Research Article

Boundedness of Weighted Hardy Operator and Its Adjoint on Triebel-Lizorkin-Type Spaces

Canqin Tang and Ruohong Zhou

Department of Mathematics, Dalian Maritime University, Liaoning, Dalian, 116026, China

Correspondence should be addressed to Canqin Tang, tangcq2000@dlmu.edu.cn

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Let $p \in [1, \infty]$, $q \in [1, \infty)$, $\tau \in (0, \infty)$, and $\alpha \in (0, 1)$ such that $\tau > 1/p - 1/q$ and $\alpha \leq n(1/p - \tau)$, let U_ψ be the weighted Hardy operator and V_ψ its adjoint operator with respect to the weight function ψ . In this paper, the authors establish a sufficient and necessary condition on weight function ψ to ensure the boundedness of U_ψ and V_ψ on the Triebel-Lizorkin-type spaces $\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$ and their predual spaces, Triebel-Lizorkin-Hausdorff spaces, which unify and generalize the known results on Q -type spaces.

1. Introduction

This paper focuses on the boundedness of the weighted Hardy operator U_ψ and its adjoint operator V_ψ on Triebel-Lizorkin-type spaces and their predual spaces. Recall that, for a fixed function $\psi : [0, 1] \rightarrow [0, \infty)$, the weighted Hardy operator U_ψ is defined by

$$U_\psi f(x) = \int_0^1 f(tx)\psi(t)dt, \quad x \in \mathbb{R}^n, \quad (1.1)$$

see Carton-Lebrun and Fosset [1]. Accordingly, the adjoint operator V_ψ of U_ψ , named by the weighted Cesàro average operator, is defined by

$$V_\psi f(x) = \int_0^1 f\left(\frac{x}{t}\right)t^{-n}\psi(t)dt, \quad x \in \mathbb{R}^n. \quad (1.2)$$

This paper is motivated by the following facts. On one hand, U_ψ is related to the Hardy-Littlewood maximal operators in harmonic analysis. If $\psi \equiv 1$ and $n = 1$, then U_ψ goes

back to the classical Hardy-Littlewood average U :

$$Uf(x) = \frac{1}{x} \int_0^x f(y) dy, \quad x \neq 0. \quad (1.3)$$

Its adjoint operator V_ψ is the classical Cesàro average operator:

$$Vf(x) = \begin{cases} \int_x^\infty \frac{f(y)}{y} dy, & x > 0, \\ -\int_{-\infty}^x \frac{f(y)}{y} dy, & x < 0. \end{cases} \quad (1.4)$$

Xiao in [2] obtained the boundedness of U_ψ on $BMO(\mathbb{R}^n)$, the boundedness of V_ψ on $H^1(\mathbb{R}^n)$, and their corresponding operator norms. Recall that $Q_\alpha(\mathbb{R}^n)$ can be viewed as a generalization of $BMO(\mathbb{R}^n)$ (cf. [3]). Recently Tang and Zhai in [4] obtained the boundedness and norm of U_ψ on $Q_\alpha(\mathbb{R}^n)$. They even work to more general spaces $Q_p^{\alpha,q}(\mathbb{R}^n)$. The boundedness of U_ψ and V_ψ on $Q_p^{\alpha,q}(\mathbb{R}^n)$ and its dual space are worked out. Recall that Q spaces (Q_α spaces) were originally defined by Aulaskari et al. [5] in 1995 as spaces of holomorphic functions on the unit disk, which are geometric in the sense that they transform naturally under conformal mappings, and later, in 2000, were extended to the n -dimensional Euclidean spaces $\mathbb{R}^n$ by Essén et al. By [3, Theorem 2.3], $Q_\alpha(\mathbb{R}^n)$ is always a subspace of $BMO(\mathbb{R}^n)$. As a generation of Q spaces, the spaces $Q_p^{\alpha,q}(\mathbb{R}^n)$ when $\alpha \in (0, 1)$ and $2 \leq q < p < \infty$ were first introduced by Cui and Yang [6] and later extended to $\alpha \in (0, 1)$, $p \in (0, \infty]$, and $q \in [1, \infty]$ in [7]. We also refer to [4, 8–11] for more studies on these spaces.

On the other hand, it is well known that Besov spaces $\dot{B}_{p,q}^\alpha(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $\dot{F}_{p,q}^\alpha(\mathbb{R}^n)$ allow a unified approach to various types of function spaces, such as Hölder-Zygmund spaces, Sobolev spaces, Hardy spaces, BMO, and Bessel-potential spaces. Over hundreds of papers and books are focused on these spaces and their applications. Very recently, Yang and Yuan [7, 12] introduced the scales of homogeneous Besov-Triebel-Lizorkin-type spaces $\dot{B}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$ for all $\alpha \in \mathbb{R}$, $\tau \in [0, \infty)$, and $p, q \in (0, \infty]$. These spaces unify and generalize the homogeneous Besov spaces $\dot{B}_{p,q}^\alpha(\mathbb{R}^n)$, Triebel-Lizorkin spaces $\dot{F}_{p,q}^\alpha(\mathbb{R}^n)$, and the spaces $Q_\alpha(\mathbb{R}^n)$ simultaneously. The following equivalent definition of the spaces $\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$ were given in [7, Section 3].

Definition 1.1. Let $\alpha \in (0, 1)$, $p \in (0, \infty)$, $q \in (1, \infty)$, and $\tau \in [0, \infty]$. The space $\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$ is defined to be the space of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)} = \sup \frac{1}{|I|^\tau} \left\{ \int_I \left(\int_{|x-y|<2l(I)} \frac{|f(x) - f(y)|^q}{|x-y|^{n+q\alpha}} dy \right)^{p/q} dx \right\}^{1/p} < \infty, \quad (1.5)$$

where the supremum is taken over all cubes I with the edges parallel to the coordinate axes in $\mathbb{R}^n$.

Motivated by the above facts, a natural question is following: under what condition on ψ the weighted Hardy operator and its adjoint are bounded on Triebel-Lizorkin-type spaces

$\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$? The main purpose of this paper is to give the sufficient and necessary condition on ψ such that the operators U_ψ and V_ψ are bounded on Triebel-Lizorkin-type spaces $\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$. We have the following main results.

Theorem 1.2. *Let $\psi : [0, 1] \rightarrow [0, \infty)$ be a function, and let $p \in [1, \infty]$, $q \in [1, \infty)$, $\tau \in (0, \infty)$, and $\alpha \in (0, 1)$ such that $\tau > 1/p - 1/q$ and $\alpha \leq n(1/p - \tau)$. Then $U_\psi : \dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n) \rightarrow \dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$ exists as a bounded operator if*

$$\int_0^1 t^{-n(1/p-\tau)+\alpha} \psi(t) dt < \infty. \quad (1.6)$$

Moreover, when (1.6) holds, the norm of U_ψ on $\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$ is given by

$$\|U_\psi\|_{\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n) \rightarrow \dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)} = \int_0^1 t^{-n(1/p-\tau)+\alpha} \psi(t) dt. \quad (1.7)$$

We give the proof of Theorem 1.2 in Section 2.

Finally, we make some conventions on notations. Throughout the paper, $\mathbb{R}^n$ denotes the n -dimensional Euclidean space, with Euclidean norm $|x|$, and Lebesgue measure dx . A cube I will always mean a cube in $\mathbb{R}^n$ with the edges parallel to the coordinate axes with sidelength $l(I)$ and volume $|I|$. The dilated cube λI , $\lambda > 0$, is the cube with the same center as I and sidelength $\lambda l(I)$. For all $q \in (0, \infty]$, q' denotes the adjoint index of q , namely, $1/q + 1/q' = 1$. C will often be used to denote a positive constant, but it may vary from line to line.

2. Proof of Main Results

We begin with the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose (1.6) holds. If $f \in \dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$, then for any cube $I \subset \mathbb{R}^n$, applying Minkowski's inequality, we have

$$\begin{aligned} & \left\{ \int_I \left(\int_{|x-y|<2l(I)} \frac{|U_\psi f(x) - U_\psi f(y)|^q}{|x-y|^{n+q\alpha}} dy \right)^{p/q} dx \right\}^{1/p} \\ &= \left\{ \int_I \left(\int_{|x-y|<2l(I)} \frac{\left| \int_0^1 [f(tx) - f(ty)] \psi(t) dt \right|^q}{|x-y|^{n+q\alpha}} dy \right)^{p/q} dx \right\}^{1/p} \\ &\leq \left\{ \int_I \left[\int_0^1 \left(\int_{|x-y|<2l(I)} \frac{|f(tx) - f(ty)|^q}{|x-y|^{n+q\alpha}} dy \right)^{1/q} \psi(t) dt \right]^p dx \right\}^{1/p} \\ &\leq \int_0^1 \left\{ \int_I \left(\int_{|x-y|<2l(I)} \frac{|f(tx) - f(ty)|^q}{|x-y|^{n+q\alpha}} dy \right)^{p/q} dx \right\}^{1/p} \psi(t) dt. \end{aligned} \quad (2.1)$$

Notice that

$$\begin{aligned} & \int_I \left(\int_{|x-y| < 2l(I)} \frac{|f(tx) - f(ty)|^q}{|x-y|^{n+q\alpha}} dy \right)^{p/q} dx \\ &= \int_{tI} \left(\int_{|u-v| < 2l(tI)} \frac{|f(u) - f(v)|^q}{|u-v|^{n+q\alpha}} dv \right)^{p/q} du \cdot t^{p\alpha-n}. \end{aligned} \quad (2.2)$$

Then, by (1.6), we have

$$\begin{aligned} & \frac{1}{|I|^\tau} \left\{ \int_I \left(\int_{|x-y| < 2l(I)} \frac{|U_\psi f(x) - U_\psi f(y)|^q}{|x-y|^{n+q\alpha}} dy \right)^{p/q} dx \right\}^{1/p} \\ & \leq \frac{1}{|tI|^\tau} \int_0^1 \left\{ \int_{tI} \left(\int_{|u-v| < 2l(tI)} \frac{|f(u) - f(v)|^q}{|u-v|^{n+q\alpha}} dv \right)^{p/q} du \right\}^{1/p} t^{\alpha-n/p+n\tau} \psi(t) dt \\ & \leq \|f\|_{\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)} \int_0^1 t^{\alpha-n(1/p-\tau)} \psi(t) dt. \end{aligned} \quad (2.3)$$

Thus, the operator U_ψ is bounded on $\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$ with operator norm no more than

$$\int_0^1 t^{\alpha-n(1/p-\tau)} \psi(t) dt. \quad (2.4)$$

Conversely, if U_ψ is bounded on $\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$, then we can choose the function

$$f_0(x) = \begin{cases} -|x|^{-n(1/p-\tau)+\alpha}, & x \in \mathbb{R}_l^n, \\ |x|^{-n(1/p-\tau)+\alpha}, & x \in \mathbb{R}_r^n, \end{cases} \quad (2.5)$$

where $\mathbb{R}_l^n$ and $\mathbb{R}_r^n$ denote, respectively, the left and right halves of $\mathbb{R}^n$, separated by the hyperplane $x_1 = 0$ (x_1 is the first coordinate of $x \in \mathbb{R}^n$).

We now show that $0 < \|f_0\|_{\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)} < \infty$. Notice that the assumption on α implies that $\tau < 1/p$. For any cube I in $\mathbb{R}^n$, we consider two cases.

Case 1. If $p < q$, then $p/q < 1$. Since $\tau > 1/p - 1/q$, then $\alpha > 0 > n(1/p - 1/q) - n\tau$, and hence

$$\begin{aligned} & \int_{I+h} |f_0(z)|^q dz \\ & \leq \int_{I \cap \{|z| \leq |h|\}} |z|^{[-n(1/p-\tau)+\alpha]q} dz + \int_{I \cap \{|z| > |h|\}} |z|^{[-n(1/p-\tau)+\alpha]q} dz \\ & \leq |h|^{[-n(1/p-\tau)+\alpha]q+n} + |h|^{[-n(1/p-\tau)+\alpha]q} |I|, \end{aligned} \quad (2.6)$$

which together with Hölder's inequality, $\tau > 1/p - 1/q$, yields that

$$\begin{aligned}
& \int_I \left(\int_{|h| < 2l(I)} \frac{|f_0(x+h)|^q}{|h|^{n+q\alpha}} dh \right)^{p/q} dx \\
& \leq \left\{ \int_I \left(\int_{|h| < 2l(I)} \frac{|f_0(x+h)|^q}{|h|^{n+q\alpha}} dh \right)^{p/q} dx \right\}^{1-p/q} |I|^{1-p/q} \\
& = \left\{ \int_{|h| < 2l(I)} \int_{I+h} |f_0(z)|^q dz \frac{1}{|h|^{n+q\alpha}} dh \right\}^{p/q} |I|^{1-p/q} \\
& \leq \left\{ \int_{|h| < 2l(I)} \frac{|h|^{[-n(1/p-\tau)+\alpha]q+n} + |h|^{[-n(1/p-\tau)+\alpha]q}|I|}{|h|^{n+q\alpha}} dh \right\}^{p/q} |I|^{1-p/q} \\
& \leq \left\{ \int_{|h| < 2l(I)} \frac{|h|^{[-n(1/p-\tau)+\alpha]q+n} + |h|^{[-n(1/p-\tau)+\alpha]q}|I|}{|h|^{n+q\alpha}} dh \right\}^{p/q} |I|^{1-p/q} \\
& \leq Cl(I)^{n\tau p}.
\end{aligned} \tag{2.7}$$

Case 2. If $p \geq q$, then $p/q \geq 1$. Using Minkowski's inequality, similar to the computation of Case 1, we also obtain

$$\begin{aligned}
& \left\{ \int_I \left(\int_{|h| < 2l(I)} \frac{|f_0(x+h)|^q}{|h|^{n+q\alpha}} dh \right)^{p/q} dx \right\}^{1/p} \\
& \leq \left\{ \int_{|h| < 2l(I)} \left[\int_I \left(\frac{|f_0(x+h)|^q}{|h|^{n+q\alpha}} \right)^{p/q} dx \right]^{q/p} dh \right\}^{1/q} \\
& \leq Cl(I)^{n\tau}.
\end{aligned} \tag{2.8}$$

Combining Cases 1 and 2 yields that

$$\begin{aligned}
& \frac{1}{|I|^\tau} \left\{ \int_I \left(\int_{|h| < 2l(I)} \frac{|f_0(x+h) - f_0(x)|^q}{|h|^{n+q\alpha}} dh \right)^{p/q} dx \right\}^{1/p} \\
& \leq \frac{1}{|I|^\tau} \left\{ \int_I \left(\int_{|h| < 2l(I)} \frac{|f_0(x+h)|^q + |f_0(x)|^q}{|h|^{n+q\alpha}} dh \right)^{p/q} dx \right\}^{1/p} \\
& \leq C,
\end{aligned} \tag{2.9}$$

namely, $f_0 \in \dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$ and $0 \neq \|f_0\|_{\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)} < \infty$.

Since

$$U_\psi f_0(x) = f_0(x) \int_0^1 t^{n(1/p-\tau)+\alpha} \psi(t) dt, \tag{2.10}$$

we then have

$$\|U_\psi\|_{\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n) \rightarrow \dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)} = \int_0^1 t^{-(n/p-\tau)+\alpha} \psi(t) dt, \quad (2.11)$$

which completes the proof of Theorem 1.2. $\square$

Remark 2.1. In fact, if we choose $p = q$ and $\tau = 1/p - 1/r$, then

$$\int_0^1 t^{\alpha-n(1/p-\tau)} \psi(t) dt = \int_0^1 t^{\alpha-n/r} \psi(t) dt. \quad (2.12)$$

Therefore, Theorem 1.2 is consistent with [4, Theorem 4.1] since $Q_r^{\alpha,p}(\mathbb{R}^n) = \dot{F}_{p,p}^{\alpha,1/p-1/r}(\mathbb{R}^n)$.

The weighted Hardy operator U_ψ and the weighted Cesàro average operator V_ψ are adjoint mutually, namely,

$$\int_{\mathbb{R}^n} g(x) U_\psi f(x) dx = \int_{\mathbb{R}^n} f(x) V_\psi g(x) dx. \quad (2.13)$$

Recall that

$$V_\psi f(x) = \int_0^1 f\left(\frac{x}{t}\right) t^{-n} \psi(t) dt, \quad x \in \mathbb{R}^n, \quad (2.14)$$

then we have

$$\begin{aligned} & \left\{ \int_I \left(\int_{|x-y|<2l(I)} \frac{|V_\psi f(x) - V_\psi f(y)|^q}{|x-y|^{n+q\alpha}} dy \right)^{p/q} dx \right\}^{1/p} \\ &= \left\{ \int_I \left(\int_{|x-y|<2l(I)} \frac{\left| \int_0^1 [f(x/t) - f(y/t)] t^{-n} \psi(t) dt \right|^q}{|x-y|^{n+q\alpha}} dy \right)^{p/q} dx \right\}^{1/p} \\ &\leq \left\{ \int_I \left[\int_0^1 \left(\int_{|x-y|<2l(I)} \frac{|f(x/t) - f(y/t)|^q}{|x-y|^{n+q\alpha}} dy \right)^{1/q} t^{-n} \psi(t) dt \right]^p dx \right\}^{1/p} \\ &\leq \int_0^1 \left\{ \int_I \left(\int_{|x-y|<2l(I)} \frac{|f(x/t) - f(y/t)|^q}{|x-y|^{n+q\alpha}} dy \right)^{p/q} dx \right\}^{1/p} t^{-n} \psi(t) dt, \end{aligned} \quad (2.15)$$

where

$$\begin{aligned} & \int_I \left(\int_{|x-y| < 2l(I)} \frac{|f(x/t) - f(y/t)|^q}{|x-y|^{n+q\alpha}} dy \right)^{p/q} dx \\ &= \int_{(1/t)I} \left(\int_{|u-v| < 2l((1/t)I)} \frac{|f(u) - f(v)|^q}{|u-v|^{n+q\alpha}} dv \right)^{p/q} du \cdot t^{-p\alpha+n}. \end{aligned} \quad (2.16)$$

Similar to the proof of Theorem 1.2, we have next theorem.

Theorem 2.2. *Let $\psi : [0, 1] \rightarrow [0, \infty)$ be a function, and let $p \in [1, \infty]$, $q \in [1, \infty)$, $\tau \in (0, \infty)$, and $\alpha \in (0, 1)$ such that $\tau > 1/p - 1/q$ and $\alpha \leq n(1/p - \tau)$. Then $V_\psi : \dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n) \rightarrow \dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$ exists as a bounded operator if*

$$\int_0^1 t^{-n(1-1/p+\tau)-\alpha} \psi(t) dt < \infty. \quad (2.17)$$

Moreover, when (2.17) holds, the norm of V_ψ on $\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$ is given by

$$\|V_\psi\|_{\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n) \rightarrow \dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)} = \int_0^1 t^{-n(1-1/p+\tau)-\alpha} \psi(t) dt. \quad (2.18)$$

Recall that the Triebel-Lizorkin-Hausdorff spaces $F\dot{H}_{p',q'}^{-\alpha,\tau}(\mathbb{R}^n)$ were originally introduced in [7, 12] and proved therein to be the predual spaces of Triebel-Lizorkin-type spaces $\dot{F}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)$. Triebel-Lizorkin-Hausdorff spaces unify and generalize Triebel-Lizorkin spaces (see, e.g., [13]) and Hardy-Hausdorff spaces $HH_{-\alpha}^1$ in [14]. These spaces were further studied in [15].

By dual argument, we have the following theorems.

Theorem 2.3. *Let $\psi : [0, 1] \rightarrow [0, \infty)$ be a function, and let $p \in [1, \infty]$, $q \in (1, \infty]$, $\tau \in (0, 1/(\max\{p, q\})')$, and $\alpha \in (0, 1)$ such that $\tau > 1/p' - 1/q'$ and $\alpha \leq n(1/p' - \tau)$. Then $U_\psi : F\dot{H}_{p,q}^{-\alpha,\tau}(\mathbb{R}^n) \rightarrow F\dot{H}_{p,q}^{-\alpha,\tau}(\mathbb{R}^n)$ exists as a bounded operator if*

$$\int_0^1 t^{-n(1/p'-\tau)-\alpha} \psi(t) dt < \infty. \quad (2.19)$$

Moreover, when (2.19) holds, the norm of U_ψ on $F\dot{H}_{p,q}^{-\alpha,\tau}(\mathbb{R}^n)$ is given by

$$\|U_\psi\|_{F\dot{H}_{p,q}^{-\alpha,\tau}(\mathbb{R}^n) \rightarrow F\dot{H}_{p,q}^{-\alpha,\tau}(\mathbb{R}^n)} = \int_0^1 t^{-n(1/p'-\tau)-\alpha} \psi(t) dt. \quad (2.20)$$

Theorem 2.4. Let $\psi : [0, 1] \rightarrow [0, \infty)$ be a function, and let $p \in [1, \infty]$, $q \in (1, \infty]$, $\tau \in (0, 1/(\max\{p, q\})']$, and $\alpha \in (0, 1)$ such that $\tau > 1/p' - 1/q'$ and $\alpha \leq n(1/p' - \tau)$. Then $V_\psi : F\dot{H}_{p,q}^{-\alpha,\tau}(\mathbb{R}^n) \rightarrow F\dot{H}_{p,q}^{-\alpha,\tau}(\mathbb{R}^n)$ exists as a bounded operator if

$$\int_0^1 t^{-n(1-1/p'+\tau)-\alpha} \psi(t) dt < \infty. \quad (2.21)$$

Moreover, when (2.21) holds, the norm of V_ψ on $F\dot{H}_{p,q}^{-\alpha,\tau}(\mathbb{R}^n)$ is given by

$$\|V_\psi\|_{F\dot{H}_{p,q}^{-\alpha,\tau}(\mathbb{R}^n) \rightarrow F\dot{H}_{p,q}^{-\alpha,\tau}(\mathbb{R}^n)} = \int_0^1 t^{-n(1-1/p'+\tau)-\alpha} \psi(t) dt. \quad (2.22)$$

Remark 2.5. When $p' = q'$ and $\tau = 1/q' - 1/r$, then

$$\int_0^1 t^{-n(1-1/p'+\tau)-\alpha} \psi(t) dt = \int_0^1 t^{-n(1-1/r)-\alpha} \psi(t) dt, \quad (2.23)$$

and Theorem 2.4 is consistent with [4, Theorem 5.8] since $F\dot{H}_{q',q'}^{-\alpha,1/q'-1/r}(\mathbb{R}^n)$ is predual space of $Q_r^{\alpha,q}(\mathbb{R}^n)$.

Our Theorems actually induce the following results:

$$\begin{aligned} \|V_\psi\|_{F\dot{H}_{p',q'}^{-\alpha,\tau}(\mathbb{R}^n) \rightarrow F\dot{H}_{p',q'}^{-\alpha,\tau}(\mathbb{R}^n)} &= \|U_\psi\|_{F\dot{H}_{p,q}^{\alpha,\tau}(\mathbb{R}^n) \rightarrow F\dot{H}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)}, \\ \|U_\psi\|_{F\dot{H}_{p',q'}^{-\alpha,\tau}(\mathbb{R}^n) \rightarrow F\dot{H}_{p',q'}^{-\alpha,\tau}(\mathbb{R}^n)} &= \|V_\psi\|_{F\dot{H}_{p,q}^{\alpha,\tau}(\mathbb{R}^n) \rightarrow F\dot{H}_{p,q}^{\alpha,\tau}(\mathbb{R}^n)}. \end{aligned} \quad (2.24)$$

In particular,

$$\begin{aligned} \|U_\psi\|_{F\dot{H}_{q',q'}^{-\alpha,1/q'-1/p}(\mathbb{R}^n) \rightarrow F\dot{H}_{q',q'}^{-\alpha,1/q'-1/p}(\mathbb{R}^n)} &= \|V_\psi\|_{Q_p^{\alpha,q}(\mathbb{R}^n) \rightarrow Q_p^{\alpha,q}(\mathbb{R}^n)}, \\ \|V_\psi\|_{F\dot{H}_{q',q'}^{-\alpha,1/q'-1/p}(\mathbb{R}^n) \rightarrow F\dot{H}_{q',q'}^{-\alpha,1/q'-1/p}(\mathbb{R}^n)} &= \|U_\psi\|_{Q_p^{\alpha,q}(\mathbb{R}^n) \rightarrow Q_p^{\alpha,q}(\mathbb{R}^n)}. \end{aligned} \quad (2.25)$$

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