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Research Article

Some New Refined General Boas-Type Inequalities

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We state and prove a new refined Boas-type inequality in a setting with a topological space and general σ-finite and finite Borel measures. As a consequence of the result obtained, we derive a new class of Hardy- and Pólya-Knopp-type inequalities related to balls in $\mathbb{R}^n$ and prove that constant factors involved in their right-hand sides are the best possible.

1. Introduction

Boas [1], proved that the inequality

$$\int_0^\infty \Phi\left(\frac{1}{M}\int_0^\infty f(tx)dm(t)\right)\frac{dx}{x} \le \int_0^\infty \Phi\big(f(x)\big)\frac{dx}{x} \tag{1.1}$$

holds for all continuous convex functions $\Phi : [0,\infty\rangle \to \mathbb{R}$, measurable nonnegative functions $f : \mathbb{R}_+ \to \mathbb{R}$, and nondecreasing bounded functions $m : [0,\infty\rangle \to \mathbb{R}$, where $M = m(\infty) - m(0) > 0$, and the inner integral on the left-hand side of (1.1) is the Lebesgue-Stieltjes integral with respect to m. After its author, inequality (1.1) was named the Boas inequality. In the case of a concave function Φ, it holds with the sign of inequality reversed.

Since its publication, the Boas inequality has been generalized in different ways. Here we mention only those that have guided us in our research. Namely, following some ideas of Kaijser et al. [2] (see also the paper [3] of Levinson), Čižmešija et al. [4] considered a general

Borel measure λ on $\mathbb{R}_+$, such that $L = \lambda(\mathbb{R}_+) < \infty$, a convex function Φ on a convex set $I \subseteq \mathbb{R}$, and a weight function u on $\mathbb{R}_+$, and proved that the inequality

$$\int_0^\infty u(x)\Phi\big(Af(x)\big)\frac{dx}{x} \le \frac{1}{L}\int_0^\infty w(x)\Phi\big(f(x)\big)\frac{dx}{x} \tag{1.2}$$

holds for all measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ with values in I, where $Af(x) = (1/L)\int_0^\infty f(tx)d\lambda(t)$ and $w(x) = \int_0^\infty u(x/t)d\lambda(t) < \infty$, $x \in \mathbb{R}_+$. In the same paper, a refinement of (1.2) is obtained.

Another generalization of (1.1) was recently given by Luor [5] in a setting with σ-finite Borel measures μ and ν on a topological space X and a Borel probability measure λ on $\mathbb{R}_+$. For a λ-balanced Borel set $\Omega \subseteq X$ and the measures $\mu_t \ll \nu$, defined for all Borel sets $S \subseteq X$ and $t \in \operatorname{supp}\lambda$ by $\mu_t(S) = \mu(t^{-1}S)$, he obtained the inequality

$$\int_\Omega \Phi\left(\int_0^\infty f(t\mathbf{x})d\lambda(t)\right)d\mu(\mathbf{x}) \le \int_\Omega \Phi\big(f(\mathbf{x})\big)\left(\int_0^\infty \frac{d\mu_t}{d\nu}(\mathbf{x})d\lambda(t)\right)d\nu(\mathbf{x}), \tag{1.3}$$

where Φ is a nonnegative convex function on a real interval I, and $f : \Omega \to \mathbb{R}$ is a Borel measurable function with values in I. A weighted version of (1.3) was given by Čižmešija et al. [6, Theorem 2.1], along with a detailed analysis of properties of the related so-called Boas functional.

Notice that the Boas inequality (1.1) unifies some well-known classical inequalities, such as Hardy's and Pólya-Knopp's inequalities. In the sequel, we state their strengthened versions, obtained independently by Debnath et al. [7, 8] and Čižmešija et al. [9, 10].

Let $0 < b \le \infty$ and $p, k \in \mathbb{R}$ such that $p/(k-1) > 0$. If $p \in \mathbb{R} \setminus [0,1]$, f is a nonnegative function, $x^{1-(k/p)}f \in L^p(0,b)$, and

$$F(x) = \int_0^x f(t)dt, \qquad x \in \langle 0, b\rangle, \tag{1.4}$$

then

$$\int_0^b x^{-k}F^p(x)dx \le \left(\frac{p}{k-1}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{(k-1)/p}\right]x^{p-k}f^p(x)dx \tag{1.5}$$

holds, while for $p \in \langle 0,1\rangle$ the sign of inequality in (1.5) is reversed. Moreover, if $0 \le b < \infty$ and parameters $p \in \mathbb{R} \setminus [0,1]$ and $k \in \mathbb{R}$ such that $p/(k-1) < 0$, then the inequality

$$\int_b^\infty x^{-k}\widetilde{F}^p(x)dx \le \left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{(1-k)/p}\right]x^{p-k}f^p(x)dx \tag{1.6}$$

holds for all nonnegative functions f such that $x^{1-(k/p)}f \in L^p(b,\infty)$, where

$$\widetilde{F}(x) = \int_x^\infty f(t)dt, \qquad x \in \langle b, \infty\rangle. \tag{1.7}$$

For $p \in \langle 0,1\rangle$ inequality (1.6) holds with the sign of inequality reversed. The constant $|p/(k-1)|^p$ is the best possible for both inequalities, that is, it cannot be replaced with any smaller constant. The classical Hardy's inequality follows by taking $b = \infty$ in (1.5), while for $b = 0$ in (1.6) we get its dual inequality.

On the other hand, if $0 < b \le \infty$, $f \in L^1(0,b)$ is a positive function, and

$$G(x) = \exp\left(\frac{1}{x}\int_0^x \log f(t)dt\right), \qquad x \in \langle 0,b\rangle, \tag{1.8}$$

then

$$\int_0^b G(x)\,dx \le e\int_0^b \left(1-\frac{x}{b}\right) f(x)dx \tag{1.9}$$

holds, while the inequality

$$\int_b^\infty \tilde{G}(x)dx \le \frac{1}{e}\int_b^\infty \left(1-\frac{b}{x}\right) f(x)dx \tag{1.10}$$

holds for $0 \le b < \infty$, $0 < f \in L^1(b,\infty)$, and

$$\tilde{G}(x) = \exp\left(x\int_x^\infty \log f(t)\frac{dt}{t^2}\right), \qquad x \in \langle b,\infty\rangle. \tag{1.11}$$

For $b = \infty$ in (1.9) and for $b = 0$ in (1.10) we, respectively, get the classical Pólya-Knopp's inequality and its dual inequality. Notice that the constant factors e and $1/e$, respectively, involved in the right-hand sides of (1.9) and (1.10), are the best possible.

In this paper, we also make use of the following n-dimensional strengthened Hardy's inequality related to the setting with balls in $\mathbb{R}^n$ centered at the origin (see [11] for details). Let $p, k, R \in \mathbb{R}$ such that $p > 1$, $k \ne 1$, and $R > 0$. Suppose that f is a nonnegative measurable function and the function F is defined on $\mathbb{R}^n$ by

$$F(\mathbf{x}) = \begin{cases} \displaystyle\int_{B(|\mathbf{x}|)} f(\mathbf{y})d\mathbf{y}, & k > 1, \\ \displaystyle\int_{\mathbb{R}^n\setminus B(|\mathbf{x}|)} f(\mathbf{y})d\mathbf{y}, & k < 1, \end{cases} \tag{1.12}$$

where $B(|\mathbf{x}|)$ is a ball in $\mathbb{R}^n$ centered at the origin and of radius $|\mathbf{x}|$, while $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$. If $p/(k-1) > 0$, then the inequality

$$\begin{aligned} &\int_{B(R)} |B(|\mathbf{x}|)|^{-k} F^p(\mathbf{x})d\mathbf{x} \\ &\quad \le \left(\frac{p}{k-1}\right)^p \int_{B(R)} \left[1-\left(\frac{|B(|\mathbf{x}|)|}{|B(R)|}\right)^{(k-1)/p}\right] |B(|\mathbf{x}|)|^{p-k} f^p(\mathbf{x})d\mathbf{x} \end{aligned} \tag{1.13}$$

holds, while for $p/(k-1) < 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(R)} &|B(|\mathbf{x}|)|^{-k} F^p(\mathbf{x})\,d\mathbf{x} \\ &\le \left(\frac{p}{1-k}\right)^p \int_{\mathbb{R}^n\setminus B(R)} \left[1-\left(\frac{|B(R)|}{|B(|\mathbf{x}|)|}\right)^{(1-k)/p}\right] |B(|\mathbf{x}|)|^{p-k} f^p(\mathbf{x})\,d\mathbf{x}.\end{aligned} \tag{1.14}$$

Terms $|B(|\mathbf{x}|)|$ and $|B(R)|$, respectively, denote the volumes of $B(|\mathbf{x}|)$ and $B(R)$. The constant $(p/|k-1|)^p$ is the best possible for both inequalities. Observe that the first natural generalization of the classical Hardy's inequality to balls in $\mathbb{R}^n$ was given by Christ and Grafakos in [12].

Finally, here we state an n-dimensional Pólya-Knopp's inequality, related to (1.13) and (1.14),

$$\int_{B(R)} G(\mathbf{x})\,d\mathbf{x} < e\int_{B(R)} \left(1-\frac{|B(|\mathbf{x}|)|}{|B(R)|}\right) f(\mathbf{x})\,d\mathbf{x} \tag{1.15}$$

for a positive function f on $B(R)$ and

$$G(\mathbf{x}) = \exp\left(\frac{1}{|B(|\mathbf{x}|)|}\int_{B(|\mathbf{x}|)} \log f(\mathbf{y})\,d\mathbf{y}\right), \qquad \mathbf{x}\in B(R), \tag{1.16}$$

as well as its dual inequality

$$\int_{\mathbb{R}^n\setminus B(R)} \widetilde{G}(\mathbf{x})\,d\mathbf{x} < \frac{1}{e}\int_{\mathbb{R}^n\setminus B(R)} \left(1-\frac{|B(R)|}{|B(|\mathbf{x}|)|}\right) f(\mathbf{x})\,d\mathbf{x} \tag{1.17}$$

for a positive function f on $\mathbb{R}^n \setminus B(R)$ and

$$\widetilde{G}(\mathbf{x}) = \exp\left(|B(|\mathbf{x}|)|\int_{\mathbb{R}^n\setminus B(|\mathbf{x}|)} \log f(\mathbf{y})\frac{d\mathbf{y}}{|B(|\mathbf{x}|)|^2}\right), \qquad \mathbf{x}\in\mathbb{R}^n\setminus B(R). \tag{1.18}$$

Observe that all above-mentioned inequalities of the Hardy and Pólya-Knopp type are just a small contribution to the rich theory of the Hardy-type inequalities. Many information about history and new developments regarding Hardy's and related Carleman's inequalities can be found, for example, in recent monographs [13–15], expository papers [2, 16, 17], and in exhaustive lists of references given therein.

Our aim in this paper is to state and prove a new weighted refined Boas-type inequality in a setting with topological spaces and finite and σ-finite Borel measures and to show that our result refines and generalizes Luor's inequality (1.3). In addition, we extend the results obtained to a case with nonnegative kernels.

Moreover, as a consequence of our new refined Boas-type inequality, we derive a new class of Hardy- and Pólya-Knopp-type inequalities related to balls in $\mathbb{R}^n$, along with their respective dual inequalities, and prove that constant factors involved in their right-hand sides

are the best possible. Finally, we show that our Hardy's and Pólya-Knopp's inequalities differ from (1.13), (1.14), (1.15) and (1.17), although for $n = 1$ both classes coincide.

Conventions. Throughout this paper, all measures are assumed to be positive, all functions are assumed to be measurable on their respective domains and expressions of the form $0 \cdot \infty$, $0/0$, a/∞ $(a \in \mathbb{R})$, and ∞/∞ are taken to be equal to zero. As usual, by dx and $d\mathbf{x}$ we denote the Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^n$ $(n \in \mathbb{N},\ n \geq 2)$, respectively, while by a weight function we mean a nonnegative measurable function on the actual set. An interval in $\mathbb{R}$ is any convex subset of $\mathbb{R}$, while Int I denotes the interior of an interval $I \subseteq \mathbb{R}$. In particular, $\mathbb{R}_+ = \langle 0, \infty \rangle$. For $R > 0$, by $B(R)$ we denote a ball in $\mathbb{R}^n$ centered at the origin and of radius R, that is, $B(R) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq R\}$, where $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$. By its dual set we mean the set $\mathbb{R}^n \setminus B(R) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| > R\}$. Finally, by S^{n-1} we denote the surface of the unit ball $B(1)$ and by $|S^{n-1}|$ its area. Using polar coordinates in $\mathbb{R}^n$, the volume of the ball $\mathrm{B}(R)$ is then

$$\begin{aligned} |B(R)| &= \int_{B(R)} d\mathbf{x} = \int_{|\mathbf{x}| \leq R} d\mathbf{x} = \int_0^R t^{n-1} \left(\int_{S^{n-1}} dS \right) dt \\ &= \int_{S^{n-1}} \left(\int_0^R t^{n-1} dt \right) dS = \frac{R^n |S^{n-1}|}{n}. \end{aligned} \tag{1.19}$$

2. A New Refined Weighted Boas-Type Inequality

To start, we recall some basic facts on convex functions. Let I be an interval in $\mathbb{R}$ and $\Phi : I \to \mathbb{R}$ be a convex function. For $x \in I$, by $\partial\Phi(x)$ we denote the subdifferential of Φ at x, that is, the set

$$\partial\Phi(x) = \left\{ \alpha \in \mathbb{R} : \Phi(y) - \Phi(x) - \alpha(y - x) \geq 0,\ y \in I \right\}. \tag{2.1}$$

It is well known that $\partial\Phi(x) \neq \emptyset$ for all $x \in \operatorname{Int} I$. More precisely, at each point $x \in \operatorname{Int} I$ we have $-\infty < \Phi'_-(x) \leq \Phi'_+(x) < \infty$, $\partial\Phi(x) = [\Phi'_-(x),\ \Phi'_+(x)]$, and the set on which Φ is not differentiable is at most countable. Moreover, every function $\varphi : I \to \mathbb{R}$ for which $\varphi(x) \in \partial\Phi(x)$, whenever $x \in \operatorname{Int} I$, is increasing on Int I.

On the other hand, if $\Phi : I \to \mathbb{R}$ is a concave function, that is, $-\Phi$ is convex, then $\partial\Phi(x) = \{\alpha \in \mathbb{R} : \Phi(x) - \Phi(y) - \alpha(x - y) \geq 0,\ y \in I\}$ denotes the superdifferential of Φ at the point $x \in I$. For all $x \in \operatorname{Int} I$, in this setting we have $-\infty < \Phi'_+(x) \leq \Phi'_-(x) < \infty$ and $\partial\Phi(x) = [\Phi'_+(x), \Phi'_-(x)] \neq \emptyset$. Notice that, although the symbol $\partial\Phi(x)$ has two different notions, it will be clear from the context whether it applies to a convex or to a concave function Φ. For more details about convex and concave functions see, for example, the recent monograph [18].

We continue by introducing some necessary notation, related to a setting with topological spaces and Borel measures, in which we state and prove a new refined weighted inequality of the Boas type.

Let λ be a finite Borel measure on $\mathbb{R}_+$. By supp λ, we denote its support, that is, the set of all $t \in \mathbb{R}_+$ such that $\lambda(N_t) > 0$ holds for all open neighbourhoods N_t of t. Hence,

$$L = \int_{\operatorname{supp}\lambda} d\lambda(t) = \int_0^\infty d\lambda(t) = \lambda(\mathbb{R}_+) < \infty. \tag{2.2}$$

Further, let X be a topological space equipped with a continuous scalar multiplication $(a, \mathbf{x}) \mapsto a\mathbf{x} \in X$, for $a \in \mathbb{R}_+$ and $\mathbf{x} \in X$, such that

$$1\mathbf{x} = \mathbf{x}, \quad a(b\mathbf{x}) = (ab)\mathbf{x}, \quad \mathbf{x} \in X, \ a, b \in \mathbb{R}_+. \tag{2.3}$$

Let a Borel set $\Omega \subseteq X$ be λ-balanced, that is, let $t\Omega = \{t\mathbf{x} : \mathbf{x} \in \Omega\} \subseteq \Omega$ hold for all $t \in \operatorname{supp} \lambda$. For a Borel measurable function $f : \Omega \to \mathbb{R}$, we define its Hardy-Littlewood average, Af, as

$$Af(\mathbf{x}) = \frac{1}{L} \int_0^\infty f(t\mathbf{x}) d\lambda(t), \quad \mathbf{x} \in \Omega. \tag{2.4}$$

Finally, suppose that μ and ν are σ-finite Borel measures on X. For $t > 0$ and a Borel set $S \subseteq X$, we define

$$\mu_t(S) = \mu\left(\frac{1}{t} S\right). \tag{2.5}$$

Obviously, μ_t is a σ-finite Borel measure on X for all $t \in \mathbb{R}_+$. Throughout this paper we assume that μ_t is absolutely continuous with respect to the measure ν, that is, $\mu_t \ll \nu$, for each $t \in \operatorname{supp} \lambda$. As usual, by $d\mu_t/d\nu$ we denote the related Radon-Nikodym derivative.

As announced, now we can state and prove the main result of this paper, a new refined weighted Boas-type inequality in the above setting.

Theorem 2.1. *Let λ be a finite Borel measure on $\mathbb{R}_+$ and L be defined by* (2.2). *Let μ and ν be σ-finite Borel measures on a topological space X, μ_t be defined by* (2.5), *and let $\mu_t \ll \nu$ for all $t \in \operatorname{supp} \lambda$. Further, let $\Omega \subseteq X$ be a λ-balanced Borel set and u be a nonnegative function on X, such that*

$$v(\mathbf{x}) = \int_0^\infty u\left(\frac{1}{t}\mathbf{x}\right) \frac{d\mu_t}{d\nu}(\mathbf{x}) d\lambda(t) < \infty, \quad \mathbf{x} \in \Omega. \tag{2.6}$$

Suppose that $\Phi : I \to \mathbb{R}$ is a nonnegative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function fulfilling $\varphi(x) \in \partial\Phi(x)$, for all $x \in \operatorname{Int} I$. If $f : \Omega \to \mathbb{R}$ is a Borel measurable function with values in I and Af is defined by (2.4), *then $Af(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$ and the inequality*

$$\begin{aligned} &\frac{1}{L} \int_\Omega v(\mathbf{x}) \Phi\big(f(\mathbf{x})\big) d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x}) \Phi\big(Af(\mathbf{x})\big) d\mu(\mathbf{x}) \\ &\quad \geq \frac{1}{L} \bigg| \int_\Omega u(\mathbf{x}) \int_0^\infty \big|\Phi\big(f(t\mathbf{x})\big) - \Phi\big(Af(\mathbf{x})\big)\big| d\lambda(t)\, d\mu(\mathbf{x}) \\ &\qquad - \int_\Omega u(\mathbf{x}) \int_0^\infty \big|\varphi\big(Af(\mathbf{x})\big)\big| \cdot \big|f(t\mathbf{x}) - Af(\mathbf{x})\big| d\lambda(t)\, d\mu(\mathbf{x}) \bigg| \end{aligned} \tag{2.7}$$

holds. For a nonpositive concave function Φ, relation (2.7) *holds with*

$$\int_\Omega u(\mathbf{x}) \Phi\big(Af(\mathbf{x})\big) d\mu(\mathbf{x}) - \frac{1}{L} \int_\Omega v(\mathbf{x}) \Phi\big(f(\mathbf{x})\big) d\nu(\mathbf{x}) \tag{2.8}$$

on its left-hand side.

Proof. Since $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$, it is not hard to see that $Af(\mathbf{x}) \in I$, for all $\mathbf{x} \in \Omega$ (see the proof of Theorem 2.1 in [6] for details). Suppose the function Φ is convex and nonnegative. To prove inequality (2.7), observe that for arbitrary $r \in \operatorname{Int} I$ and $s \in I$ we have $\Phi(s) - \Phi(r) - \varphi(r)(s-r) \geq 0$, so

$$\begin{aligned}\Phi(s) - \Phi(r) - \varphi(r)(s-r) &= \left|\Phi(s) - \Phi(r) - \varphi(r)(s-r)\right| \\ &\geq \left||\Phi(s) - \Phi(r)| - \left|\varphi(r)\right| \cdot |s-r|\right|.\end{aligned} \tag{2.9}$$

In particular, for $\mathbf{x} \in \Omega$, such that $Af(\mathbf{x}) \in \operatorname{Int} I$, and for $t \in \operatorname{supp} \Omega$, from (2.9) we get

$$\begin{aligned}&\Phi\big(f(t\mathbf{x})\big) - \Phi\big(Af(\mathbf{x})\big) - \varphi\big(Af(\mathbf{x})\big) \cdot \big(f(t\mathbf{x}) - Af(\mathbf{x})\big) \\ &\quad \geq \left|\left|\Phi\big(f(t\mathbf{x})\big) - \Phi\big(Af(\mathbf{x})\big)\right| - \left|\varphi\big(Af(\mathbf{x})\big)\right| \cdot \left|f(t\mathbf{x}) - Af(\mathbf{x})\right|\right|.\end{aligned} \tag{2.10}$$

On the other hand, if I is not an open interval and $Af(\mathbf{x})$ is an endpoint of I for some $\mathbf{x} \in \Omega$, then either $f(t\mathbf{x}) - Af(\mathbf{x}) \geq 0$ for all $t \in \operatorname{supp} \lambda$ or $f(t\mathbf{x}) - Af(\mathbf{x}) \leq 0$ for all $t \in \operatorname{supp} \lambda$. Since

$$\int_0^\infty \big(f(t\mathbf{x}) - Af(\mathbf{x})\big) d\lambda(t) = 0, \tag{2.11}$$

we conclude that $f(t\mathbf{x}) - Af(\mathbf{x}) = 0$ for λ-a.e. $t \in \operatorname{supp} \lambda$, so in that case both sides of (2.10) are equal to 0. Hence, (2.10) holds for all $\mathbf{x} \in \Omega$ and λ-a.e. $t \in \operatorname{supp} \lambda$. Multiplying it by $u(\mathbf{x})$ and then integrating over $\mathbb{R}_+$ and Ω, we obtain the following sequence of inequalities:

$$\begin{aligned}&\int_\Omega \int_0^\infty u(\mathbf{x}) \Phi\big(f(t\mathbf{x})\big) d\lambda(t)\, d\mu(\mathbf{x}) - \int_\Omega \int_0^\infty u(\mathbf{x}) \Phi\big(Af(\mathbf{x})\big) d\lambda(t)\, d\mu(\mathbf{x}) \\ &\qquad - \int_\Omega \int_0^\infty u(\mathbf{x}) \varphi\big(Af(\mathbf{x})\big)\big(f(t\mathbf{x}) - Af(\mathbf{x})\big) d\lambda(t)\, d\mu(\mathbf{x}) \\ &\quad \geq \int_\Omega \int_0^\infty u(\mathbf{x}) \left|\left|\Phi\big(f(t\mathbf{x})\big) - \Phi\big(Af(\mathbf{x})\big)\right| - \left|\varphi\big(Af(\mathbf{x})\big)\right| \cdot \left|f(t\mathbf{x}) - Af(\mathbf{x})\right|\right| d\lambda(t)\, d\mu(\mathbf{x}) \\ &\quad \geq \int_\Omega u(\mathbf{x}) \left| \int_0^\infty \left|\Phi\big(f(t\mathbf{x})\big) - \Phi\big(Af(\mathbf{x})\big)\right| d\lambda(t) - \left|\varphi\big(Af(\mathbf{x})\big)\right| \int_0^\infty \left|f(t\mathbf{x}) - Af(\mathbf{x})\right| d\lambda(t) \right| d\mu(\mathbf{x}) \\ &\quad \geq \left| \int_\Omega u(\mathbf{x}) \int_0^\infty \left|\Phi\big(f(t\mathbf{x})\big) - \Phi\big(Af(\mathbf{x})\big)\right| d\lambda(t)\, d\mu(\mathbf{x}) \right. \\ &\qquad \left. - \int_\Omega u(\mathbf{x}) \left|\varphi\big(Af(\mathbf{x})\big)\right| \int_0^\infty \left|f(t\mathbf{x}) - Af(\mathbf{x})\right| d\lambda(t)\, d\mu(\mathbf{x}) \right|.\end{aligned} \tag{2.12}$$

By using Fubini's and the Radon-Nikodym theorems, the substitution $\mathbf{y} = t\mathbf{x}$, and the fact that the set Ω is λ-balanced and the function Φ is nonnegative, the first integral on the left-hand side of (2.12) becomes

$$\begin{aligned}
&\int_\Omega \int_0^\infty u(\mathbf{x})\Phi\big(f(t\mathbf{x})\big)d\lambda(t)\, d\mu(\mathbf{x}) \\
&\quad = \int_0^\infty \int_\Omega u(\mathbf{x})\Phi\big(f(t\mathbf{x})\big)d\mu(\mathbf{x})\, d\lambda(t) \\
&\quad = \int_0^\infty \int_{t\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\Phi\big(f(\mathbf{y})\big)d\mu_t(\mathbf{y})\, d\lambda(t) \\
&\quad \le \int_0^\infty \int_\Omega u\left(\frac{1}{t}\mathbf{y}\right)\Phi\big(f(\mathbf{y})\big)d\mu_t(\mathbf{y})\, d\lambda(t) \\
&\quad = \int_0^\infty \int_\Omega u\left(\frac{1}{t}\mathbf{y}\right)\Phi\big(f(\mathbf{y})\big)\frac{d\mu_t}{d\nu}(\mathbf{y})d\nu(\mathbf{y})\, d\lambda(t) \\
&\quad = \int_\Omega \left(\int_0^\infty u\left(\frac{1}{t}\mathbf{y}\right)\frac{d\mu_t}{d\nu}(\mathbf{y})d\lambda(t)\right)\Phi\big(f(\mathbf{y})\big)d\nu(\mathbf{y}) \\
&\quad = \int_\Omega v(\mathbf{y})\Phi\big(f(\mathbf{y})\big)d\nu(\mathbf{y}).
\end{aligned} \tag{2.13}$$

Further, the second integral on the left-hand side in (2.12) reduces to

$$\int_\Omega \int_0^\infty u(\mathbf{x})\Phi\big(Af(\mathbf{x})\big)d\lambda(t)\, d\mu(\mathbf{x}) = L\int_\Omega u(\mathbf{x})\Phi\big(Af(\mathbf{x})\big)d\mu(\mathbf{x}), \tag{2.14}$$

while the corresponding third integral is equal to 0 since by (2.11) we have

$$\begin{aligned}
&\int_\Omega \int_0^\infty u(\mathbf{x})\varphi\big(Af(\mathbf{x})\big)\big(f(t\mathbf{x}) - Af(\mathbf{x})\big)d\lambda(t)\, d\mu(\mathbf{x}) \\
&\quad = \int_\Omega u(\mathbf{x})\varphi\big(Af(\mathbf{x})\big)\left(\int_0^\infty \big(f(t\mathbf{x}) - Af(\mathbf{x})\big)d\lambda(t)\right)d\mu(\mathbf{x}) = 0.
\end{aligned} \tag{2.15}$$

Finally, (2.7) holds by combining (2.12), (2.13), (2.14), and (2.15).

It remains to prove the last part of the statement of Theorem 2.1. If Φ is a nonpositive concave function, then $-\Phi$ is a nonnegative convex function and (2.9) becomes

$$\begin{aligned}
\Phi(r) - \Phi(s) - \varphi(r)(r-s) &= \big|\Phi(r) - \Phi(s) - \varphi(r)(r-s)\big| \\
&\ge \big||\Phi(s) - \Phi(r)| - |\varphi(r)||s-r|\big|,
\end{aligned} \tag{2.16}$$

where $\varphi : I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x) = [\Phi'_+(x), \Phi'_-(x)]$ for all $x \in \operatorname{Int} I$. Following the same lines as in the proof for a convex function, we get (2.7) with swapped order of two integrals on its left-hand side. □

Remark 2.2. Observe that a pair of inequalities interpolated between the left-hand side and the right-hand side of (2.12) provides other new refinements of (2.7).

It is important to notice that the condition on nonnegativity of the convex function Φ in Theorem 2.1 can be omitted only in a particular setting with cones in X. More precisely, the following corollary holds.

Corollary 2.3. *If in Theorem 2.1 one has $t\Omega = \Omega$ for λ-a.e. $t \in \operatorname{supp} \lambda$, then* (2.7) *holds for all convex functions Φ on an interval $I \subseteq \mathbb{R}$. In this setting, inequality* (2.7) *holds also for all concave functions Φ on $I \subseteq \mathbb{R}$, but with swapped order of the integrals on its left-hand side.*

As a consequence of Theorem 2.1, we get a weighted general Boas-type inequality obtained in [6]. It is given in the following corollary.

Corollary 2.4. *Suppose that $\lambda, L, X, \mu, \nu, \mu_t, \Omega, u$, and v are as in Theorem 2.1. If Φ is a nonnegative convex function on an interval $I \subseteq \mathbb{R}$, then the inequality*

$$\int_\Omega u(\mathbf{x})\Phi\big(Af(\mathbf{x})\big)d\mu(\mathbf{x}) \le \frac{1}{L}\int_\Omega v(\mathbf{x})\Phi\big(f(\mathbf{x})\big)d\nu(\mathbf{x}) \tag{2.17}$$

holds for all measurable functions $f : \Omega \to \mathbb{R}$ such that $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$, where Af is defined by (2.4). *For a nonpositive concave function Φ, the sign of inequality in* (2.7) *is reversed. Moreover, if in Theorem 2.1 one has $t\Omega = \Omega$ for λ-a.e. $t \in \operatorname{supp} \lambda$, then* (2.7) *holds for all convex functions Φ on an interval $I \subseteq \mathbb{R}$. In that case, for all concave functions Φ, inequality* (2.7) *holds with the sign of inequality reversed.*

Remark 2.5. Observe that Theorem 2.1 generalizes and refines the Boas-type inequality (1.3) obtained by Luor [5]. Namely, since the right-hand side of (2.7) is nonnegative, for $u(\mathbf{x}) \equiv 1$, $\mathbf{x} \in \Omega$, inequality (2.17) reduces to (1.3).

3. New Multidimensional Hardy- and Pólya-Knopp-Type Inequalities

In this section, we apply Theorem 2.1 to a particular multidimensional setting, namely, to balls in $\mathbb{R}^n$ centered at the origin and to their dual sets. The results obtained represent a new class of n-dimensional Hardy and Pólya-Knopp-type inequalities, different from the existing inequalities (1.13), (1.14), (1.15), and (1.17). Moreover, the constant factors appearing on the right-hand sides of our relations are the best possible.

Our first result in this direction is a refinement of an inequality by Luor [5, relation (1.14)], related to cones in $\mathbb{R}^n$.

Theorem 3.1. *Let λ be a finite Borel measure on $\mathbb{R}_+$, L be defined by* (2.2), *Ω be a λ-balanced Borel set in $\mathbb{R}^n$, and B be a Borel subset of the unit sphere S^{n-1}. Let u be a nonnegative function on $\mathbb{R}^n$, Φ be a nonnegative convex function on an interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ be any function fulfilling $\varphi(x) \in \partial\Phi(x)$, for all $x \in \operatorname{Int} I$. Finally, let $f : \Omega \to \mathbb{R}$ be a measurable function with values in I.*

(i) *If* $\operatorname{supp}\lambda \subseteq \langle 0,1]$, $0 < R \le \infty$, *and* $\Omega = \Omega_1 = \{\mathbf{x} = \xi S : S \in B,\ 0 \le \xi < R\}$, *then*

$$\begin{aligned}
&\frac{1}{L}\int_{\Omega_1}\Phi(f(\mathbf{x}))\left(\int_{|\mathbf{x}|/R}^{1} u\left(\frac{1}{t}\mathbf{x}\right)t^{1-n}d\lambda(t)\right)\frac{d\mathbf{x}}{|\mathbf{x}|} - \int_{\Omega_1} u(\mathbf{x})\Phi(A_1 f(\mathbf{x}))\frac{d\mathbf{x}}{|\mathbf{x}|} \\
&\quad \ge \frac{1}{L}\left|\int_{\Omega_1} u(\mathbf{x})\int_0^1 \left|\Phi(f(t\mathbf{x})) - \Phi(A_1 f(\mathbf{x}))\right| d\lambda(t)\,\frac{d\mathbf{x}}{|\mathbf{x}|}\right. \\
&\qquad \left. - \int_{\Omega_1} u(\mathbf{x})\int_0^1 \left|\varphi(A_1 f(\mathbf{x}))\right| \cdot \left|f(t\mathbf{x}) - A_1 f(\mathbf{x})\right| d\lambda(t)\,\frac{d\mathbf{x}}{|\mathbf{x}|}\right|,
\end{aligned} \tag{3.1}$$

where $A_1 f(\mathbf{x}) = \int_0^1 f(t\mathbf{x})d\lambda(t),\ \mathbf{x} \in \Omega_1$.

(ii) *If* $\operatorname{supp}\lambda \subseteq [1,\infty\rangle$, $0 \le R < \infty$, *and* $\Omega = \Omega_2 = \{\mathbf{x} = \xi S : S \in B, R \le \xi < \infty\}$, *then*

$$\begin{aligned}
&\frac{1}{L}\int_{\Omega_2}\Phi(f(\mathbf{x}))\left(\int_{1}^{|\mathbf{x}|/R} u\left(\frac{1}{t}\mathbf{x}\right)t^{1-n}d\lambda(t)\right)\frac{d\mathbf{x}}{|\mathbf{x}|} - \int_{\Omega_2} u(\mathbf{x})\Phi(A_2 f(\mathbf{x}))\frac{d\mathbf{x}}{|\mathbf{x}|} \\
&\quad \ge \frac{1}{L}\left|\int_{\Omega_2} u(\mathbf{x})\int_1^\infty \left|\Phi(f(t\mathbf{x})) - \Phi(A_2 f(\mathbf{x}))\right| d\lambda(t)\,\frac{d\mathbf{x}}{|\mathbf{x}|}\right. \\
&\qquad \left. - \int_{\Omega_2} u(\mathbf{x})\int_1^\infty \left|\varphi(A_2 f(\mathbf{x}))\right| \cdot \left|f(t\mathbf{x}) - A_2 f(\mathbf{x})\right| d\lambda(t)\,\frac{d\mathbf{x}}{|\mathbf{x}|}\right|,
\end{aligned} \tag{3.2}$$

where $A_2 f(\mathbf{x}) = \int_1^\infty f(t\mathbf{x})d\lambda(t),\ \mathbf{x} \in \Omega_2$.

Proof. Relation (3.1) is a direct consequence of Theorem 2.1, rewritten with $X = \mathbb{R}^n$, $\Omega = \Omega_1$, $d\mu(\mathbf{x}) = \chi_{\Omega_1}(\mathbf{x})d\mathbf{x}$, and $d\nu(\mathbf{x}) = d\mathbf{x}$, as well as with the function u replaced with $\mathbf{x} \mapsto |\mathbf{x}|^{-1}u(\mathbf{x})$. Then we have $(d\mu_t/d\nu)(\mathbf{x}) = t^{-n}\chi_{t\Omega_1}(\mathbf{x}),\ t \in \langle 0,1]$,

$$v(\mathbf{x}) = \int_{|\mathbf{x}|/R}^{1} u\left(\frac{1}{t}\mathbf{x}\right)t^{1-n}d\lambda(t), \qquad \mathbf{x} \in \Omega_1, \tag{3.3}$$

and $Af(\mathbf{x}) = A_1 f(\mathbf{x}),\ x \in \Omega_1$, so relation (1.13) reduces to (3.1). The proof of (3.2) follows the same lines, by considering $\Omega = \Omega_2$. In such setting we get

$$v(\mathbf{x}) = \int_{1}^{|\mathbf{x}|/R} u\left(\frac{1}{t}\mathbf{x}\right)t^{1-n}d\lambda(t), \qquad \mathbf{x} \in \Omega_2, \tag{3.4}$$

and $Af = A_2 f$. □

By taking $B = S^{n-1}$, that is, by setting a λ-balanced set Ω to be a ball in $\mathbb{R}^n$ centered at the origin or its corresponding dual set, and by choosing a suitable measure λ and a weight function u, we obtain the following sequence of new refined strengthened inequalities of the Hardy type.

Theorem 3.2. *Let $n \in \mathbb{N}$, $p \in \mathbb{R} \setminus [0,1\rangle$, and $k \in \mathbb{R}$, $k \neq n$.*

(i) *If $0 < R \leq \infty$, $p/(k-n) > 0$, and f is a nonnegative measurable function on $B(R)$, then the inequality*

$$\begin{aligned}
&\left(\frac{p}{k-n}\right)^p \int_{B(R)} |\mathbf{x}|^{p-k}\left[1-\left(\frac{|\mathbf{x}|}{R}\right)^{(k-n)/p}\right] f^p(\mathbf{x})d\mathbf{x} - \int_{B(R)} |\mathbf{x}|^{-k}(Hf(\mathbf{x}))^p d\mathbf{x} \\
&\quad \geq \left|\left(\frac{p}{k-n}\right)^{p-1} \int_{B(R)} |\mathbf{x}|^{-k} \int_0^1 t^{((k-n)/p)-1}\left|\,|\mathbf{x}|^p t^{n-k+p} f^p(t\mathbf{x}) - \left(\frac{k-n}{p}\right)^p (Hf(\mathbf{x}))^p\right| dt\, d\mathbf{x}\right. \\
&\qquad \left. -|p| \int_{B(R)} |\mathbf{x}|^{-k}(Hf(\mathbf{x}))^{p-1} \cdot \int_0^1 \left|\,|\mathbf{x}| f(t\mathbf{x}) - \frac{k-n}{p} t^{((k-n)/p)-1} Hf(\mathbf{x})\right| dt\, d\mathbf{x}\right|
\end{aligned} \tag{3.5}$$

holds, where

$$Hf(\mathbf{x}) = |\mathbf{x}| \int_0^1 f(t\mathbf{x})dt, \quad \mathbf{x} \in B(R). \tag{3.6}$$

(ii) *If $0 \leq R < \infty$, $p/(k-n) < 0$, and f is a nonnegative measurable function on $\mathbb{R}^n \setminus B(R)$, then*

$$\begin{aligned}
&\left(\frac{p}{n-k}\right)^p \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{p-k}\left[1-\left(\frac{R}{|\mathbf{x}|}\right)^{(n-k)/p}\right] f^p(\mathbf{x})d\mathbf{x} - \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-k}\left(\widetilde{H}f(\mathbf{x})\right)^p d\mathbf{x} \\
&\quad \geq \left|\left(\frac{p}{n-k}\right)^{p-1} \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-k} \int_1^\infty t^{((k-n)/p)-1}\left|\,|\mathbf{x}|^p t^{n-k+p} f^p(t\mathbf{x}) - \left(\frac{n-k}{p}\right)^p \left(\widetilde{H}f(\mathbf{x})\right)^p\right| dt\, d\mathbf{x}\right. \\
&\qquad \left. -|p| \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-k}\left(\widetilde{H}f(\mathbf{x})\right)^{p-1} \cdot \int_1^\infty \left|\,|\mathbf{x}| f(t\mathbf{x}) - \frac{n-k}{p} t^{((k-n)/p)-1} \widetilde{H}f(\mathbf{x})\right| dt\, d\mathbf{x}\right|,
\end{aligned} \tag{3.7}$$

where

$$\widetilde{H}f(\mathbf{x}) = |\mathbf{x}| \int_1^\infty f(t\mathbf{x})dt, \quad \mathbf{x} \in \mathbb{R}^n \setminus B(R). \tag{3.8}$$

For $p \in \langle 0,1]$ relations (3.5) *and* (3.7) *hold with swapped order of the integrals on their respective left-hand sides.*

Proof. Follows from Theorems 2.1 and 3.1 by rewriting (2.7), that is, (3.1) and (3.2), with some particular parameters. Namely, let $X = \mathbb{R}^n$, $I = [0,\infty\rangle$, $u(\mathbf{x}) = |\mathbf{x}|^{-n}$, $d\nu(\mathbf{x}) = d\mathbf{x}$, and $\Phi(x) = x^p$,

$p \neq 0$, that is, $\varphi(x) = px^{p-1}$. In the case (i), let also $\Omega = B(R^{(k-n)/p})$, $d\lambda(t) = \chi_{\langle 0,1\rangle}(t)dt$, and $d\mu(\mathbf{x}) = \chi_{B(R^{(k-n)/p})}(\mathbf{x})d\mathbf{x}$. Then we have $L = 1$, $(d\mu_t/d\nu)(\mathbf{x}) = t^{-n}\chi_{B(tR^{(k-n)/p})}(\mathbf{x})$, and

$$v(\mathbf{x}) = \int_0^1 \left|\frac{1}{t}\mathbf{x}\right|^{-n} t^{-n}\chi_{B(tR^{(k-n)/p})}(\mathbf{x})dt = |\mathbf{x}|^{-n}\left(1 - \frac{|\mathbf{x}|}{R^{(k-n)/p}}\right), \quad \mathbf{x} \in B\left(R^{(k-n)/p}\right). \tag{3.9}$$

Replace the function f in (2.7) with the function $g : B(R^{(k-n)/p}) \to \mathbb{R}$, $g(\mathbf{x}) = |\mathbf{x}|^{(p/(k-n))-1} f(|\mathbf{x}|^{(p/(k-n))-1}\mathbf{x})$. Then

$$\begin{aligned} Ag(\mathbf{x}) &= |\mathbf{x}|^{(p/(k-n))-1}\int_0^1 t^{(p/(k-n))-1} f\left(t^{p/(k-n)}|\mathbf{x}|^{p/(k-n)}\frac{\mathbf{x}}{|\mathbf{x}|}\right)dt \\ &= \frac{k-n}{p}\frac{1}{|\mathbf{x}|}\int_0^{|\mathbf{x}|^{p/(k-n)}} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right)dr, \end{aligned} \tag{3.10}$$

where we applied the substitution $r = (t|\mathbf{x}|)^{p/(k-n)}$. In this setting, by using polar coordinates, the first integral on the left-hand side of inequality (2.7) becomes

$$\begin{aligned} &\int_{B(R^{(k-n)/p})} |\mathbf{x}|^{p((p/(k-n))-1)-n}\left(1 - \frac{|\mathbf{x}|}{R^{(k-n)/p}}\right) f^p\left(|\mathbf{x}|^{(p/(k-n))-1}\mathbf{x}\right)d\mathbf{x} \\ &\quad = \int_{S^{n-1}} dS \int_0^{R^{(k-n)/p}} r^{p((p/(k-n))-1)-1}\left(1 - \frac{r}{R^{(k-n)/p}}\right) f^p\left(r^{p/(k-n)}S\right)dr \\ &\quad = \frac{k-n}{p}\int_{S^{n-1}} dS \int_0^R t^{n-1+p-k}\left[1 - \left(\frac{t}{R}\right)^{(k-n)/p}\right] f^p(tS)dt \\ &\quad = \frac{k-n}{p}\int_{B(R)} |\mathbf{x}|^{p-k}\left[1 - \left(\frac{|\mathbf{x}|}{R}\right)^{(k-n)/p}\right] f^p(\mathbf{x})d\mathbf{x}, \end{aligned} \tag{3.11}$$

while the second integral on the left-hand side of (2.7) reduces to

$$\begin{aligned} &\left(\frac{k-n}{p}\right)^p \int_{B(R^{(k-n)/p})} |\mathbf{x}|^{-n-p}\left(\int_0^{|\mathbf{x}|^{p/(k-n)}} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right)dr\right)^p d\mathbf{x} \\ &\quad = \left(\frac{k-n}{p}\right)^p \int_{S^{n-1}} dS \int_0^{R^{(k-n)/p}} t^{-p-1}\left(\int_0^{t^{p/(k-n)}} f(rS)dr\right)^p dt \\ &\quad = \left(\frac{k-n}{p}\right)^{p+1} \int_{S^{n-1}} dS \int_0^R s^{n-k-1}\left(\int_0^s f(rS)dr\right)^p ds \\ &\quad = \left(\frac{k-n}{p}\right)^{p+1} \int_{B(R)} |\mathbf{x}|^{-k}(Hf(\mathbf{x}))^p d\mathbf{x}. \end{aligned} \tag{3.12}$$

Analogously, on the right-hand side of (2.7) we get

$$\begin{aligned}
&\Bigg|\int_{B(R^{(k-n)/p})} |\mathbf{x}|^{-n-p}\int_0^1 \Bigg| t^{p((p/(k-n))-1)}\,|\mathbf{x}|^{p(p/(k-n))} f^p\left(t^{p/(k-n)}|\mathbf{x}|^{p/(k-n)}\frac{\mathbf{x}}{|\mathbf{x}|}\right) \\
&\qquad\qquad - \left(\frac{k-n}{p}\right)^p \left(\int_0^{|\mathbf{x}|^{p/(k-n)}} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right)dr\right)^p\Bigg| dt\,d\mathbf{x} \\
&\quad - \left(\frac{k-n}{p}\right)^{p-1}|p|\int_{B(R^{(k-n)/p})}|\mathbf{x}|^{-n-p}\left(\int_0^{|\mathbf{x}|^{p/(k-n)}} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right)dr\right)^{p-1} \\
&\quad \cdot \int_0^1 \left| t^{(p/(k-n))-1}|\mathbf{x}|^{p/(k-n)} f\left(t^{p/(k-n)}|\mathbf{x}|^{p/(k-n)}\frac{\mathbf{x}}{|\mathbf{x}|}\right) - \frac{k-n}{p}\int_0^{|\mathbf{x}|^{p/(k-n)}} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right)dr\right| dt\,d\mathbf{x}\Bigg| \\
&= \Bigg|\frac{k-n}{p}\int_{S^{n-1}} dS\int_0^R z^{n-k-1}\int_0^1 \Bigg| z^p t^{p((p/(k-n))-1)} f^p\left(t^{p/(k-n)}zS\right) \\
&\qquad\qquad - \left(\frac{k-n}{p}\right)^p\left(\int_0^z f(rS)dr\right)^p\Bigg| dt\,dz \\
&\quad - \left(\frac{k-n}{p}\right)^p|p|\int_{S^{n-1}} dS\int_0^R z^{n-k-1}\left(\int_0^z f(rS)\ dr\right)^{p-1} \\
&\quad \cdot \int_0^1\left| t^{(p/(k-n))-1} z f\left(t^{p/(k-n)}zS\right) - \frac{k-n}{p}\int_0^z f(rS)dr\right| dt\,dz\Bigg| \\
&= \Bigg|\left(\frac{k-n}{p}\right)^2\int_{B(R)}|\mathbf{x}|^{-k}\int_0^1 t^{((k-n)/p)-1}\left||\mathbf{x}|^p t^{n-k+p} f^p(t\mathbf{x}) - \left(\frac{k-n}{p}\right)^p (Hf(\mathbf{x}))^p\right| dt\,d\mathbf{x} \\
&\quad - \left(\frac{k-n}{p}\right)^{p+1}|p|\int_{B(R)}|\mathbf{x}|^{-k}(Hf(\mathbf{x}))^{p-1} \\
&\quad \cdot \int_0^1\left||\mathbf{x}|f(t\mathbf{x}) - \frac{k-n}{p}\ \ t^{((k-n)/p)-1}Hf(\mathbf{x})\right| dt\,d\mathbf{x}\Bigg|.
\end{aligned} \tag{3.13}$$

Finally, (3.5) holds by combining (3.11), (3.12), and (3.13).

To obtain relation (3.7), that is, the case (ii), we consider $\Omega = \mathbb{R}^n \setminus B(R^{(n-k)/p})$, $d\lambda(t) = \chi_{\langle 1,\infty\rangle}(t)(dt/t^2)$, and $d\mu(\mathbf{x}) = \chi_{\mathbb{R}^n\setminus B(R^{(n-k)/p})}(\mathbf{x})d\mathbf{x}$. As in the case (i), then we have $L = 1$, $(d\mu_t/d\nu)(\mathbf{x}) = t^{-n}\chi_{\mathbb{R}^n\setminus B(tR^{(n-k)/p})}(\mathbf{x})$, and

$$v(\mathbf{x}) = |\mathbf{x}|^{-n}\left(1 - \frac{R^{(n-k)/p}}{|\mathbf{x}|}\right), \qquad \mathbf{x}\in\mathbb{R}^n\setminus B\left(R^{(n-k)/p}\right). \tag{3.14}$$

Inequality (3.7) now follows by rewriting (2.7) with the above parameters and with the function $g : \mathbb{R}^n \setminus B(R^{(n-k)/p}) \to \mathbb{R}$, $g(\mathbf{x}) = |\mathbf{x}|^{(p/(n-k))+1} f(|\mathbf{x}|^{(p/(n-k))-1}\mathbf{x})$ instead of f, and by using analogous techniques as in the proof of inequality (3.5). □

Observe that the right-hand sides of inequalities (3.5) and (3.7) are nonnegative. Moreover, the constants involved in their left-hand sides are shown to be the best possible. That result is given in the following theorem.

Theorem 3.3. *Let $n \in \mathbb{N}$, $p \in \mathbb{R} \setminus [0,1\rangle$, and $k \in \mathbb{R}$, $k \neq n$.*

(i) *If $0 < R \leq \infty$, $p/(k-n) > 0$, f is a nonnegative function on $B(R)$, and Hf is defined by* (3.6), *then*

$$\int_{B(R)} |\mathbf{x}|^{-k} (Hf(\mathbf{x}))^p d\mathbf{x} \leq \left(\frac{p}{k-n}\right)^p \int_{B(R)} |\mathbf{x}|^{p-k} \left[1 - \left(\frac{|\mathbf{x}|}{R}\right)^{(k-n)/p}\right] f^p(\mathbf{x}) d\mathbf{x}. \tag{3.15}$$

(ii) *If $0 \leq R < \infty$, $p/(k-n) < 0$, f is a nonnegative function on $\mathbb{R}^n \setminus B(R)$, and $\widetilde{H}f$ is given by* (3.8), *then*

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-k} \left(\widetilde{H}f(\mathbf{x})\right)^p d\mathbf{x} \\ &\quad \leq \left(\frac{p}{n-k}\right)^p \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{p-k} \left[1 - \left(\frac{R}{|\mathbf{x}|}\right)^{(n-k)/p}\right] f^p(\mathbf{x}) d\mathbf{x}. \end{aligned} \tag{3.16}$$

The constant $|p/(k-n)|^p$ is the best possible for both inequalities. For $p \in \langle 0,1]$, the signs of inequality in (3.15) *and* (3.16) *are reversed.*

Proof. We only need to prove that $|p/(k-n)|^p$ is the best possible constant for inequalities (3.15) and (3.16). Consider the case (i) first. For a sufficiently small $\varepsilon > 0$, and the function $f_\varepsilon : B(R) \to \mathbb{R}$ defined by $f_\varepsilon(\mathbf{x}) = |\mathbf{x}|^{((k-n+\varepsilon)/p)-1}$, the left-hand side of (3.15) is equal to

$$\begin{aligned} L_\varepsilon &= \int_{B(R)} |\mathbf{x}|^{-n+\varepsilon} \left(\int_0^1 t^{((k-n+\varepsilon)/p)-1} dt\right)^p d\mathbf{x} = \left(\frac{p}{k-n+\varepsilon}\right)^p \int_{B(R)} |\mathbf{x}|^{-n+\varepsilon} d\mathbf{x} \\ &= \left(\frac{p}{k-n+\varepsilon}\right)^p \int_{S^{n-1}} dS \int_0^R r^{\varepsilon-1} dr = \left(\frac{p}{k-n+\varepsilon}\right)^p \left|S^{n-1}\right| \cdot \frac{R^\varepsilon}{\varepsilon}, \end{aligned} \tag{3.17}$$

while on the right-hand side of (3.15) we get

$$\begin{aligned} R_\varepsilon &= \left(\frac{p}{k-n}\right)^p \int_{B(R)} |\mathbf{x}|^{-n+\varepsilon} \left[1 - \left(\frac{|\mathbf{x}|}{R}\right)^{(k-n)/p}\right] d\mathbf{x} \\ &\leq \left(\frac{p}{k-n}\right)^p \int_{B(R)} |\mathbf{x}|^{-n+\varepsilon} d\mathbf{x} = \left(\frac{p}{k-n}\right)^p \left|S^{n-1}\right| \cdot \frac{R^\varepsilon}{\varepsilon}. \end{aligned} \tag{3.18}$$

Therefore, $1 \le R_\varepsilon / L_\varepsilon \le ((k-n+\varepsilon)/(k-n))^p$. Since $((k-n+\varepsilon)/(k-n))^p \searrow 1$, as $\varepsilon \searrow 0$, the constant $(p/(k-n))^p$ is the best possible for (3.15). The proof that the constant $(p/(n-k))^p$ is the best possible for (3.16) follows the same lines, considering the function $f_\varepsilon : \mathbb{R}^n \backslash B(R) \to \mathbb{R}$, $f_\varepsilon(\mathbf{x}) = |\mathbf{x}|^{((k-n-\varepsilon)/p)-1}$. □

If $R = \infty$ in (3.15) and $R = 0$ in (3.16), we immediately get the following new multidimensional Hardy-type inequality.

Corollary 3.4. *Let $n \in \mathbb{N}$, $p \in \mathbb{R} \setminus [0,1\rangle$, and $k \in \mathbb{R}$, $k \neq n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function and Hf and $\widetilde{H}f$ be defined by* (3.6) *and* (3.8)*, respectively. If $p/(k-n) > 0$, then the inequality*

$$\int_{\mathbb{R}} |\mathbf{x}|^{-k} (Hf(\mathbf{x}))^p d\mathbf{x} \le \left(\frac{p}{k-n}\right)^p \int_{\mathbb{R}^n} |\mathbf{x}|^{p-k} f^p(\mathbf{x}) d\mathbf{x} \tag{3.19}$$

holds, while for $p/(k-n) < 0$ one has

$$\int_{\mathbb{R}^n} |\mathbf{x}|^{-k} \left(\widetilde{H}f(\mathbf{x})\right)^p d\mathbf{x} \le \left(\frac{p}{n-k}\right)^p \int_{\mathbb{R}^n} |\mathbf{x}|^{p-k} f^p(\mathbf{x}) d\mathbf{x}. \tag{3.20}$$

The constant $|p/(k-n)|^p$ is the best possible for both inequalities. Moreover, for $p \in \langle 0,1]$, the signs of inequality in (3.19) *and* (3.20) *are reversed.*

We continue our analysis by obtaining the corresponding refined Pólya-Knopp-type inequality.

Theorem 3.5. *Let $n \in \mathbb{N}$.*

(i) *If $0 < R \le \infty$, f is a positive measurable function on $B(R)$, and*

$$Gf(\mathbf{x}) = \exp\left(\int_0^1 \log f(t\mathbf{x}) dt\right), \quad \mathbf{x} \in B(R), \tag{3.21}$$

then the inequality

$$\begin{aligned} &e^n \int_{B(R)} \left(1 - \frac{|\mathbf{x}|}{R}\right) f(\mathbf{x}) d\mathbf{x} - \int_{B(R)} Gf(\mathbf{x}) d\mathbf{x} \\ &\quad \ge \left| \int_{B(R)} \int_0^1 \left|(et)^n f(t\mathbf{x}) - Gf(\mathbf{x})\right| dt\, d\mathbf{x} - \int_{B(R)} Gf(\mathbf{x}) \int_0^1 \left|\log \frac{(et)^n f(t\mathbf{x})}{Gf(\mathbf{x})}\right| dt\, d\mathbf{x} \right| \end{aligned} \tag{3.22}$$

holds.

(ii) *If $0 \le R < \infty$, f is a positive measurable function on $\mathbb{R}^n \setminus B(R)$, and*

$$\widetilde{G}f(\mathbf{x}) = \exp\left(\int_1^\infty \log f(t\mathbf{x}) \frac{dt}{t^2}\right), \quad \mathbf{x} \in \mathbb{R}^n \setminus B(R), \tag{3.23}$$

then

$$
\begin{aligned}
&e^{-n}\int_{\mathbb{R}^n\setminus B(R)}\left(1-\frac{R}{|\mathbf{x}|}\right)f(\mathbf{x})d\mathbf{x}-\int_{\mathbb{R}^n\setminus B(R)}\widetilde{G}f(\mathbf{x})d\mathbf{x}\\
&\quad\geq\left|\int_{\mathbb{R}^n\setminus B(R)}\int_1^\infty\left|\left(\frac{t}{e}\right)^n f(t\mathbf{x})-\widetilde{G}f(\mathbf{x})\right|\frac{dt}{t^2}\,d\mathbf{x}\right.\\
&\qquad\left.-\int_{\mathbb{R}^n\setminus B(R)}\widetilde{G}f(\mathbf{x})\int_1^\infty\left|\log\frac{(et)^n f(t\mathbf{x})}{\widetilde{G}f(\mathbf{x})}\right|\frac{dt}{t^2}\,d\mathbf{x}\right|.
\end{aligned}
\tag{3.24}
$$

Proof. Follows from Theorems 2.1 and 3.1 by considering $X=\mathbb{R}^n$, $I=\mathbb{R}$, $u(\mathbf{x})=|B(|\mathbf{x}|)|^{-1}$, $d\nu(\mathbf{x})=d\mathbf{x}$, and $\Phi(x)=\varphi(x)=e^x$. To get (3.22), we also set $\Omega=B(R)$, $d\lambda(t)=\chi_{\langle 0,1\rangle}(t)dt$, and $d\mu(\mathbf{x})=\chi_{B(R)}(\mathbf{x})\quad d\mathbf{x}$. In that case, we have $L=1$, $(d\mu_t/d\nu)(\mathbf{x})=t^{-n}\chi_{B(tR)}(\mathbf{x})$, and

$$
\begin{aligned}
v(\mathbf{x})&=\int_0^1\frac{1}{|B(|(1/t)\mathbf{x}|)|}t^{-n}\chi_{B(tR)}(\mathbf{x})dt=\frac{1}{|B(|\mathbf{x}|)|}\int_0^1\chi_{B(tR)}(\mathbf{x})dt\\
&=\frac{1}{|B(|\mathbf{x}|)|}\left(1-\frac{|\mathbf{x}|}{R}\right),\quad \mathbf{x}\in B(R).
\end{aligned}
\tag{3.25}
$$

Further, replace the function f in (2.7) with the function $\mathbf{x}\mapsto\log(|B(|\mathbf{x}|)|f(\mathbf{x}))$. The first integral on the left-hand side of (2.7) then becomes

$$
\int_{B(R)}\frac{1}{|B(|\mathbf{x}|)|}\left(1-\frac{|\mathbf{x}|}{R}\right)|B(|\mathbf{x}|)|f(\mathbf{x})d\mathbf{x}=\int_{B(R)}\left(1-\frac{|\mathbf{x}|}{R}\right)f(\mathbf{x})d\mathbf{x},
\tag{3.26}
$$

the corresponding second integral reduces to

$$
\begin{aligned}
&\int_{B(R)}\frac{1}{|B(|\mathbf{x}|)|}\exp\left(\int_0^1\log\left(t^n|B(|\mathbf{x}|)|f(t\mathbf{x})\right)dt\right)d\mathbf{x}\\
&\quad=\int_{B(R)}\frac{1}{|B(|\mathbf{x}|)|}\exp\left(-n+\log|B(|\mathbf{x}|)|+\log\int_0^1\log f(t\mathbf{x})dt\right)d\mathbf{x}\\
&\quad=e^{-n}\int_{B(R)}\exp\left(\int_0^1\log f(t\mathbf{x})dt\right)d\mathbf{x}=e^{-n}\int_{B(R)}Gf(\mathbf{x})d\mathbf{x},
\end{aligned}
\tag{3.27}
$$

while on the right-hand side of (2.7) we obtain

$$\begin{aligned}
&\left|\int_{B(R)} \frac{1}{|B(|\mathbf{x}|)|} \int_0^1 \left|t^n |B(|\mathbf{x}|)| f(t\mathbf{x}) - e^{-n} |B(|\mathbf{x}|)| Gf(\mathbf{x})\right| dt\, d\mathbf{x}\right. \\
&\quad - \int_{B(R)} \frac{1}{|B(|\mathbf{x}|)|} \int_0^1 \left|e^{-n} |B(|\mathbf{x}|)| Gf(\mathbf{x})\right| \\
&\quad \cdot \left|\log\left(t^n |B(|\mathbf{x}|)| f(t\mathbf{x})\right) + n - \log|B(|\mathbf{x}|)| - \int_0^1 \log f(t\mathbf{x}) dt\right| dt\, d\mathbf{x}\Bigg| \\
&= \left|\int_{B(R)} \int_0^1 \left|t^n f(t\mathbf{x}) - e^{-n} Gf(\mathbf{x})\right| dt\, d\mathbf{x}\right. \\
&\quad \left. - e^{-n} \int_{B(R)} Gf(\mathbf{x}) \int_0^1 \left|n \log t + \log|B(|\mathbf{x}|)| + \log\left(f(t\mathbf{x})\right) + n - \log|B(|\mathbf{x}|)| - \log Gf(\mathbf{x})\right| dt\, d\mathbf{x}\right| \\
&= \left|\int_{B(R)} \int_0^1 \left|t^n f(t\mathbf{x}) - e^{-n} Gf(\mathbf{x})\right| dt\, d\mathbf{x} - e^{-n} \int_{B(R)} Gf(\mathbf{x}) \int_0^1 \left|\log \frac{(et)^n f(t\mathbf{x})}{Gf(\mathbf{x})}\right| dt\, d\mathbf{x}\right|.
\end{aligned} \tag{3.28}$$

Finally, (3.22) holds by combining (3.26), (3.27), and (3.28) and then multiplying the whole inequality by e^n.

Inequality (3.24) can be derived by a similar technique by taking $\Omega = \mathbb{R}^n \setminus B(R)$, $d\lambda(t) = \chi_{[1,\infty\rangle}(t)(dt/t^2)$, and $d\mu(\mathbf{x}) = \chi_{\mathbb{R}^n \setminus B(R)}(\mathbf{x}) d\mathbf{x}$. Then $L = 1$, $(d\mu_t/d\nu)(\mathbf{x}) = t^{-n} \chi_{\mathbb{R}^n \setminus B(tR)}(\mathbf{x})$ and

$$\begin{aligned}
v(\mathbf{x}) &= \int_1^\infty \left|B\left(\frac{1}{t}|\mathbf{x}|\right)\right|^{-1} t^{-n} \chi_{\mathbb{R}^n \setminus B(tR)}(\mathbf{x}) \frac{dt}{t^2} = \frac{1}{|B(|\mathbf{x}|)|} \int_1^{|\mathbf{x}|/R} \frac{dt}{t^2} \\
&= \frac{1}{|B(|\mathbf{x}|)|}\left(1 - \frac{R}{|\mathbf{x}|}\right), \qquad \mathbf{x} \in \mathbb{R}^n \setminus B(R),
\end{aligned} \tag{3.29}$$

so (3.24) follows by replacing the function f in (2.7) with $\mathbf{x} \mapsto \log(|B(|\mathbf{x}|)| f(\mathbf{x}))$. □

As a direct consequence of Theorem 3.5, we obtain the following strengthened Pólya-Knopp-type inequality.

Theorem 3.6. *Let $n \in \mathbb{N}$.*

(i) *If $0 < R \leq \infty$, f is a positive measurable function on $B(R)$, and Gf is defined by* (3.21), *then the inequality*

$$\int_{B(R)} Gf(\mathbf{x}) d\mathbf{x} \leq e^n \int_{B(R)} \left(1 - \frac{|\mathbf{x}|}{R}\right) f(\mathbf{x}) d\mathbf{x} \tag{3.30}$$

holds and the constant e^n is the best possible.

(ii) *If* $0 \le R < \infty$, f *is a positive measurable function on* $\mathbb{R}^n \setminus B(R)$, *and* $\widetilde{G}f$ *is defined by* (3.23), *then the inequality*

$$\int_{\mathbb{R}^n \setminus B(R)} \widetilde{G}f(\mathbf{x})d\mathbf{x} \le e^{-n} \int_{\mathbb{R}^n \setminus B(R)} \left(1 - \frac{R}{|\mathbf{x}|}\right) f(\mathbf{x})d\mathbf{x} \tag{3.31}$$

holds and the constant e^{-n} *is the best possible.*

Proof. Since the right-hand sides of (3.22) and (3.24) are nonnegative, inequalities (3.30) and (3.31) are their respective direct consequences. Now, we discuss the best possible constant for (3.30). For arbitrary $\varepsilon > 0$, let the function $f_\varepsilon : B(R) \to \mathbb{R}$ be defined by $f_\varepsilon(\mathbf{x}) = e^{-n}|B(|\mathbf{x}|)|^{\varepsilon-1}$. Calculating the left-hand side of (3.30) for f_ε, we obtain

$$\begin{aligned} L_\varepsilon &= \int_{B(R)} \exp\left(\int_0^1 \log\left(e^{-n}|B(|t\mathbf{x}|)|^{\varepsilon-1}\right)dt\right)d\mathbf{x} \\ &= e^{-n} \int_{B(R)} \exp\left(n(\varepsilon-1)\int_0^1 \log t\, dt + (\varepsilon-1)\log|B(|\mathbf{x}|)|\right)d\mathbf{x} \\ &= e^{-n\varepsilon} \int_{B(R)} |B(|\mathbf{x}|)|^{\varepsilon-1}d\mathbf{x} = e^{-n\varepsilon}\frac{|B(R)|^\varepsilon}{\varepsilon}. \end{aligned} \tag{3.32}$$

On the other hand, the right-hand side of (3.30), rewritten for f_ε, can be estimated as

$$R_\varepsilon = e^n \cdot e^{-n} \int_{B(R)} \left(1 - \frac{|\mathbf{x}|}{R}\right)|B(|\mathbf{x}|)|^{\varepsilon-1}d\mathbf{x} \le \int_{B(R)} |B(|\mathbf{x}|)|^{\varepsilon-1}d\mathbf{x} = \frac{|B(R)|^\varepsilon}{\varepsilon}. \tag{3.33}$$

Since $1 \le (R_\varepsilon/L_\varepsilon) \le e^{n\varepsilon} \searrow 1$, as $\varepsilon \searrow 0$, e^n is the best possible constant for inequality (3.30). The proof that e^{-n} is the best possible constant for (3.31) is similar, if the function $f_\varepsilon : \mathbb{R}^n \setminus B(R) \to \mathbb{R}$, $f_\varepsilon = e^n|B(|\mathbf{x}|)|^{-\varepsilon-1}$, is considered. □

For $R = \infty$ in (3.30) and $R = 0$ in (3.31), we get a new multidimensional Pólya-Knopp-type inequality.

Corollary 3.7. *If* f *is a positive measurable function on* $\mathbb{R}^n$, *and* Gf, $\widetilde{G}f$ *are, respectively, defined by* (3.21) *and* (3.23), *then the inequalities*

$$\begin{aligned} \int_{\mathbb{R}^n} Gf(\mathbf{x})d\mathbf{x} &\le e^n \int_{\mathbb{R}^n} f(\mathbf{x})d\mathbf{x}, \\ \int_{\mathbb{R}^n} \widetilde{G}f(\mathbf{x})d\mathbf{x} &\le e^{-n} \int_{\mathbb{R}^n} f(\mathbf{x})d\mathbf{x} \end{aligned} \tag{3.34}$$

hold. The constants e^n *and* e^{-n} *are the best possible.*

Remark 3.8. Notice that (3.30) and (3.31), respectively, follow from (3.15) and (3.16) by rewriting those inequalities for $k = p > n$, and f replaced with $f^{1/p}$, and by letting $p \to \infty$. In particular, observe that $\lim_{p\to\infty}(p/(p-n))^p = e^n$.

Although being related to the setting with balls in $\mathbb{R}^n$ centered at the origin, inequalities (3.15), (3.16) and (3.30), (3.31) are not equivalent with the previously obtained Hardy- and Pólya-Knopp-type inequalities (1.13), (1.14), (1.15), and (1.17). Therefore, our inequalities can be considered as a new class of generalizations of the classical Hardy's and Pólya-Knopp's inequalities in a multidimensional setting. However, for $n = 1$ inequalities of both type coincide. In particular, inequality (3.5) for $n = 1$ was obtained in [4]. It is given in the following corollary.

Corollary 3.9. *Let $0 < b \leq \infty$, f be a nonnegative function on $\langle 0, b\rangle$, and $p, k \in \mathbb{R}$ be such that $0 \neq p \neq 1$, $k \neq 1$, and $p/(k-1) > 0$. If $p \in \mathbb{R} \setminus [0,1]$, then the inequality*

$$\begin{aligned}
&\left(\frac{p}{k-1}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{(k-1)/p}\right] x^{p-k} f^p(x)dx - \int_0^\infty x^{-k} F^p(x)dx \\
&\quad \geq \left| \left(\frac{p}{k-1}\right)^{p-1} \int_0^b x^{((1-k)/p)-1} \int_0^x t^{((k-1)/p)-1} \right. \\
&\qquad \cdot \left| t^{p-k+1} f^p(t) - \left(\frac{k-1}{p}\right)^p x^{1-k} F^p(x) \right| dt\, dx \\
&\qquad \left. -|p| \int_0^b x^{-k} F^{p-1}(x) \int_0^x \left| f(t) - \frac{k-1}{p} \cdot \frac{1}{t} \left(\frac{t}{x}\right)^{(k-1)/p} F(x) \right| dt\, dx \right|
\end{aligned} \tag{3.35}$$

holds, where $F(x)$ is defined by (1.4). *In the case when $p \in \langle 0, 1\rangle$, the order of integrals on the left-hand side in* (3.35) *is reversed.*

An inequality dual to (3.35) can be derived from (3.7) for $n = 1$ and it can be found in [4].

Finally, for $n = 1$, inequalities (3.22) and (3.24), respectively, reduce to the following result from [4].

Corollary 3.10. *Let $0 < b \leq \infty$, f be a positive function on $\langle 0, b\rangle$, and $G(x)$ be defined by* (1.8). *Then*

$$\begin{aligned}
&e \int_0^b \left(1 - \frac{x}{b}\right) f(x)dx - \int_0^b G(x)dx \\
&\quad \geq \left| \int_0^b \int_0^x |etf(t) - xG(x)| dt \frac{dx}{x^2} - \int_0^b G(x) \int_0^x \left| \log \frac{etf(t)}{xG(x)} \right| dt \frac{dx}{x} \right|
\end{aligned} \tag{3.36}$$

holds. On the other hand, if $0 \leq b < \infty$, f is a positive function on $\langle b, \infty\rangle$, and $\widetilde{G}(x)$ is defined by (1.11), *then*

$$\begin{aligned}
&\frac{1}{e} \int_b^\infty \left(1 - \frac{b}{x}\right) f(x)dx - \int_b^\infty \widetilde{G}(x)dx \\
&\quad \geq \left| \int_b^\infty \int_x^\infty \left| \frac{1}{e} tf(t) - x\widetilde{G}(x) \right| \frac{dt}{t^2} dx - \int_b^\infty x\widetilde{G}(x) \int_x^b \left| \log \frac{tf(t)}{ex\widetilde{G}(x)} \right| \frac{dt}{t^2} dx \right|
\end{aligned} \tag{3.37}$$

holds.

Having in mind Corollaries 3.9 and 3.10, our results in this paper can be considered as generalizations of the inequalities obtained in [4].

4. Concluding Remarks

To conclude this paper, we analyze Boas-type inequalities with kernels. Let the setting be as in Section 2, except that λ is a σ-finite Borel measure on $\mathbb{R}_+$. By a kernel we mean a nonnegative measurable function $k : X \times X \to \mathbb{R}$, such that

$$K(\mathbf{x}) = \int_0^\infty k(\mathbf{x}, t\mathbf{x}) d\lambda(t) < \infty \tag{4.1}$$

for μ-a.e. $\mathbf{x} \in X$. For a λ-balanced set $\Omega \subseteq X$ and a Borel measurable function $f : \Omega \to \mathbb{R}$, we define its Hardy-Littlewood average with the kernel k, denoted by $A_k f$, as

$$A_k f(\mathbf{x}) = \frac{1}{K(\mathbf{x})} \int_0^\infty k(\mathbf{x}, t\mathbf{x}) f(t\mathbf{x}) d\lambda(t), \quad \mathbf{x} \in \Omega. \tag{4.2}$$

A related Boas-type inequality is given as follows.

Theorem 4.1. *Let λ be a σ-finite Borel measure on $\mathbb{R}_+$, let μ and ν be σ-finite Borel measures on a topological space X, and let μ_t, defined by* (2.5)*, be absolutely continuous with respect to the measure ν for all $t \in \operatorname{supp} \lambda$. Let $\Omega \subseteq X$ be a λ-balanced set and u be a nonnegative function on X, such that*

$$v(\mathbf{x}) = \int_0^\infty u(\mathbf{x}) \frac{k((1/t)\mathbf{x}, \mathbf{x})}{K((1/t)\mathbf{x})} \cdot \frac{d\mu_t}{d\nu}(\mathbf{x}) d\lambda(t) < \infty, \quad \mathbf{x} \in \Omega, \tag{4.3}$$

where $k : X \times X \to \mathbb{R}$ is a nonnegative measurable function satisfying (4.1)*. Further, let Φ be a nonnegative convex function on an interval $I \subseteq \mathbb{R}$. If $f : \Omega \to \mathbb{R}$ is a measurable function, such that $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$, and $A_k f$ is defined by* (4.2)*, then $A_k f(\mathbf{x}) \in I$, for all $\mathbf{x} \in \Omega$, and the inequality*

$$\int_\Omega u(\mathbf{x}) \Phi\big(A_k f(\mathbf{x})\big) d\mu(\mathbf{x}) \le \int_\Omega v(\mathbf{x}) \Phi\big(f(\mathbf{x})\big) d\nu(\mathbf{x}) \tag{4.4}$$

holds. For a nonpositive concave function Φ, relation (4.4) *holds with the sign of inequality reversed.*

Proof. First, we need to prove that $A_k f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$. Otherwise, there exists $\mathbf{x}_0 \in \Omega$ such that $A_k f(\mathbf{x}_0) \notin I$. In that case, we have either $f(t\mathbf{x}_0) - A_k f(\mathbf{x}_0) < 0$ for all $t \in \operatorname{supp} \lambda$, or $f(t\mathbf{x}_0) - A_k f(\mathbf{x}_0) > 0$ for all $t \in \operatorname{supp} \lambda$. On the other hand, the identity

$$\frac{1}{K(\mathbf{x}_0)} \int_0^\infty k(\mathbf{x}_0, t\mathbf{x}_0) \big(f(t\mathbf{x}_0) - A_k f(\mathbf{x}_0)\big) d\lambda(t) = 0 \tag{4.5}$$

and nonnegativity of $k(\mathbf{x}_0, t\mathbf{x}_0)$ for all $t \in \operatorname{supp} \lambda$ yield that

$$k(x_0, t\mathbf{x}_0) \big(f(t\mathbf{x}_0) - A_k f(\mathbf{x}_0)\big) = 0, \quad t \in \operatorname{supp} \lambda. \tag{4.6}$$

Since $K(\mathbf{x}_0) > 0$, there exists a set $J \subseteq \operatorname{supp} \lambda$ such that $\lambda(J) > 0$ and $k(\mathbf{x}_0, t\mathbf{x}_0) > 0$ for all $t \in J$. Hence, $f(t\mathbf{x}_0) - A_k f(\mathbf{x}_0) = 0$, $t \in J$, so we came to a contradiction. Therefore, $A_k f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$.

By using Jensen's inequality, Fubini's theorem, the Radon-Nikodym theorem, the substitution $\mathbf{y} = t\mathbf{x}$, and the properties of the set Ω and the function Φ, we now obtain

$$\begin{aligned}
&\int_\Omega u(\mathbf{x})\Phi\big(A_k f(\mathbf{x})\big)d\mu(\mathbf{x}) \\
&\quad \le \int_\Omega \frac{u(\mathbf{x})}{K(\mathbf{x})} \int_0^\infty k(\mathbf{x}, t\mathbf{x})\Phi\big(f(t\mathbf{x})\big)d\lambda(t)\, d\mu(\mathbf{x}) \\
&\quad = \int_0^\infty \int_\Omega u(\mathbf{x}) \frac{k(\mathbf{x}, t\mathbf{x})}{K(\mathbf{x})}\, \Phi\big(f(t\mathbf{x})\big)d\mu(\mathbf{x})\, d\lambda(t) \\
&\quad = \int_0^\infty \int_{t\Omega} u\left(\frac{1}{t}\mathbf{y}\right) \frac{k((1/t)\mathbf{y}, \mathbf{y})}{K((1/t)\mathbf{y})}\, \Phi\big(f(\mathbf{y})\big)d\mu_t(\mathbf{y})\, d\lambda(t) \\
&\quad \le \int_0^\infty \int_\Omega u\left(\frac{1}{t}\mathbf{y}\right) \frac{k((1/t)\mathbf{y}, \mathbf{y})}{K((1/t)\mathbf{y})}\, \Phi\big(f(\mathbf{y})\big)d\mu_t(\mathbf{y})\, d\lambda(t) \\
&\quad = \int_0^\infty \int_\Omega u\left(\frac{1}{t}\mathbf{y}\right) \frac{k((1/t)\mathbf{y}, \mathbf{y})}{K((1/t)\mathbf{y})}\, \Phi\big(f(\mathbf{y})\big)\frac{d\mu_t}{d\nu}(\mathbf{y})d\nu(\mathbf{y})\, d\lambda(t) \\
&\quad = \int_\Omega \left(\int_0^\infty u\left(\frac{1}{t}\mathbf{y}\right) \frac{k((1/t)\mathbf{y}, \mathbf{y})}{K((1/t)\mathbf{y})} \cdot \frac{d\mu_t}{d\nu}(\mathbf{y})d\lambda(t)\right)\Phi\big(f(\mathbf{y})\big)d\nu(\mathbf{y}) \\
&\quad = \int_\Omega v(\mathbf{y})\Phi\big(f(\mathbf{y})\big)d\nu(\mathbf{y}),
\end{aligned} \tag{4.7}$$

so the proof is completed. □

Moreover, applying similar reasoning as in Section 2, we get a refinement of the Boas-type inequality (4.4).

Theorem 4.2. *Suppose that λ is a σ-finite Borel measure on $\mathbb{R}_+$, μ and ν are σ-finite Borel measures on a topological space X, and the measures μ_t, defined by* (2.5)*, are absolutely continuous with respect to the measure ν for all $t \in \operatorname{supp} \lambda$. Further, suppose that $\Omega \subseteq X$ is a λ-balanced set, u is a nonnegative function on X, and v is defined on Ω by* (4.3)*, where $k : X \times X \to R$ is a nonnegative measurable function satisfying* (4.1)*. If Φ is a nonnegative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality*

$$\begin{aligned}
&\int_\Omega v(\mathbf{x})\Phi\big(f(\mathbf{x})\big)d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\Phi\big(A_k f(\mathbf{x})\big)d\mu(\mathbf{x}) \\
&\quad \ge \left| \int_\Omega \frac{u(\mathbf{x})}{K(\mathbf{x})} \int_0^\infty k(\mathbf{x}, t\mathbf{x})\big|\Phi\big(f(t\mathbf{x})\big) - \Phi\big(A_k f(\mathbf{x})\big)\big|d\lambda(t)\, d\mu(\mathbf{x}) \right. \\
&\qquad \left. - \int_\Omega \frac{u(\mathbf{x})}{K(\mathbf{x})} \int_0^\infty k(\mathbf{x}, t\mathbf{x})\big|\varphi\big(A_k f(\mathbf{x})\big)\big| \cdot \big|f(t\mathbf{x}) - A_k f(\mathbf{x})\big|d\lambda(t)\, d\mu(\mathbf{x}) \right|
\end{aligned} \tag{4.8}$$

holds for all measurable functions $f : \Omega \to \mathbb{R}$, *such that* $f(\mathbf{x}) \in I$ *for all* $\mathbf{x} \in \Omega$, *and* $A_k f$ *defined by* (4.2). *For a nonpositive concave function* Φ, *relation* (4.8) *holds with*

$$\int_{\Omega} u(\mathbf{x})\Phi\big(A_k f(\mathbf{x})\big)\,d\mu(\mathbf{x}) - \int_{\Omega} v(\mathbf{x})\Phi\big(f(\mathbf{x})\big)\,d\nu(\mathbf{x}) \tag{4.9}$$

on its left-hand side.

Proof. The proof follows the same lines as the proof of Theorem 2.1, considering the setting from Theorem 4.1, so we omit further details. □

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