Research Article

Smoothness and Function Spaces Generated by Homogeneous Multipliers

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Differential operators generated by homogeneous functions φ of an arbitrary real order $s > 0$ (φ -derivatives) and related spaces of s -smooth periodic functions of d variables are introduced and systematically studied. The obtained scale is compared with the scales of Besov and Triebel-Lizorkin spaces. Explicit representation formulas for φ -derivatives are obtained in terms of the Fourier transform of their generators. Some applications to approximation theory are discussed.

1. Introduction

Smoothness is one of the basic concepts of analysis, having a long history. A fundamental observation is its strong connection with the decay of the Fourier coefficients of a given function. This paved the way to apply methods of Fourier analysis to the further development of smoothness concepts. Let us consider two directions of the Fourier analytic approach to describe the differentiability properties of functions. The first one is related to the scales of Besov spaces $B_{p,q}^s$ and Triebel-Lizorkin spaces $F_{p,q}^s$ which are constructed by means of decomposition of the Fourier series into dyadic blocks with the help of an appropriate resolution of unity (see, e.g., [1–5] for periodic and nonperiodic setting). The second direction is based on the interpretation of a derivative as an operator of multiplier type. In this case one also deals with the Fourier coefficients, but not with decomposition into blocks. Following this way the concept of classical derivative was essentially extended to fractional derivatives such as Riesz and Weyl derivatives (see, e.g., [6]) and later on to the concept of generalized derivatives and the corresponding scale of the Stepanets classes in the one-dimensional case (see [7, 8]).

Both the theory of function spaces as it has been developed by the Russian (S. M. Nikol'skij) or German (H. Triebel) school and the theory of generalized smoothness elaborated by Stepanets and his coworkers have found many applications in various fields of modern mathematics. More precisely, the first direction is mainly applied to the theory of (partial) differential equations, computational mathematics, stochastic processes, and fractal and nonlinear analysis. The second one turned out to be important for many problems of approximation theory, in particular, constructing optimal linear approximation methods on various classes of smooth functions and obtaining approximation relations with (asymptotically) sharp constants.

It is well known (see [5], Ch. 3, or [4], Ch. 2, in the nonperiodic case), for $1 < p < +\infty$ and $s > 0$, that the space $F_{p,2}^s$ coincides with the (fractional) Sobolev H_p^s which is related to the operator $(I - \Delta)^{s/2}$, where Δ is the Laplace operator and I is the identity operator. Here "relation to operator" means that H_p^s consists of periodic functions f in L_p such that $(I - \Delta)^{s/2}f$ belongs to L_p as well. In other words, we have $H_p^s = (I - \Delta)^{-s/2}(L_p)$ which means that H_p^s is characterized as the image of L_p by the operator $(I - \Delta)^{-s/2}$. With this exception both Besov spaces $B_{p,q}^s$ and Triebel-Lizorkin spaces $F_{p,q}^s$ are not related to any operator acting on L_p in general. For this reason the scales of Besov and Triebel-Lizorkin are not well adapted to the study of problems which are directly connected with concrete operators. In contrast to these function spaces, the classes of Stepanets are related to certain operators of multiplier type which are connected to the so-called (ψ, β) -derivatives; see, for example [7]. In Stepanets' theory the one-dimensional case is considered only. The smoothness generator ψ is an arbitrary function satisfying some natural conditions. On the one hand, it gives quite a lot of freedom and allows the description of many interesting properties of functions which are smooth in this sense. On the other hand, such a "poor" information on ψ does not enable us to obtain explicit representation formulas for related derivatives in terms of the functions under consideration itself in place of their Fourier coefficients. This fact prevents the application of this approach to many important problems of numerical approximation.

In the present paper, we will introduce and study operators $\mathfrak{D}(\psi)$ of multiplier type generated by homogeneous functions and related spaces $X_p(\psi)$ of periodic functions of d variables. On the one hand, homogeneity seems to be a rather general assumption. Practically all known differential operators as, for instance, the classical derivatives, Weyl and Riesz derivatives, mixed derivatives, and the Laplace operator and its (fractional) powers are generated by homogeneous multipliers. On the other hand, taking into account that the Fourier transform (in the sense of distributions) of a homogeneous function of order s is also a homogeneous function of order $-(d + s)$ (see, e.g., [9, Theorem 7.1.6]), one can derive quite substantial statements concerning the corresponding operators and related function spaces. In particular, we will prove that there is unique space $X_p(\psi)$ which coincides with the fractional Sobolev space H_p^s if ψ is homogeneous of degree $s > 0$ and $1 < p < +\infty$ (Theorems 3.1 and 4.1). However, we get infinitely many new spaces in the cases $p = 1$ and $p = +\infty$ (Theorem 3.4). Moreover, we find an explicit representation formula for $\mathfrak{D}(\psi)f$ for functions f belonging to the periodic Besov space $B_{p,1}^s$ if ψ is homogeneous of degree s , $0 < s < 1$ (Theorem 5.1).

Let us also mention that it is aimed to extend the approach to generalized smoothness based on the introduction of operators $\mathfrak{D}(\psi)$ and spaces $X_p(\psi)$ in further works by

- (i) finding representation formulae for operators generated by homogeneous functions of degree $s \geq 1$ using higher-order differences,

- (ii) studying generalized K -functionals and their realizations,
- (iii) constructing new moduli of smoothness related to $\mathfrak{D}(\psi)$,
- (iv) studying the same concept in nonperiodic case,
- (v) investigating generalized differential equations.

The paper is organized as follows. Section 2 deals with notations and preliminaries. The basic properties of operators $\mathfrak{D}(\psi)$ and spaces $X_p(\psi)$ are described in Section 3. Some relations between spaces $X_p(\psi)$ and Besov and Triebel-Lizorkin spaces are discussed in Section 4. Finally, Section 5 is devoted to the derivation of an explicit formula for $\mathfrak{D}(\psi)$ in terms of the Fourier transform of the generator ψ .

2. Notations and Preliminaries

2.1. Numbers and Vectors

By the symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}_0^d$, $\mathbb{Z}^d$, and $\mathbb{R}^d$ we denote the sets of natural, nonnegative integer, integer, real and complex numbers, d -dimensional vectors with non-negative integer, integer and real components, respectively. The symbol $\mathbb{T}^d$ is reserved for the d -dimensional torus $[0, 2\pi)^d$. We will also use the notations

$$xy = x_1y_1 + \cdots + x_dy_d, \quad |x| \equiv |x|_2 = \left(x_1^2 + \cdots + x_d^2\right)^{1/2}, \quad (2.1)$$

for the scalar product and the l_2 -norm of vectors and

$$B_r = \{x \in \mathbb{R}^d : |x| < r\}, \quad \bar{B}_r = \{x \in \mathbb{R}^d : |x| \leq r\} \quad (2.2)$$

for the open and closed balls, respectively.

2.2. L_p -Spaces

As usual, $L_p \equiv L_p(\mathbb{T}^d)$, where $1 \leq p < +\infty$, $\mathbb{T}^d = [0, 2\pi)^d$, is the space of measurable real-valued functions $f = f(x)$, $x = (x_1, \dots, x_d)$ which are 2π -periodic with respect to each variable satisfying

$$\|f\|_p = \left(\int_{\mathbb{T}^d} |f(x)|^p dx\right)^{1/p} < +\infty. \quad (2.3)$$

In the case $p = +\infty$, we consider the space $C \equiv C(\mathbb{T}^d)$ ($p = +\infty$) of real-valued 2π -periodic continuous functions equipped with the Chebyshev norm

$$\|f\|_\infty = \max_{x \in \mathbb{T}^d} |f(x)|. \quad (2.4)$$

For L_p -spaces of non-periodic functions defined on a measurable set $\Omega \subseteq \mathbb{R}^d$ we will use the notation $L_p(\Omega)$.

2.3. Fourier Coefficients and Fourier Transform

The Fourier coefficients of $f \in L_1$ are defined by

$$f^\wedge(k) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x) e^{-ikx} dx, \quad k \in \mathbb{Z}^d. \quad (2.5)$$

Let $\mathcal{E}_{\text{per}}$ and $\mathcal{E}'_{\text{per}}$ be the space of infinitely differentiable periodic functions and its dual the space of periodic distributions, respectively. The Fourier coefficients of $f \in \mathcal{E}'_{\text{per}}$ are given by

$$f^\wedge(k) = (2\pi)^{-d} \langle f, e^{-ik\cdot} \rangle, \quad k \in \mathbb{Z}^d, \quad (2.6)$$

where, as usual, $\langle f, g \rangle$ means the value of the functional f at $g \in \mathcal{E}_{\text{per}}$.

The Fourier transform and its inverse are given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx, \quad f^\vee(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} f(\xi) e^{ix\xi} d\xi, \quad f \in L_1(\mathbb{R}^d). \quad (2.7)$$

For an element f belonging to the space of tempered distributions $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$, which is the dual of the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing infinitely differentiable functions, the Fourier transform is defined by setting

$$\langle \hat{f}, g \rangle = \langle f, \hat{g} \rangle, \quad g \in \mathcal{S}. \quad (2.8)$$

2.4. Trigonometric Polynomials

Let σ be a real nonnegative real number. By $\mathcal{T}_\sigma$ we denote the space of all real-valued trigonometric polynomials of (spherical) order σ . It means

$$\mathcal{T}_\sigma = \left\{ t(x) = \sum_{|k| \leq \sigma} c_k e^{ikx} : c_{-k} = \overline{c_k}, |k| \leq \sigma \right\}, \quad (2.9)$$

where $\bar{c}$ is the complex conjugate to $c \in \mathbb{C}$. Further, $\mathcal{T}$ stands for the space of all real-valued trigonometric polynomials of arbitrary order. Let $1 \leq p \leq +\infty$. As usual, we put

$$E_\sigma(f)_p = \inf_{t \in \mathcal{T}_\sigma} \|f - t\|_p, \quad \sigma > 0, \quad (2.10)$$

for the best approximation of f in L_p ($f \in C$ if $p = +\infty$) by trigonometric polynomials of order at most σ in the metric of L_p .

2.5. Homogeneous Functions

A complex-valued function ψ defined on $\mathbb{R}^d \setminus \{0\}$ is called homogeneous of order $s \in \mathbb{R}$ if

$$\psi(t\xi) = t^s \psi(\xi) \quad (2.11)$$

for $t > 0$ and $\xi \in \mathbb{R}^d \setminus \{0\}$. An element ψ of the space $\mathcal{S}'$ is called homogeneous distribution of order s (see, e.g., [9, Def. 3.2.2, page 74]) if for any $t > 0$

$$\langle \psi, g(t \cdot) \rangle = t^{-(s+d)} \langle \psi, g \rangle, \quad g \in \mathcal{S}. \quad (2.12)$$

It is well known (see, e.g., [9, Theorem 7.1.16, page 167]) that the Fourier transform of a homogeneous distribution ψ of order s is also a homogeneous distribution of order $-(s + d)$.

Let now $s > 0$. By the symbol $\mathcal{H}_s$ we denote the class of functions ψ satisfying the following conditions:

- (i) ψ is continuous on $\mathbb{R}^d$ and complex valued;
- (ii) ψ is infinitely differentiable on $\mathbb{R}^d \setminus \{0\}$;
- (iii) ψ is homogeneous of order s ;
- (iv) $\psi(-\xi) = \overline{\psi(\xi)}$ for each $\xi \in \mathbb{R}^d$;
- (v) $\psi(\xi) \neq 0$ for $\xi \in \mathbb{R}^d \setminus \{0\}$.

It follows that $\psi \in \mathcal{S}'$ and that the restriction of $\widehat{\psi}$ to $\mathbb{R}^d \setminus \{0\}$ belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$ (see [9, Theorem 7.1.18, page 168]).

2.6. Periodic Besov and Triebel-Lizorkin Spaces

The Fourier analytical definition is based on dyadic resolutions of unity (see, e.g., [5, Chapter 3], or [4] for the nonperiodic case). Let φ_0 be a real-valued centrally symmetric ($\varphi_0(-\xi) = \varphi_0(\xi)$ for all $\xi \in \mathbb{R}^d$) infinitely differentiable function satisfying

$$\varphi_0(\xi) = \begin{cases} 1, & |\xi| \leq \frac{1}{2}, \\ 0, & |\xi| \geq 1. \end{cases} \quad (2.13)$$

We put

$$\theta(\xi) = \varphi_0(\xi) - \varphi_0(2\xi), \quad \varphi_j(\xi) = \theta(2^{-j}\xi), \quad j \in \mathbb{N}. \quad (2.14)$$

Clearly, these functions are also centrally symmetric and infinitely differentiable with compact support. We have

$$\text{supp } \theta \subset \frac{\overline{B_1}}{B_{1/4}}; \quad \text{supp } \varphi_j \subset \frac{\overline{B_{2^j}}}{B_{2^{j-2}}}, \quad j \in \mathbb{N}. \quad (2.15)$$

By (2.14) we obtain

$$\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi), \quad j \in \mathbb{N}. \quad (2.16)$$

Combining (2.13) and (2.16), one has

$$\sum_{j=0}^{+\infty} \varphi_j(\xi) = 1, \quad \xi \in \mathbb{R}^d. \quad (2.17)$$

In view of (2.15) and (2.17), the system $\Phi = \{\varphi_j\}_{j=0}^{+\infty}$ is called a smooth dyadic resolution of unity.

Let $s > 0$, $1 \leq p \leq +\infty$, and $1 \leq q \leq +\infty$. The periodic Besov space $B_{p,q}^s$ and the periodic Triebel-Lizorkin space $F_{p,q}^s$ are given by (cf., [5, Chapter 3])

$$B_{p,q}^s = \left\{ f \in L_p : \|f\|_{B_{p,q}^s} \equiv \left(\sum_{j=0}^{+\infty} 2^{jsq} \|f_j\|_p^q \right)^{1/q} < +\infty \right\} \quad (2.18)$$

if $q < \infty$,

$$B_{p,\infty}^s = \left\{ f \in L_p : \|f\|_{B_{p,\infty}^s} \equiv \sup_{j \in \mathbb{N}_0} 2^{js} \|f_j\|_p < +\infty \right\}, \quad (2.19)$$

and by

$$F_{p,q}^s = \left\{ f \in L_p : \|g\|_{F_{p,q}^s} \equiv \left\| \left(\sum_{j=0}^{+\infty} 2^{jsq} |f_j(x)|^q \right)^{1/q} \right\|_p < +\infty \right\} \quad (2.20)$$

if $p < +\infty, q < +\infty$,

$$F_{p,\infty}^s = \left\{ f \in L_p : \|f\|_{F_{p,\infty}^s} \equiv \left\| \sup_{j \in \mathbb{N}_0} 2^{js} |f_j(x)| \right\|_p < +\infty \right\}, \quad (2.21)$$

if $p < +\infty$, where the function $f_j, j \in \mathbb{N}_0$, are defined by

$$f_j(x) = \sum_{k \in \mathbb{Z}^d} \varphi_j(k) f^\wedge(k) e^{ikx}, \quad x \in \mathbb{R}^d, \quad j \in \mathbb{N}_0, \quad (2.22)$$

and $\varphi_j, j \in \mathbb{N}_0$, are given by (2.13) and (2.14). It is well known that Definitions (2.18)–(2.21) are independent of the choice of the resolution of unity Φ . The associated norms are mutually equivalent. Therefore, we do not indicate Φ in the notation of norms and spaces. For the

details, further properties, and natural extensions to parameters $s \in \mathbb{R}$, $0 < p$, and $q \leq +\infty$, we refer to [5, Chapter 3].

2.7. Fourier Means

Let φ be a real-valued centrally symmetric continuous function with a compact support. It generates the operators $\mathcal{F}_\sigma^{(\varphi)}$ which are given by

$$\mathcal{F}_\sigma^{(\varphi)}(f; x) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(h) W_\sigma(\varphi)(x - h) dh, \quad \sigma \geq 0, \quad (2.23)$$

for $f \in L_1$. The function $\mathcal{F}_\sigma^{(\varphi)}(f)$ is called Fourier mean of f generated by φ . The functions $W_\sigma(\varphi)$ in (2.23) are defined as

$$W_0(\varphi)(h) = 1; \quad W_\sigma(\varphi)(h) = \sum_{k \in \mathbb{Z}^d} \varphi\left(\frac{k}{\sigma}\right) e^{ikx}, \quad \sigma > 0. \quad (2.24)$$

The Fourier means describe classical methods of trigonometric approximation which are well defined for functions in L_p , where $1 \leq p \leq +\infty$. They are well studied and investigated in detail in many books and papers on approximation theory; see, for example, [6, 10]. Let us also mention [5, Chapter 3], for a treatment within the framework of periodic Besov and Triebel-Lizorkin spaces. Following [11], we recall and state the following properties.

(i) The set of the norms

$$\left\| \mathcal{F}_\sigma^{(\varphi)}(f) \right\|_{(p)} = \sup_{\|f\|_p \leq 1} \left\| \mathcal{F}_\sigma^{(\varphi)} \right\|_p \quad (2.25)$$

of operators defined on L_p by (2.23) is uniformly bounded, and we have

$$\sup_{\sigma \geq 0} \left\| \mathcal{F}_\sigma^{(\varphi)} \right\|_{(p)} < +\infty, \quad (2.26)$$

for $p = 1, p = +\infty$ or for all $1 \leq p \leq +\infty$ if and only if $\hat{\varphi}$ belongs to $L_1(\mathbb{R}^d)$;

(ii) for $f \in L_1$ it holds that

$$\mathcal{F}_\sigma^{(\varphi)}(f; x) = \sum_{k \in \mathbb{Z}^d} \varphi\left(\frac{k}{\sigma}\right) f^\wedge(k) e^{ikx}, \quad \sigma > 0; \quad (2.27)$$

(iii) if $\hat{\varphi} \in L_1(\mathbb{R}^d)$ and $\varphi(0) = 1$, then the Fourier means $\mathcal{F}_\sigma^{(\varphi)}$ converge in L_p for all $1 \leq p \leq +\infty$, and we have

$$\lim_{\sigma \rightarrow +\infty} \left\| f - \mathcal{F}_\sigma^{(\varphi)}(f) \right\|_p = 0, \quad f \in L_p; \quad (2.28)$$

(iv) if $\widehat{\varphi} \in L_1(\mathbb{R}^d)$ and

$$\varphi(\xi) = \begin{cases} 1, & |\xi| \leq \rho, \\ 0, & |\xi| > \rho, \end{cases} \quad (2.29)$$

for some $\rho > 0$, then for $1 \leq p \leq +\infty$

$$\left\| f - \mathcal{F}_\sigma^{(\varphi)}(f) \right\|_p \leq c E_{\rho\sigma}(f)_p, \quad f \in L_p, \sigma \geq 0, \quad (2.30)$$

where the positive constant c does not depend on f and σ .

If θ is given by (2.14), then in view of (2.13) and (2.27) the functions f_j which are well defined for $f \in L_1$ by (2.22) can be represented as

$$f_0(x) = f^\wedge(0), \quad f_j(x) = \mathcal{F}_{2^j}^{(\theta)}(f; x), \quad j \in \mathbb{N}. \quad (2.31)$$

Moreover, using (2.16) and (2.27) we get

$$\begin{aligned} \sum_{j=0}^J f_j(x) &= \sum_{k \in \mathbb{Z}^d} \left(\sum_{j=0}^J \varphi_j(k) \right) f^\wedge(k) e^{ikx} \\ &= \sum_{k \in \mathbb{Z}^d} \varphi_0(2^{-J}k) f^\wedge(k) e^{ikx} = \mathcal{F}_{2^J}^{(\varphi_0)}(f; x) \end{aligned} \quad (2.32)$$

for $J \in \mathbb{N}$. Taking into account that $\widehat{\varphi}_0 \in L_1(\mathbb{R}^d)$, we obtain therefrom by the convergence property (2.28) of the Fourier means the decomposition

$$f = \sum_{j=0}^{+\infty} f_j, \quad f \in L_p, \quad (2.33)$$

into a series of trigonometric polynomials being convergent in the space L_p , $1 \leq p \leq +\infty$.

2.8. Inequalities of Multiplier Type for Trigonometric Polynomials

Let $\psi \in \mathcal{H}_s$ for some $s > 0$. It generates the family of operators $(A_\sigma(\psi))_\sigma$ defined by

$$A_\sigma(\psi)t(x) = \sum_{k \in \mathbb{Z}^d} \psi\left(\frac{k}{\sigma}\right) t^\wedge(k) e^{ikx}, \quad \sigma > 0, t \in \mathcal{T}, \quad (2.34)$$

on the space $\mathcal{T}$ of trigonometric polynomials. We have proved in [12, 13] that the inequality

$$\|A_\sigma(\psi_1)t\|_p \leq c \|A_\sigma(\psi_2)t\|_p, \quad (2.35)$$

where $A_\sigma(\psi_1)$ is generated by $\psi_1 \in \mathcal{H}_{s_1}$ and $A_\sigma(\psi_2)$ is generated by $\psi_2 \in \mathcal{H}_{s_2}$, is valid for all $t \in \mathcal{T}_\sigma$ and $\sigma \geq 1$ with a certain constant c independent of t and σ for all $1 \leq p \leq +\infty$ if $s_1 > s_2$. If $s_1 = s_2$, then (2.35) is valid for $1 < p < +\infty$ and for arbitrary generators ψ_1, ψ_2 . If $s_1 = s_2$ and $p = 1$ or $p = +\infty$, then the validity of (2.35) implies that the functions ψ_1 and ψ_2 are proportional.

3. ψ -Smoothness and Basic Properties

Let $\psi \in \mathcal{H}_s$ for some $s > 0$. It generates an operator $\mathfrak{D}(\psi)$ by setting

$$\mathfrak{D}(\psi)t(x) = \sum_{k \in \mathbb{Z}^d} \psi(k)t^\wedge(k)e^{ikx}, \quad t \in \mathcal{T}, \quad (3.1)$$

which is initially defined on the space of trigonometric polynomials. The domain of definition can be extended within the spaces L_p , $1 \leq p \leq +\infty$. To this end we introduce the space $X_p(\psi)$ which consists of functions f in L_p having the property that the set

$$\left\{ \psi(k)f^\wedge(k), k \in \mathbb{Z}^d \right\} \quad (3.2)$$

is the system of the Fourier coefficients of a certain function in L_p . This function will be called ψ -derivative of f in the following. By this definition we have

$$(\mathfrak{D}(\psi)f)^\wedge(k) = \psi(k)f^\wedge(k), \quad k \in \mathbb{Z}^d, \quad (3.3)$$

for the Fourier coefficients of the ψ -derivative of f . Its uniqueness follows from the well-known fact that each L_1 -function is uniquely determined by the set of its Fourier coefficients. If $f \in X_p(\psi)$, then the series in (3.1) with f in place of t converges in L_p (see the proof of Theorem 3.1). In this sense we can reformulate

$$X_p(\psi) = \left\{ f \in L_p : \mathfrak{D}(\psi)f = \sum_{k \in \mathbb{Z}^d} \psi(k)f^\wedge(k)e^{ikx} \in L_p \right\}. \quad (3.4)$$

We give some examples. Let $d = 1$, $s \in \mathbb{N}$, $1 < p < +\infty$, and $\psi(\xi) = (i\xi)^s$. Then, $\mathfrak{D}(\psi)$ is the operator of the usual derivative of order s . In this case $X_p(\psi)$ is the Sobolev space W_p^s of $(s-1)$ -times differentiable functions f such that $f^{(s-1)}$ is absolutely continuous, $f^{(s)}$ exists almost everywhere and belongs to L_p . If $d = 1$, $s \notin \mathbb{N}$, and $\psi(\xi) = (i\xi)^s = |\xi|^s e^{\text{sgn } \xi \cdot (s\pi i)/2}$, then $\mathfrak{D}(\psi)$ is the Weyl derivative $(\cdot)^{(s)}$ of fractional order s and $X_p(\psi)$ is the corresponding Weyl class. For $d = 1$ and $\psi(\xi) = |\xi|$, the operator $\mathfrak{D}(\psi)$ is the Riesz derivative $(\cdot)^{(\cdot)}$. Taking into account that it is the composition of the usual derivative of the first order and the operator of conjugation, we see that in this case $X_p(\psi)$ coincides with the space $\widetilde{W}_p^1$ of those functions where both the function itself and its conjugate belong to W_p^1 . It is well known that this space coincides with W_p^1 if and only if $1 < p < +\infty$. For more details concerning the derivatives and spaces mentioned above, we refer to [5–8].

In the multivariate case ($d > 1$) the classical Laplace operator Δ and its (fractional) power $(-\Delta)^{s/2}$, $s > 0$, are operators of type $\mathfrak{D}(\psi)$ associated with $\psi(\xi) = -|\xi|^2$ and $\psi(\xi) = |\xi|^s$, respectively. In this case $X_p(\psi) \equiv X_p((-\Delta)^{s/2})$ coincides with the periodic version of the Bessel-potential space (the (fractional) Sobolev space if $1 < p < +\infty$) considered in [14, Chapter 5] (see also the next subsection).

Now we study the properties of the spaces $X_p(\psi)$. Recall that we have to replace $L_p(\mathbb{T}^d)$ by $C(\mathbb{T}^d)$ if $p = +\infty$.

Theorem 3.1. *Let $\psi \in \mathcal{H}_s$, $s > 0$, and $1 \leq p \leq +\infty$. Then,*

(i) $X_p(\psi)$ is a Banach space with respect to the norm

$$\|f\|_{X_p(\psi)} = \|f\|_p + \|\mathfrak{D}(\psi)f\|_p; \quad (3.5)$$

(ii) $\mathcal{T}$ is dense in $X_p(\psi)$;

(iii) all spaces $X_p(\psi)$ with $\psi \in \mathcal{H}_s$ coincide and their norms are equivalent if $1 < p < +\infty$.

Proof. In order to prove (i) we only have to check completeness because all other properties of Banach space are obviously fulfilled. Let $\{f_n\}$ be a Cauchy sequence in $X_p(\psi)$. In view of (3.5)

$$\|f_n - f_m\|_p, \quad \|\mathfrak{D}(\psi)f_n - \mathfrak{D}(\psi)f_m\|_p \longrightarrow 0 \quad (n, m \longrightarrow +\infty). \quad (3.6)$$

By completeness of L_p there exist functions $f, F \in L_p$, such that

$$f_n \xrightarrow{L_p} f, \quad \mathfrak{D}(\psi)f_n \xrightarrow{L_p} F \quad (n \longrightarrow +\infty). \quad (3.7)$$

Using (3.3) and Hölder's inequality, we get

$$\begin{aligned} |F^\wedge(k) - \psi(k)f^\wedge(k)| &\leq |(F - \mathfrak{D}(\psi)f_n)^\wedge(k)| + |\psi(k)| \cdot |(f - f_n)^\wedge(k)| \\ &\leq (2\pi)^{-d/p} \left(\|F - \mathfrak{D}(\psi)f_n\|_p + |\psi(k)| \cdot \|f - f_n\|_p \right) \end{aligned} \quad (3.8)$$

for any $k \in \mathbb{Z}^d$ and $n \in \mathbb{N}$. In view of (3.7) the right-hand side of this inequality tends to 0 if $n \rightarrow +\infty$. This implies

$$F^\wedge(k) = \psi(k)f^\wedge(k), \quad k \in \mathbb{Z}^d. \quad (3.9)$$

Hence, f is ψ -differentiable, $\mathfrak{D}(\psi)f = F$, and completeness is proved.

In order to show part (ii) we consider the Fourier means $\mathcal{F}_\sigma^{(\varphi_0)}$ of type (2.23)-(2.24), where φ_0 satisfies the conditions described in Section 2 (see, in particular, (2.13)). Let $f \in X_p(\psi)$. Taking into account that

$$\mathfrak{D}(\psi) \left(\mathcal{F}_\sigma^{(\varphi_0)}(f) \right) = \mathcal{F}_\sigma^{(\varphi_0)}(\mathfrak{D}(\psi)f), \quad \sigma \geq 0, \quad (3.10)$$

by (2.27) and (3.3) we get

$$\begin{aligned} \|f - \mathcal{F}_\sigma^{(\varphi_0)}(f)\|_{X_p(\varphi)} &= \|f - \mathcal{F}_\sigma^{(\varphi_0)}(f)\|_p + \|\mathfrak{D}(\varphi)f - \mathfrak{D}(\varphi)(\mathcal{F}_\sigma^{(\varphi_0)}(f))\|_p \\ &= \|f - \mathcal{F}_\sigma^{(\varphi_0)}(f)\|_p + \|\mathfrak{D}(\varphi)f - \mathcal{F}_\sigma^{(\varphi_0)}(\mathfrak{D}(\varphi)f)\|_p \end{aligned} \quad (3.11)$$

for $\sigma \geq 0$. The terms on the right-hand side tend to 0 if $\sigma \rightarrow +\infty$ by (2.28). This yields the desired density of $\mathcal{T}$ in $X_p(\varphi)$.

Now we prove part (iii). Let $1 < p < +\infty$, $\varphi_1, \varphi_2 \in \mathcal{H}_s$, and let f be an arbitrary function in $X_p(\varphi_2)$. First we observe that in view of (2.34) and (3.1), inequality (2.35) can be rewritten as

$$\|\mathfrak{D}(\varphi_1)t\|_p \leq c\|\mathfrak{D}(\varphi_2)t\|_p, \quad t \in \mathcal{T}. \quad (3.12)$$

It is valid for $1 < p < +\infty$ (see the comment at the end of Section 2). Therefore, we obtain

$$\begin{aligned} \|\mathfrak{D}(\varphi_1)(\mathcal{F}_n^{(\varphi_0)}(f) - \mathcal{F}_m^{(\varphi_0)}(f))\|_p &\leq c\|\mathfrak{D}(\varphi_2)(\mathcal{F}_n^{(\varphi_0)}(f) - \mathcal{F}_m^{(\varphi_0)}(f))\|_p \\ &= c\|\mathcal{F}_n^{(\varphi_0)}(\mathfrak{D}(\varphi_2)f) - \mathcal{F}_m^{(\varphi_0)}(\mathfrak{D}(\varphi_2)f)\|_p \end{aligned} \quad (3.13)$$

for $n, m \in \mathbb{N}$. Hence, $\{\mathfrak{D}(\varphi_1)(\mathcal{F}_n^{(\varphi_0)}(f))\}$ is a Cauchy sequence in L_p by (2.28), and there exists $F \in L_p$ such that

$$\mathfrak{D}(\varphi_1)(\mathcal{F}_n^{(\varphi_0)}(f)) \xrightarrow{L_p} F \quad (n \rightarrow +\infty). \quad (3.14)$$

By the help of (3.3), (2.27), and (3.14) we get

$$F^\wedge(k) = \lim_{n \rightarrow +\infty} \left[\mathfrak{D}(\varphi_1)(\mathcal{F}_n^{(\varphi_0)}(f)) \right]^\wedge(k) = \varphi_1(k)f^\wedge(k). \quad (3.15)$$

For each $k \in \mathbb{Z}^d$. Hence, f belongs to $X_p(\varphi_1)$ and $\mathfrak{D}(\varphi_1)f = F$. In order to prove that the embedding

$$X_p(\varphi_2) \subset X_p(\varphi_1) \quad (3.16)$$

is continuous, it is enough to notice that

$$\begin{aligned} \|\mathfrak{D}(\varphi_1)f\|_p &= \lim_{n \rightarrow +\infty} \|\mathfrak{D}(\varphi_1)(\mathcal{F}_n^{(\varphi_0)}(f))\|_p \\ &\leq c \lim_{n \rightarrow +\infty} \|\mathfrak{D}(\varphi_2)(\mathcal{F}_n^{(\varphi_0)}(f))\|_p = \|\mathfrak{D}(\varphi_2)f\|_p \end{aligned} \quad (3.17)$$

as a consequence of (2.35). This completes the proof. $\square$

In other words part (iii) of the theorem means that in the case $1 < p < +\infty$ there is only one space $X_p^{(s)}$ of s -smooth functions. It can be characterized as

$$X_p^{(s)} = \left\{ f \in L_p : \mathfrak{D}(|\cdot|^s) f = \sum_{k \in \mathbb{Z}^d} |k|^s f^\wedge(k) e^{ikx} \in L_p \right\} \quad (3.18)$$

and may be equipped with the norm

$$\|f\|_{X_p^{(s)}} = \|f\|_p + \|\mathfrak{D}(|\cdot|^s) f\|_p. \quad (3.19)$$

Theorem 3.2. Suppose $1 \leq p \leq +\infty$, $\psi_1 \in \mathcal{H}_{s_1}$, and $\psi_2 \in \mathcal{H}_{s_2}$. If $s_1 > s_2 > 0$, then

$$X_p(\psi_1) \subset X_p(\psi_2) \quad (3.20)$$

and this embedding is continuous.

Proof. Let $f \in X_p(\psi_1)$. In view of (2.15) the function $\varphi = ((\theta\psi_2)/\psi_1)$, where θ is given by (2.14), is infinitely differentiable on $\mathbb{R}^d$ and has compact support. Hence, its Fourier transform belongs to $L_1(\mathbb{R}^d)$ and the operators $\mathfrak{F}_\sigma^{(\varphi)}$ are uniformly bounded in L_p as stated in Section 2. Using this fact as well as (3.3), (2.31), and the homogeneity property of ψ_1 and ψ_2 , we get

$$\begin{aligned} \left\| \sum_{j=m}^n \mathfrak{D}(\psi_2) f_j \right\|_p &\leq \sum_{j=m}^n \|\mathfrak{D}(\psi_2) f_j\|_p \\ &= \sum_{j=m}^n \left\| \sum_{k \in \mathbb{Z}^d} \theta(2^{-j}k) \psi_2(k) f^\wedge(k) e^{ikx} \right\|_p \\ &= \sum_{j=m}^n 2^{-(s_1-s_2)j} \left\| \sum_{k \neq 0} \varphi(2^{-j}k) \psi_1(k) f^\wedge(k) e^{ikx} \right\|_p \\ &= \sum_{j=m}^n 2^{-(s_1-s_2)j} \left\| \mathfrak{F}_{2^j}^{(\varphi)} (\mathfrak{D}(\psi_1) f) \right\|_p \\ &\leq c \|\mathfrak{D}(\psi_1) f\|_p \sum_{j=m}^n 2^{-(s_1-s_2)j} \end{aligned} \quad (3.21)$$

for $n > m \geq 1$. Here, f_j , $j \in \mathbb{N}$, has the meaning of (2.22). Because of $s_1 > s_2$ the sequence of the partial sums of the series $\sum_{j=1}^{+\infty} \mathfrak{D}(\psi_2) f_j$ is fundamental in L_p and there exists $F \in L_p$ such that

$$F \stackrel{L_p}{=} \sum_{j=1}^{+\infty} \mathfrak{D}(\psi_2) f_j. \quad (3.22)$$

By (2.13) we have $\varphi_0(k) \neq 0$ only for $k = 0$. Thus, we obtain

$$\sum_{j=1}^J \mathfrak{D}(\psi_2) f_j(x) = \sum_{k \in \mathbb{Z}^d} \varphi_0\left(\frac{k}{2^J}\right) \psi_2(k) f^\wedge(k) e^{ikx}, \quad x \in \mathbb{R}^d \quad (3.23)$$

for any $J \in \mathbb{N}$ in analogy to (2.32). Considering the limit process $J \rightarrow +\infty$ we find

$$F^\wedge(\nu) = \lim_{J \rightarrow +\infty} \left(\sum_{j=1}^J \mathfrak{D}(\psi_2) f_j \right)^\wedge(\nu) = \psi_2(\nu) f^\wedge(\nu) \quad (3.24)$$

for any $\nu \in \mathbb{Z}^d$ with the help of (2.13) and (3.22). Hence, f belongs to $X_p(\psi_2)$ and $\mathfrak{D}(\psi_2)f = F$. In order to prove that embedding (3.20) is continuous, it is enough to put $m = 1$ in estimate (3.21) and to consider n to $+\infty$. This completes the proof. $\square$

Theorem 3.3. Let $1 \leq p \leq +\infty$ and $\psi \in \mathcal{H}_s, s > 0$. Then,

$$\mathfrak{D}(\psi) : X_p(\psi) \longrightarrow L_p^0 \equiv \{f \in L_p : f^\wedge(0) = 0\} \quad (3.25)$$

is a surjective mapping.

Proof. Let $f \in L_p^0$. We introduce functions $g_j, j \in \mathbb{N}$, by setting

$$g_j(x) = \sum_{k \neq 0} \theta(2^{-j}k) (\psi(k))^{-1} f^\wedge(k) e^{ikx}, \quad j \in \mathbb{N}, \quad (3.26)$$

where θ has the meaning of (2.14), and we put $\varphi = \theta/\psi$. By the same arguments as in the proof of Theorem 3.2 (see, in particular, (3.21)) we get

$$\begin{aligned} \left\| \sum_{j=m}^n g_j \right\|_p &\leq \sum_{j=m}^n \|g_j\|_p = \sum_{j=m}^n 2^{-js} \|\mathfrak{T}_{2^j}^{(\varphi)}(f)\|_p \\ &\leq c \|f\|_p \sum_{j=m}^n 2^{-js} \end{aligned} \quad (3.27)$$

for $n > m \geq 1$. Hence, there exists $g \in L_p$ such that

$$g \stackrel{L_p}{=} \sum_{j=1}^{+\infty} g_j. \quad (3.28)$$

Taking into account that $\varphi_0(k) \neq 0$ only for $k = 0$ in view of (2.13), we obtain analogously to (2.32)

$$\sum_{j=1}^J g_j = \sum_{k \neq 0} \varphi_0(2^{-J}k) (\varphi(k))^{-1} f^\wedge(k) e^{ikx} \quad (3.29)$$

by (3.26). As a consequence of (3.28) and (3.29) we get

$$g^\wedge(v) = \lim_{J \rightarrow +\infty} \left(\sum_{j=1}^J g_j \right)^\wedge(v) = (\varphi(v))^{-1} f^\wedge(v) \quad (3.30)$$

for $v \neq 0$. Finally, because of $f^\wedge(0) = 0$ we obtain

$$f^\wedge(v) = \varphi(v) g^\wedge(v), \quad v \in \mathbb{Z}^d. \quad (3.31)$$

From (3.30). Hence, $g \in X_p(\varphi)$, $\mathfrak{D}(\varphi)g = f$, and the proof is complete. $\square$

Theorem 3.3 shows that for any $\varphi \in \mathcal{L}_s$ an operation which is inverse to $\mathfrak{D}(\varphi)$ is well defined on L_p^0 . The operator

$$I(\varphi) = (\mathfrak{D}(\varphi))^{-1} : L_p^0 \longrightarrow (X_p(\varphi)) \cap L_p^0 \quad (3.32)$$

is called *operator of φ -integration*. The Fourier series of the function $I(\varphi)f$, $f \in L_p^0$, is given by

$$\sum_{k \neq 0} (\varphi(k))^{-1} f^\wedge(k) e^{ikx}. \quad (3.33)$$

As it follows from the proof of Theorem 3.3 the series (3.33) converges in the sense of (3.26) and (3.28). Taking in (3.27) $m = 1$ and considering n to $+\infty$, we obtain the boundedness of the operator $I(\varphi)$ in L_p^0 .

With the help of Theorem 3.3 we will see that in contrast to the case $1 < p < +\infty$, where in view of part (iii) of Theorem 3.1 only one space of s -smooth functions exists, in the cases $p = 1, +\infty$ there are infinitely many ways to define smoothness spaces of order s associated with homogeneous generators by means of operators of multiplier type.

Theorem 3.4. *Suppose that φ_1 and φ_2 are linear independent homogeneous functions belonging to $\mathcal{L}_s$, $s > 0$. Then,*

$$X_1(\varphi_1) \not\subset X_1(\varphi_2), \quad X_\infty(\varphi_1) \not\subset X_\infty(\varphi_2). \quad (3.34)$$

Proof. We consider the case $p = 1$. For $p = +\infty$ the proof is similar. Let us assume (to the contrary) that

$$X_1(\varphi_1) \subset X_1(\varphi_2). \quad (3.35)$$

Then, by Theorem 3.3 the operators

$$\mathcal{L}_n = \mathcal{F}_n^{(\varphi_0)} \circ \mathfrak{D}(\psi_2) \circ I(\psi_1), \quad n \in \mathbb{N}, \quad (3.36)$$

where φ_0 has the meaning of Section 2 (see also (2.13)), are well defined on the space L_1^0 . The operator $\mathcal{L}_n$ acts in accordance with the following chain of mappings and inclusions:

$$L_1^0 \xrightarrow{I(\psi_1)} X_1(\psi_1) \subset X_1(\psi_2) \xrightarrow{\mathfrak{D}(\psi_2)} L_1^0 \xrightarrow{\mathcal{F}_n^{(\varphi_0)}} \mathcal{T}_n. \quad (3.37)$$

By (2.23), (2.27), (3.3), and (3.33) we get

$$\begin{aligned} \mathcal{F}_n^{(\varphi_0)} \circ \mathfrak{D}(\psi_2) \circ I(\psi_1)(f; x) &= \sum_{k \neq 0} \varphi_0\left(\frac{k}{n}\right) \psi_2(k) (\psi_1(k))^{-1} f^\wedge(k) e^{ikx} \\ &= (2\pi)^{-d} \int_{\mathbb{T}^d} f(x-h) \Phi_n(h) dh \end{aligned} \quad (3.38)$$

for $n \in \mathbb{N}$. Here,

$$\Phi_n(h) = \sum_{k \neq 0} \varphi_0\left(\frac{k}{n}\right) \psi_2(k) (\psi_1(k))^{-1} e^{ikx}, \quad n \in \mathbb{N}. \quad (3.39)$$

In view of (3.38) each operator $\mathcal{L}_n$ is bounded in L_1^0 . Moreover, the boundedness of $\mathcal{F}_n^{(\varphi_0)}$ (see Section 2) yields

$$\|\mathcal{L}_n(f)\|_1 \leq c \|\mathfrak{D}(\psi_2) \circ I(\psi_1)f\|_1 \quad (3.40)$$

for any $f \in L_1^0$. Applying now the Banach-Steinhaus principle we conclude that the operators $\mathcal{L}_n$ are uniformly bounded in L_1^0 . This leads to the estimate

$$\left\| \sum_{k \neq 0} \varphi_0\left(\frac{k}{n}\right) \psi_2(k) (\psi_1(k))^{-1} f^\wedge(k) e^{ikx} \right\|_1 \leq c \|f\|_1, \quad (3.41)$$

where the constant c does not depend on $f \in L_1^0$ and $n \in \mathbb{N}$. Let now

$$t(x) = \sum_{|k| \leq m} c_k e^{ikx} \quad (3.42)$$

be an arbitrary trigonometric polynomial. We choose $n > 2m$. Then, it holds that $\varphi_0(k/n)\psi_2(k) = \psi_2(k)$ for $|k| \leq m$ in view of (2.13). Applying (3.41) to $f = \mathfrak{D}(\psi_1)t$, we obtain

$$\|\mathfrak{D}(\psi_2)t\|_1 \leq c \|\mathfrak{D}(\psi_1)t\|_1. \quad (3.43)$$

This contradicts the statement on the nonvalidity of inequality (2.35) for $p = 1$ pointed out at the end of Section 2. Changing the roles of ψ_1 and ψ_2 completes the proof. $\square$

4. ψ -Smoothness and Besov and Triebel-Lizorkin Spaces

The aim of this section is to compare the spaces $X_p(\psi)$, $\psi \in \mathcal{A}_s$, with periodic Besov and Triebel-Lizorkin spaces. As we have seen already in part (iii) of Theorem 3.1 there is a unique space $X_p^{(s)}$ if $1 < p < +\infty$ which has been characterized in (3.18) and (3.19). The following theorem shows that it coincides with the classical fractional Sobolev space defined by $(s > 0, 1 < p < +\infty)$

$$H_p^s = \left\{ f \in L_p : \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^{s/2} f^\wedge(k) e^{ikx} \in L_p \right\} \quad (4.1)$$

and equipped with the norm

$$\|f\|_{H_p^s} = \left\| \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^{s/2} f^\wedge(k) e^{ikx} \right\|_p. \quad (4.2)$$

Note that

$$H_p^s = W_p^s = \{ f \in L_p : D^\alpha f \in L_p \text{ for } |\alpha| \leq s \} \quad (4.3)$$

if $s \in \mathbb{N}$ (all derivatives in the sense of periodic distributions, see [5, Subsection 3.5.4]).

Theorem 4.1. *Let $1 < p < +\infty$, and let $s > 0$. Then, one has*

$$X_p^{(s)} = H_p^s = F_{p,2}^s \quad (4.4)$$

with equivalence of norms.

Proof. Both statements are known. The second identity can be found in [5, Theorem 3.5.4, page 169]. To prove the first identity and to show the equivalence of norms one can use the Fourier multipliers and the theorem of Mikhlin-Hörmander (see, e.g., [15, Theorem 5.2.7, page 367], for the non-periodic version) which can be transferred to the periodic case using [16, Chapter 7, Theorem 3.1]. The arguments are similar to [17, Subsections 6.2.2, 6.2.3], or [18, Theorem 6.3.2], where equivalent characterizations and connections between nonperiodic homogeneous and inhomogeneous Sobolev spaces are treated. We omit the details. $\square$

Next we consider the cases $p = 1$ and $p = +\infty$.

Theorem 4.2. *Let $1 \leq p \leq +\infty$, and let $\psi \in \mathcal{A}_s$ for some $s > 0$. Then,*

$$B_{p,1}^s \subset X_p(\psi) \subset B_{p,\infty}^s \quad (4.5)$$

and these embeddings are continuous.

Proof. Let η be an infinitely differentiable positive function defined on $\mathbb{R}^d$ such that

$$\eta(\xi) = 1 \quad \text{if } |\xi| \leq \frac{1}{8}, \quad \eta(\xi) = 0 \quad \text{if } |\xi| \geq \frac{1}{4}. \quad (4.6)$$

We put $\psi_* = \psi(1 - \eta)$. By (2.15) we have

$$\psi(2^{-j}k)\theta(2^{-j}k) = \psi_*(2^{-j}k)\theta(2^{-j}k), \quad k \in \mathbb{Z}^d, j \in \mathbb{N}, \quad (4.7)$$

where θ is given by (2.14).

Let f be an arbitrary function in $B_{p,1}^s$, and let $f_j, j \in \mathbb{N}_0$, be given by (2.22). Using (4.7), the homogeneity property of ψ , and (2.27) and applying a Fourier multiplier theorem which can be found in [5, Theorem 3.3.4, page 150], we obtain

$$\begin{aligned} \left\| \sum_{j=m}^n \mathfrak{D}(\psi) f_j \right\|_p &\leq \sum_{j=m}^n \|\mathfrak{D}(\psi) f_j\|_p \\ &= \sum_{j=m}^n \left\| \sum_{k \in \mathbb{Z}^d} \psi(k) \theta(2^{-j}k) f^\wedge(k) e^{ikx} \right\|_p \\ &= \sum_{j=m}^n 2^{js} \left\| \sum_{k \in \mathbb{Z}^d} \psi_*(2^{-j}k) \theta(2^{-j}k) f^\wedge(k) e^{ikx} \right\|_p \\ &\leq c \sum_{j=m}^n 2^{js} \left\| \mathcal{F}_{2^j}^{(\theta)}(f) \right\|_p = \sum_{j=m}^n 2^{js} \|f_j\|_p \end{aligned} \quad (4.8)$$

for $n > m \geq 1$. In view of (2.18) and (4.8) we can conclude that the series $\sum_{j=0}^{+\infty} \mathfrak{D}(\psi) f_j$ converges in L_p . Now, by standard arguments we see as in the proof of Theorem 3.1 that f belongs to $X_p(\psi)$ and that the first embedding in (4.5) is continuous.

In order to prove the second embedding we introduce an infinitely differentiable function $\psi^*(\xi)$ which coincides with $\psi(\xi)$ for $|\xi| \geq 1/4$ and which is not equal to 0 on $\mathbb{R}^d$. By (2.15) we have

$$\psi(2^{-j}k)\theta(2^{-j}k) = \psi^*(2^{-j}k)\theta(2^{-j}k), \quad k \in \mathbb{Z}^d, j \in \mathbb{N}, \quad (4.9)$$

where θ has the meaning of (2.14).

Let f be an arbitrary function in $X_p(\psi)$, and let f_j , $j \in \mathbb{N}_0$, be given by (2.22). Applying (4.9), the homogeneity property of ψ , and (2.27), we obtain

$$\begin{aligned}
 f_j(x) &= \sum_{k \in \mathbb{Z}^d} \theta(2^{-j}k) f^\wedge(k) e^{ikx} \\
 &= \sum_{k \in \mathbb{Z}^d} \left(\psi^*(2^{-j}k) \right)^{-1} \psi(2^{-j}k) \theta(2^{-j}k) f^\wedge(k) e^{ikx} \\
 &= 2^{-js} \sum_{k \in \mathbb{Z}^d} \left(\psi^*(2^{-j}k) \right)^{-1} \psi(k) \theta(2^{-j}k) f^\wedge(k) e^{ikx} \\
 &= 2^{-js} \sum_{k \in \mathbb{Z}^d} \left(\psi^*(2^{-j}k) \right)^{-1} \left(\mathcal{F}_{2^j}^{(\theta)}(\mathfrak{D}(\psi)f) \right)^\wedge(k) e^{ikx}
 \end{aligned} \tag{4.10}$$

for any $j \in \mathbb{N}$. The right-hand side can be estimated again by means of the Fourier multiplier theorem which can be found in [5, Theorem 3.3.4, page 150]. Using in addition the uniform boundedness of the Fourier means $\mathcal{F}_{2^j}^{(\theta)}$ in L_p , we obtain the inequalities

$$\|f_j\|_p \leq c 2^{-js} \left\| \mathcal{F}_{2^j}^{(\theta)}(\mathfrak{D}(\psi)f) \right\|_p \leq c' 2^{-js} \|\mathfrak{D}(\psi)f\|_p \tag{4.11}$$

for $j \in \mathbb{N}$ from (4.10). Obviously,

$$\|f_0\|_p \leq c \|f\|_p. \tag{4.12}$$

Hence, by (4.11) and (3.5)

$$\|f\|_{B_{p,\infty}^s} \leq c \|f\|_{X_p(\psi)} \tag{4.13}$$

for some constant $c > 0$ and all $f \in X_p(\psi)$. This completes the proof. $\square$

Having in mind Theorem 4.1 the embeddings (4.5) are well known in the case $1 < p < +\infty$. Even a better result holds. This follows also from Theorem 4.1 and the elementary embeddings

$$B_{p,1}^s \subset B_{p,\min(p,2)}^s \subset F_{p,2}^s \subset B_{p,\max(p,2)}^s \subset B_{p,\infty}^s \tag{4.14}$$

(see [5, Remark 3.5.1/4, page 164]).

5. A Representation Formula for ψ -Derivatives

The following observations give some motivation and pave the way to find explicit representations of the operator $\mathfrak{D}(\psi)$ in terms of the Fourier transform of its generator $\psi \in \mathcal{L}_s$. For the sake of simplicity we restrict ourselves to the case $0 < s < 1$.

Let $\psi \in \mathcal{L}_s$. Recall that $g \in \mathcal{E}_{\text{per}}$ if and only if

$$|g^\wedge(k)| \leq C(m)|k|^{-m}, \quad k \neq 0, \quad (5.1)$$

for each $m \in \mathbb{N}$ and note that

$$|\psi(\xi)| \leq c|\xi|^s, \quad \xi \in \mathbb{R}^d, \quad (5.2)$$

by homogeneity. Hence, it makes sense to consider the periodic distribution

$$\Psi(x) = \sum_{k \in \mathbb{Z}^d} \psi(k) e^{-ikx} \quad (5.3)$$

(convergence in $\mathcal{E}'_{\text{per}}$). Let $g \in \mathcal{E}_{\text{per}}$. Expansion into the Fourier series leads to the representation

$$\Psi(x) = \sum_{v \in \mathbb{Z}^d} \hat{\psi}(x + 2\pi v) \quad (5.4)$$

(convergence in $\mathcal{E}'_{\text{per}}$). By definition of the Fourier coefficients, the definition of the ψ -derivative and, (5.3) we get

$$\mathfrak{D}(\psi)(g)(x) = (2\pi)^{-d} \langle \Psi(\cdot), g(x + \cdot) - g(x) \rangle, \quad g \in \mathcal{E}_{\text{per}}. \quad (5.5)$$

Using (5.4) and (5.5), we see, at least formally, that

$$\begin{aligned} \mathfrak{D}(\psi)(g)(x) &= (2\pi)^{-d} \sum_{v \in \mathbb{Z}^d} \langle \hat{\psi}(\cdot + 2\pi v), g(x + \cdot) - g(x) \rangle \\ &= (2\pi)^{-d} \sum_{v \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \hat{\psi}(y + 2\pi v) (g(x + y) - g(x)) dy \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} (g(x + h) - g(x)) \hat{\psi}(h) dh, \quad g \in \mathcal{E}_{\text{per}}. \end{aligned} \quad (5.6)$$

We claim that the integral on the right-hand side of (5.6) exists for all $x \in \mathbb{T}^d$. To this end we first recall that (see Section 2) $\hat{\psi} \in \mathcal{S}'$ is a homogeneous distribution of order $-(d + s)$. Its restriction to $\mathbb{R}^d \setminus \{0\}$ can be identified with a function $\hat{\psi} \in C^\infty(\mathbb{R}^d \setminus \{0\})$ by [9, Theorem 7.1.18]. Hence, we have the estimate

$$|\hat{\psi}(h)| \leq c|h|^{-(d+s)}, \quad h \in \mathbb{R}^d \setminus \{0\}. \quad (5.7)$$

For brevity we use the standard notation $\Delta_h g(x) = g(x + h) - g(x)$. Now, we split

$$\int_{\mathbb{R}^d} \Delta_h g(x) \hat{\psi}(h) dh = \int_{B_1} \Delta_h g(x) \hat{\psi}(h) dh + \int_{\mathbb{R}^d \setminus B_1} \Delta_h g(x) \hat{\psi}(h) dh. \quad (5.8)$$

The second summand is finite because of (5.7) and the boundedness of $|\Delta_h g(x)|$ on $\mathbb{R}^d$. As for the first one we use the estimate

$$|\Delta_h g(x)| \leq c'|h|h \in B_1, \quad x \in \mathbb{T}^d, \quad (5.9)$$

and (5.7) to see that

$$\begin{aligned} \left| \int_{B_1} \Delta_h g(x) \widehat{\varphi}(h) dh \right| &\leq c' \int_{B_1} |h| |\widehat{\varphi}(h)| dh \\ &\leq c'' \int_{B_1} |h|^{-d+(1-s)} dh < +\infty. \end{aligned} \quad (5.10)$$

The above arguments suggest that formula (5.6) might be true in a stronger sense. Indeed, the following theorem shows that the representation for the derivative $\mathfrak{D}(\varphi)(g)$ holds pointwise almost everywhere under much weaker assumptions with respect to g .

Theorem 5.1. *Suppose $1 \leq p \leq +\infty$ and $\varphi \in \mathcal{L}_s$, where $0 < s < 1$. Then,*

$$\mathfrak{D}(\varphi)(f)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} (f(x+h) - f(x)) \widehat{\varphi}(h) dh, \quad f \in B_{p,1}^s, \quad (5.11)$$

for almost all $x \in \mathbb{T}^d$ (for all, if $p = +\infty$).

Proof. First we recall that $B_{p,1}^s \subset X_p(\varphi)$ by Theorem 4.2. Hence, the left-hand side of (5.11) makes sense. Let $f \in B_{p,1}^s$. As is well known (see, e.g., [5, Theorem 3.5.4]),

$$\|f\|_{B_{p,1}^s}^* = \|f\|_p + \int_{\mathbb{R}^d} |h|^{-s} \|\Delta_h f\|_p \frac{dh}{|h|^d} \quad (5.12)$$

is an equivalent norm in the Besov space $B_{p,1}^s$. Using (5.7), (5.12), and the generalized Minkowski inequality we obtain for the integral $I(x)$ at the right-hand side of (5.11) the estimates

$$\|I\|_p \leq \int_{\mathbb{R}^d} \|\Delta_h f(x)\|_p \cdot |\widehat{\varphi}(h)| dh \leq c \int_{\mathbb{R}^d} \|\Delta_h f\|_p \cdot |h|^{-(d+s)} dh \leq c \|f\|_{B_{p,1}^s}^*. \quad (5.13)$$

Hence, the function $I(x)$ belongs to L_p . To prove (5.11) it is sufficient to show that the Fourier coefficients of the functions on both sides coincide. We have

$$\begin{aligned} (I(\cdot))^\wedge(k) &= (2\pi)^{-d} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f(x+h) - f(x)) \widehat{\varphi}(h) dh e^{-ikx} dx \\ &= f^\wedge(k) \int_{\mathbb{R}^d} (e^{ikh} - 1) \widehat{\varphi}(h) dh \end{aligned} \quad (5.14)$$

by Fubini's theorem. It remains to show that

$$\int_{\mathbb{R}^d} (e^{ikh} - 1) \widehat{\psi}(h) dh = (2\pi)^{-d} \psi(k), \quad k \in \mathbb{Z}^d. \quad (5.15)$$

This is obvious if $k = 0$ because of $\psi(0) = 0$. If $k \neq 0$, we have to use appropriate limiting arguments to circumvent the difficulty caused by the fact that $\widehat{\psi}$ is not integrable in a neighbourhood of 0. We do not go into details. $\square$

We give some remarks. It is known (see [15, Theorem 2.4.6, page 128]) that for any s , $0 < s < 2$, the restriction of the Fourier transform of $\psi(\xi) = |\xi|^s$ to $\mathbb{R}^d \setminus \{0\}$ can be identified with

$$(|\cdot|^s)^\wedge(x) = c(d, s)|x|^{-d-s}, \quad \left(c(d, s) = 2^{d+s} \pi^{d/2} \frac{\Gamma(s/2 + d/2)}{\Gamma(-s/2)} \right), \quad (5.16)$$

combining (5.7) with (5.16) in the one-dimensional case, we obtain the well-known formula for the Riesz derivative (see, e.g., [6])

$$f^{(s)}(x) = (2\pi)^{-1} c(1, s) \int_{-\infty}^{+\infty} \frac{f(x+h) - f(x)}{|h|^{s+1}} dh, \quad f \in B_{p,1}^s, \quad (5.17)$$

which is valid for $0 < s < 1$ and $1 \leq p \leq +\infty$. In the multivariate case we get under the same conditions with respect to s and p the representation formula

$$(-\Delta)^{s/2} f(x) = (2\pi)^{-d} c(d, s) \int_{\mathbb{R}^d} \frac{f(x+h) - f(x)}{|h|^{d+s}} dh, \quad f \in B_{p,1}^s \quad (5.18)$$

for the fractional power of the Laplace operator.

Let us mention that formulas for ψ -derivatives with $s \geq 1$ can be achieved using differences of higher order. Suppose, for example, that $1 \leq s < 2$ and, in addition to the previous conditions, that ψ is real valued. Analogously to (5.6) we find

$$\mathfrak{D}(\psi)(g)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{g(x+h) - 2g(x) + g(x-h)}{2} \widehat{\psi}(h) dh, \quad (5.19)$$

and, in particular,

$$(-\Delta)^{s/2} g(x) = (2\pi)^{-d} c(d, s) \int_{\mathbb{R}^d} \frac{g(x+h) - 2g(x) + g(x-h)}{2|h|^{d+s}} dh \quad (5.20)$$

for $g \in \mathcal{E}_{\text{per}}$. Similarly to the proof of Theorem 5.1 one can show that formulas (5.19) and (5.20) are valid at least for functions belonging to the Besov spaces $B_{p,1}^s$.

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