we have

\[ \lim_{n \to \infty} \lambda_n(n) = \frac{1}{2} + \delta \]  

(1.12)

for some \( \delta > 0 \). Note that inequalities \( 0 \leq \lambda_n(n) \leq 1 \) which hold for every natural \( n \) imply \( 1/2 + \delta \leq 1 \), that is, \( \delta \leq 1/2 \).

So, \( 0 < \delta \leq 1/2 \) holds.

According to (1.12) there exists a natural number \( n_0 \) such that for every natural number \( n > n_0 \) we have

\[ \lambda_n(n) > \frac{1}{2} + \frac{\delta}{2}. \]  

(1.13)

So according to the Lemma, for every \( n > n_0 \) an inequality

\[ |s_n - \sigma_n| < 2 \cdot \left( 1 - \left( \frac{1}{2} + \frac{\delta}{2} \right) \right) \cdot \max_{1 \leq k \leq n} |s_k| \]  

(1.14)

holds true; that is, if \( n > n_0 \), then

\[ |s_n - \sigma_n| < (1 - \delta) \cdot \max_{1 \leq k \leq n} |s_k|. \]  

(1.15)
Thus for every $n > n_0$ an inequality
\[
\|s_n\| - |\sigma_n| < (1 - \delta) \cdot \max_{1 \leq k \leq n} |s_k| \tag{1.16}
\]
holds.

So for every $n > n_0$ we have
\[
|\sigma_n| > |s_n| - (1 - \delta) \cdot \max_{1 \leq k \leq n} |s_k|. \tag{1.17}
\]

Note that for every natural $n$ there exists at least one natural number $1 \leq q \leq n$, such that the partial sums of the series (1.3) satisfy the following condition:
\[
|s_q| = \max_{1 \leq k \leq n} |s_k|. \tag{1.18}
\]

We define $p_n$ by a formula:
\[
p_n = \max \left\{ q : 1 \leq q \leq n \& |s_q| = \max_{1 \leq k \leq n} |s_k| \right\}. \tag{1.19}
\]

So $p_n$ is maximal number among the above-mentioned natural $q$ numbers. Consequently,
\[
1 \leq p_n \leq n, \quad |s_{p_n}| = \max_{1 \leq k \leq n} |s_k|, \tag{1.20}
\]
\[
p_n \leq p_{n+1}, \quad |s_{p_n}| \leq |s_{p_{n+1}}|. \tag{1.21}
\]

According to the condition of Theorem 1.1,
\[
\lim_{n \to \infty} |s_n| = +\infty. \tag{1.22}
\]

Therefore,
\[
\lim_{n \to \infty} |s_{p_n}| = +\infty, \tag{1.23}
\]
that is,
\[
\lim_{n \to \infty} p_n = +\infty. \tag{1.24}
\]

A consequence of (1.24) is that there exists such natural $n_1$ that if $n > n_1$ then $p_n > n_0$ and since (1.17) holds for every $n > n_0$, then (1.17) remains true for every $p_n$, where $n > n_1$.

So
\[
|\sigma_{p_n}| > |s_{p_n}| - (1 - \delta) \cdot \max_{1 \leq k \leq s_{p_n}} |s_k|. \tag{1.25}
\]
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Since $1 \leq p_n \leq n$, therefore,

$$\max_{1 \leq k \leq p_n} |s_k| \leq \max_{1 \leq k \leq n} |s_k|. \quad (1.26)$$

Note that the last one and (1.25) imply

$$|\sigma_{p_n}| > |s_{p_n}| - (1 - \delta) \cdot \max_{1 \leq k \leq n} |s_k|. \quad (1.27)$$

So according to (1.21) we have

$$|\sigma_{p_n}| > |s_{p_n}| - (1 - \delta) \cdot |s_{p_n}|, \quad (1.28)$$

that is, for every $n > n_1$ an inequality

$$|\sigma_{p_n}| > \delta \cdot |s_{p_n}| \text{ holds, where } 0 < \delta \leq \frac{1}{2}. \quad (1.29)$$

Also, (1.23) and (1.29) imply

$$\lim_{n \to \infty} |\sigma_{p_n}| = +\infty. \quad (1.30)$$

So we have finished the proof of Theorem 1.1.

Below we consider some consequences of Theorem 1.1.

Let $\Lambda = \|\lambda_n(k)\|$ be a triangular matrix, where the sequence $\{\alpha_n\}$ is from $[0, 1]$ and for every $0 \leq k \leq n$ number $\lambda_n(k)$ is defined by the formula:

$$\lambda_n(k) = \frac{A^{\alpha_n}_{n-k}}{A^{\alpha_n}_n}, \quad \text{where } A^{\alpha_n}_n = \frac{(\alpha_n + 1) (\alpha_n + 2) \cdots (\alpha_n + n)}{n!}. \quad (1.31)$$

If $\alpha_n = \alpha$, for every $n \geq 0$ and (1.31) holds true, then the $\Lambda$ method is Cesaro $(C, \alpha)$ summability method, and if $\alpha_n \equiv 0$, then the $\Lambda$ method coincides with convergence.

We introduce Cesaro summability method with variable orders, denoted by a symbol $(C, \{\alpha_n\})$, which coincides with $\Lambda$ summability method defined by (1.31). Means of this method for series (1.3) we denoted by $\sigma_n^{\alpha_n}$.

For $(C, \{\alpha_n\})$ we have the following.

**Theorem 1.3.** Let a sequences $\{\alpha_n\}$ be such that for some positive number $m$ we have

$$\alpha_n \leq \frac{c}{\ln n}, \quad \text{where } 0 \leq c < \ln 2 \text{ and } n > m. \quad (1.32)$$

Then for any series (1.3) which satisfies the following condition:

$$\lim_{n \to \infty} |s_n| = +\infty, \quad (1.33)$$
an equality

$$\lim_{n \to \infty} |\sigma_n^\alpha| = +\infty$$

(1.34)

holds.

Proof of Theorem 1.3. Note that every $\lambda_n(k)$ satisfies condition (1.1) and condition (1.3). Indeed,

$$\frac{\lambda_n(k + 1)}{\lambda_n(k)} = \frac{A_n^{\alpha_k}}{A_n^{\alpha_{n-k}}} = \frac{n - k}{\alpha_n + n - k} \leq 1$$

(1.35)

and $\lambda_n(0) = 1$, when $n \geq 0$.

For every $n \geq 1$ we have

$$\lambda_n(n) = \frac{1}{A_n^\alpha}, \quad \text{where} \quad A_n^\alpha = \frac{(\alpha_n + 1)(\alpha_n + 2) \cdots (\alpha_n + n)}{n!},$$

(1.36)

that is,

$$A_n^\alpha = \left(1 + \frac{\alpha_n}{1}\right)\left(1 + \frac{\alpha_n}{2}\right) \cdots \left(1 + \frac{\alpha_n}{n}\right).$$

(1.37)

Therefore,

$$\ln A_n^\alpha = \sum_{k=1}^{n} \ln \left(1 + \frac{\alpha_k}{n}\right) < \sum_{k=1}^{n} \frac{\alpha_k}{k} = \alpha_n \cdot \sum_{k=1}^{n} \frac{1}{k} < \alpha_n(1 + \ln n).$$

(1.38)

Note that the last one and (1.32) imply that

$$c = \ln \frac{2}{1 + \gamma}, \quad \text{for some} \quad 0 < \gamma \leq 1,$$

(1.39)

and if $n > m$, we have

$$A_n^{\alpha_n} < e^{\alpha_n(1 + \ln n)} = e^{\alpha_n \cdot e^{\alpha_n \ln n}}$$

$$\leq e^{\alpha_n \cdot e^{c}} = e^{\alpha_n \cdot e^{\ln(2/1+\gamma)}} = e^{\alpha_n \cdot \frac{2}{1+\gamma}},$$

(1.40)

that is,

$$\lambda_n(n) = \frac{1}{A_n^\alpha} > \frac{1}{e^{\alpha_n} \cdot \left(\frac{1}{2} + \frac{\gamma}{2}\right)}, \quad \text{where} \quad \gamma > 0.$$
Note that $\alpha_n \to 0$ implies the existence of such $\gamma_1 > 0$ and natural $n_2$, that if $n > n_2$, then

$$\frac{1}{e^{\alpha_n}} \cdot \left( \frac{1}{2} + \frac{\gamma_1}{2} \right) > \frac{1}{2} + \gamma_1,$$

(1.42)

that is, if $n > n_2$, then

$$\lambda_n(n) > \frac{1}{2} + \gamma_1.$$  

(1.43)

A consequence of (1.43) is that if (1.32) holds, then the $\Lambda$ matrix satisfies conditions of Theorem 1.1. This completes the proof of Theorem 1.3. \qed

Theorem 1.3 directly implies the following.

**Theorem 1.4.** Let $\{\alpha_n\}$ be such sequence that

$$\{\alpha_n\} = o\left(\frac{1}{\ln n}\right).$$  

(1.44)

Then for every series (1.3) which satisfies

$$\lim_{n \to \infty} |s_n| = +\infty,$$

(1.45)

we have

$$\lim_{n \to \infty} |\sigma_n^{\alpha_n}| = +\infty.$$  

(1.46)

2. **On Divergence of Fourier Series**

It is well known the following.

**Theorem A** (Kolmogorov [1]). There exists such summable function $f$ that Fourier trigonometric series of $f$

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

(2.1)

unboundedly diverges everywhere.

Let $W = \{w_n(t)\}_{n=1}^{\infty}$ be the Walsh system. Below we formulate Theorem B which is analogous of Theorem A and holds for Fourier-Walsh series.
Theorem B (Schipp [2, 3]). There exists such summable function $g$ that Fourier-Walsh series of $g$

\[ \sum_{n=1}^{\infty} a_n \omega_n(t) \]  

unboundedly diverges everywhere.

Let $\Phi = \{ \varphi_n(t) \}$ be orthonormal functions system defined on $[0, 1]$, such that

\[ |\varphi_n(t)| \leq M, \quad t \in [0, 1], \quad n = 1, 2, \ldots \]  

Then below-mentioned theorem holds.

Theorem C (Bočkarev [4]). For every orthonormal system $\Phi$ which satisfies (2.3), there exists such summable function $h$ defined on $[0, 1]$ that its Fourier series constructed by $\Phi$ system

\[ \sum_{n=1}^{\infty} a_n \varphi_n(t) \]  

unboundedly diverges in any point of some set $E \subset [0, 1]$ with positive measure.

Denote by $\sigma_{n}^{\alpha_n}(x; f)$, $\sigma_{n}^{\alpha_n}(t, g, W)$, and $\sigma_{n}^{\alpha_n}(t, h, \Phi)$ means of series (2.1), (2.2), and (2.4), respectively.

Theorem 1.3 implies that if $\{\alpha_n\}$ satisfies (1.32), then Theorems A, B, and C hold for $(C, \{\alpha_n\})$ summability method.

Namely, the following Theorems hold true.

**Theorem 2.1.** Let a sequence $\{\alpha_n\}$ satisfies (1.32). Then there exists such summable function $f$, that sequence $\{\sigma_{n}^{\alpha_n}(x; f)\}$ unboundedly diverges everywhere.

**Theorem 2.2.** Let a sequence $\{\alpha_n\}$ satisfies (1.32). Then there exists such summable function $g$ that sequence $\{\sigma_{n}^{\alpha_n}(t, g, W)\}$ unboundedly diverges everywhere.

**Theorem 2.3.** If orthonormal system $\Phi$ satisfies (2.3) and a sequence $\{\alpha_n\}$ satisfies (1.32), then there exists such summable function $h$, defined on $[0, 1]$, that sequence $\{\sigma_{n}^{\alpha_n}(t; h; \Phi)\}$ unboundedly diverges at every point of some set $E \subset [0, 1]$ with positive measure.

It is obvious that a consequence of Theorem 1.4 is that Theorems 2.1, 2.2, and 2.3 hold true if

\[ \alpha_n = o\left(\frac{1}{\ln n}\right). \]  

**Remark 2.4.** If every number $\lambda_n(k)$ will be replaced by $(1 - k/(n + 1))^\alpha_n$ in (1.31), then we get a summability method defined by $\Lambda = \|\lambda_n(k)\|$ matrix, which we call Riesz summability method with variable orders and denote it by symbol $(R, \{\alpha_n\})$. 

It can be proved analogously that Theorems 2.1, 2.2, and 2.3 remain true for Riesz summability method with variable orders, that is, for \((R, \{\alpha_n\})\) method, where \(\{\alpha_n\}\) satisfies (1.32).

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References

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