Research Article

Uniqueness of Positive Solutions for a Perturbed Fractional Differential Equation

Chen Yang and Jieming Zhang

Department of Mathematics, Business College of Shanxi University, Shanxi, Taiyuan 030031, China

Correspondence should be addressed to Chen Yang, yangchen0809@126.com

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We are concerned with the existence and uniqueness of positive solutions for the following nonlinear perturbed fractional two-point boundary value problem: $D_{0+}^{\alpha}u(t) + f(t, u, u', \dots, u^{(n-2)}) + g(t) = 0$, $0 < t < 1$, $n-1 < \alpha \leq n$, $n \geq 2$, $u(0) = u'(0) = \dots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0$, where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. Our analysis relies on a fixed-point theorem of generalized concave operators. An example is given to illustrate the main result.

1. Introduction

In this paper, we are interested in the existence and uniqueness of positive solutions for the following nonlinear perturbed fractional two-point boundary value problem:

$$\begin{aligned} D_{0+}^{\alpha}u(t) + f(t, u, u', \dots, u^{(n-2)}) + g(t) &= 0, \quad 0 < t < 1, \quad n-1 < \alpha \leq n, \quad n \geq 2, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) &= u^{(n-2)}(1) = 0, \end{aligned} \quad (1.1)$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative and $g : [0, 1] \rightarrow [0, +\infty)$ is continuous.

Fractional differential equations arise in many fields, such as physics, mechanics, chemistry, economics, engineering, and biological sciences; see [1–15], for example. In recent years, the study of positive solutions for fractional differential equation boundary value problems has attracted considerable attention, and fruits from research into it emerge continuously. For a small sample of such work, we refer the reader to [16–26] and the references therein. On the other hand, the uniqueness of positive solutions for nonlinear

fractional differential equation boundary value problems has been studied by some authors; see [19–22, 25], for example.

When $g(t) \equiv 0$, Yang and Chen [22] investigated the existence and uniqueness of positive solutions for the problem (1.1) by means of a fixed-point theorem for u_0 concave operators. They present the following result.

Theorem 1.1 (see [22]). *Assume that*

(H₁) $f \in C([0, 1] \times [0, +\infty) \times R^{n-2} \rightarrow [0, +\infty))$, $f(t, y_1, y_2, \dots, y_{n-1})$ is increasing for $y_i \geq 0$, $i = 1, 2, \dots, n-1$, and f is not identically vanishing;

(H₂) for any $t \in [0, 1]$, $y_i \geq 0$, $i = 1, 2, \dots, n-1$, there exist constants m_1, m_2 , $m_1 \leq 0 \leq m_2 < 1$ such that

$$c^{m_2} f(t, y_1, y_2, \dots, y_{n-1}) \leq f(t, cy_1, cy_2, \dots, cy_{n-1}) \leq c^{m_1} f(t, y_1, y_2, \dots, y_{n-1}), \quad \forall 0 < c \leq 1; \quad (1.2)$$

(H₃) $m_2 < (2m_1 + 1)/3$.

Then the problem (1.1) with $g(t) \equiv 0$ has a unique positive solution.

In this paper, we will remove the condition (H₃) and improve on (H₂). And we will use a fixed-point theorem of generalized concave operators to show the existence and uniqueness of positive solutions for the problem (1.1). Our main result is summarized in the following theorem.

Theorem 1.2. *Assume that (H₁) holds and*

(H₄) for any $c \in (0, 1)$ and $y_i \geq 0$, $i = 1, 2, \dots, n-1$, there exists a number $\varphi(c) \in (c, 1)$ such that

$$f(t, cy_1, cy_2, \dots, cy_{n-1}) \geq \varphi(c) f(t, y_1, y_2, \dots, y_{n-1}). \quad (1.3)$$

Then the problem (1.1) has a unique positive solution $u^* = I_{0+}^{n-2} v^* \in C([0, 1], [0, +\infty))$ which satisfies $\lambda(v^*) t^{\alpha-n+1} (1-t) \leq v^*(t) \leq \mu(v^*) t^{\alpha-n+1} (1-t)$, $t \in [0, 1]$, where $\mu(v^*) \geq \lambda(v^*) > 0$.

Remark 1.3. Some examples of $\varphi(t)$ which satisfy the condition (H₄) are

- (1) $\varphi(t) = t^r$, for all $t \in (0, 1)$, where $r \in (0, 1)$,
- (2) $\varphi(t) = t(1 + \eta(t))$ with $0 < \eta(t) \leq 1/t - 1$, for all $t \in (0, 1)$.

Remark 1.4. It is easy to see that the condition (H₄) is weaker than the condition (H₂). Moreover, we do not need the condition (H₃) in our main result.

2. Preliminaries and Previous Results

For the convenience of the reader, we present here some definitions, lemmas, and basic results that will be used in the proof of our theorem.

Definition 2.1 (see [4, Definition 2.1]). The integral

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0, \quad (2.1)$$

is called the Riemann-Liouville fractional integral of order α , where $\alpha > 0$ and $\Gamma(\alpha)$ denotes the gamma function.

Definition 2.2 (see [4, page 36-37]). For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \quad (2.2)$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α , is called the Riemann-Liouville fractional derivative of order α .

By using the same method in [21], the problem (1.1) can be transformed into the following boundary value problem:

$$\begin{aligned} D_{0+}^{\alpha-n+2} v(t) + f\left(t, I_{0+}^{n-2} v(t), I_{0+}^{n-3} v(t), \dots, I_{0+}^1 v(t), v(t)\right) + g(t) &= 0, \quad 0 < t < 1, \\ v(0) &= v(1) = 0. \end{aligned} \quad (2.3)$$

Moreover, from Lemma 2.5 and Lemma 2.7 in [21], we can easily obtain the following result.

Lemma 2.3. *If $v \in C([0, 1], [0, +\infty))$ is a positive solution of the problem (2.3), then $u(t) = I_{0+}^{n-2} v(t)$ is a positive solution of the problem (1.1). On the other hand, if $v \in C([0, 1], [0, +\infty))$ is a positive solution of the problem (2.3), then the solution is*

$$v(t) = \int_0^1 G(t, s) \left[f\left(s, I_{0+}^{n-2} v(s), I_{0+}^{n-3} v(s), \dots, I_{0+}^1 v(s), v(s)\right) + g(s) \right] ds, \quad t \in [0, 1], \quad (2.4)$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha - n + 2)} \begin{cases} (t(1-s))^{\alpha-n+1} - (t-s)^{\alpha-n+1}, & 0 \leq s \leq t \leq 1, \\ (t(1-s))^{\alpha-n+1}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (2.5)$$

Here $G(t, s)$ is called the Green function of the problem (2.3). Evidently, $G(t, s) \geq 0$ for $t, s \in [0, 1]$.

The following property of the Green function plays important roles in this paper.

Lemma 2.4 (see [22]). *The Green function $G(t, s)$ in Lemma 2.3 has the following property:*

$$\begin{aligned} & \frac{\alpha - n + 1}{\Gamma(\alpha - n + 2)} s(1 - s)^{\alpha - n + 1} t^{\alpha - n + 1} (1 - t) \\ & \leq G(t, s) \leq \frac{1}{\Gamma(\alpha - n + 2)} (1 - s)^{\alpha - n} t^{\alpha - n + 1} (1 - t) \quad \text{for } t, s \in [0, 1]. \end{aligned} \quad (2.6)$$

In the sequel, we present some basic concepts in ordered Banach spaces for completeness and a fixed-point theorem which we will be used later. For convenience of readers, we suggest that one refers to [27, 28] for details.

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, that is, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E . Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P, -x \in P \Rightarrow x = \theta$.

P is called normal if there exists a constant $M > 0$ such that, for all $x, y \in E, \theta \leq x \leq y$ implies $\|x\| \leq M\|y\|$; in this case M is called the normality constant of P . If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2 .

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$ is convex and $\lambda P_h = P_h$ for all $\lambda > 0$.

In a recent paper [28], Zhai et al. considered the following operator equation:

$$x = Ax + x_0. \quad (2.7)$$

They established the existence and uniqueness of positive solutions for the above equation, and they present the following interesting result.

Theorem 2.5 (see Theorem 2.1 in [28]). *Let $h > \theta$ and P be a normal cone. Assume that*

(D₁) $A : P \rightarrow P$ is increasing;

(D₂) $x_0 \in P$ satisfies $Ah + x_0 \in P_h$;

(D₃) for any $x \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1)$ such that $A(tx) \geq \varphi(t)Ax$.

Then the operator equation (2.7) has a unique solution in P_h .

Remark 2.6. An operator A is said to be *generalized concave* if A satisfies condition (D₃).

3. Proof of Theorem 1.2

In this section, we apply Theorem 2.5 to study the problem (1.1), and we obtain a new result on the existence and uniqueness of positive solutions. The method used here is new to the literature and so is the existence and uniqueness result to the fractional differential equations.

In our considerations we will work in the Banach space $C[0, 1] = \{x : [0, 1] \rightarrow \mathbf{R} \text{ is continuous}\}$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0, 1]\}$. Notice that this space can be equipped with a partial order given by

$$x, y \in C[0, 1], \quad x \leq y \iff x(t) \leq y(t) \quad \text{for } t \in [0, 1]. \quad (3.1)$$

Set $P = \{x \in C[0, 1] \mid x(t) \geq 0, t \in [0, 1]\}$, the standard cone. It is clear that P is a normal cone in $C[0, 1]$ and the normality constant is 1.

Proof of Theorem 1.2. Let $h(t) = t^{\alpha-n+1}(1-t)$, $t \in [0, 1]$. Then

$$P_h = \{x \in P \mid \text{and there exist } \lambda(x), \mu(x) > 0 \text{ such that } \lambda(x)h \leq x \leq \mu(x)h\}. \quad (3.2)$$

For any $v \in P$, we define

$$Av(t) = \int_0^1 G(t, s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^1v(s), v(s)\right) ds, \quad x_0(t) = \int_0^1 G(t, s) g(s) ds, \quad (3.3)$$

where $G(t, s)$ is given as in Lemma 2.3. Noting that $I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^1v(s), v(s) \geq 0$ and $G(t, s) \geq 0$, it follows from (H_1) that $A : P \rightarrow P$. In the sequel we check that A and x_0 satisfy all assumptions of Theorem 2.5.

Firstly, we prove that $A : P \rightarrow P$ is an increasing operator. In fact, for $v_i \in P$, $i = 1, 2$ with $v_1 \leq v_2$, we know that $v_1(t) \leq v_2(t)$, $t \in [0, 1]$, by the monotonicity of Riemann-Liouville fractional integral I_{0+}^δ , $\delta > 0$ and (H_1) ,

$$\begin{aligned} Av_1(t) &= \int_0^1 G(t, s) f\left(s, I_{0+}^{n-2}v_1(s), I_{0+}^{n-3}v_1(s), \dots, I_{0+}^1v_1(s), v_1(s)\right) ds \\ &\leq \int_0^1 G(t, s) f\left(s, I_{0+}^{n-2}v_2(s), I_{0+}^{n-3}v_2(s), \dots, I_{0+}^1v_2(s), v_2(s)\right) ds = Av_2(t). \end{aligned} \quad (3.4)$$

That is $Av_1 \leq Av_2$. Hence, the condition (D_1) in Theorem 2.5 is satisfied.

Next we show that the condition (D_3) holds. From (H_4) , for any $\gamma \in (0, 1)$ and $v \in P$, we obtain

$$\begin{aligned} A(\gamma v)(t) &= \int_0^1 G(t, s) f\left(s, I_{0+}^{n-2}\gamma v(s), I_{0+}^{n-3}\gamma v(s), \dots, I_{0+}^1\gamma v(s), \gamma v(s)\right) ds \\ &= \int_0^1 G(t, s) f\left(s, \gamma I_{0+}^{n-2}v(s), \gamma I_{0+}^{n-3}v(s), \dots, \gamma I_{0+}^1v(s), \gamma v(s)\right) ds \\ &\geq \int_0^1 G(t, s) \varphi(\gamma) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^1v(s), v(s)\right) ds \\ &= \varphi(\gamma) Av(t), \quad t \in [0, 1]. \end{aligned} \quad (3.5)$$

That is $A(\gamma v) \geq \varphi(\gamma)Av$, for all $v \in P$, $\gamma \in (0, 1)$. So the condition (D_3) in Theorem 2.5 is satisfied. Now we show that the condition (D_2) is also satisfied. On one hand, it follows from (H_1) and Lemma 2.4 that

$$\begin{aligned} Ah(t) &= \int_0^1 G(t, s) f\left(s, I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), \dots, I_{0+}^1h(s), h(s)\right) ds \\ &\geq h(t) \int_0^1 \frac{\alpha - n + 1}{\Gamma(\alpha - n + 2)} s(1-s)^{\alpha-n+1} f\left(s, I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), \dots, I_{0+}^1h(s), h(s)\right) ds \\ &= \frac{\alpha - n + 1}{\Gamma(\alpha - n + 2)} h(t) \\ &\quad \times \int_0^1 s(1-s)^{\alpha-n+1} f\left(s, I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), \dots, I_{0+}^1h(s), h(s)\right) ds, \quad t \in [0, 1]. \end{aligned} \quad (3.6)$$

On the other hand, also from (H_1) and Lemma 2.4, we obtain

$$Ah(t) \leq \frac{1}{\Gamma(\alpha - n + 2)} h(t) \int_0^1 (1-s)^{\alpha-n} f\left(s, I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), \dots, I_{0+}^1h(s), h(s)\right) ds, \quad t \in [0, 1]. \quad (3.7)$$

Let

$$\begin{aligned} \lambda(h) &= \frac{\alpha - n + 1}{\Gamma(\alpha - n + 2)} \int_0^1 s(1-s)^{\alpha-n+1} f\left(s, I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), \dots, I_{0+}^1h(s), h(s)\right) ds, \\ \mu(h) &= \frac{1}{\Gamma(\alpha - n + 2)} \int_0^1 (1-s)^{\alpha-n} f\left(s, I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), \dots, I_{0+}^1h(s), h(s)\right) ds. \end{aligned} \quad (3.8)$$

Since f is continuous and $f \not\equiv 0$, we can get $0 < \lambda(h) \leq \mu(h)$. Consequently,

$$\lambda(h)h(t) \leq Ah(t) \leq \mu(h)h(t). \quad (3.9)$$

Next we consider x_0 . If $g(t) \equiv 0$, then $x_0(t) \equiv 0$; if $g(t) \not\equiv 0$, let $l = \max_{t \in [0, 1]} g(t)$, then $l > 0$. It is easy to prove that

$$0 \leq x_0(t) \leq \frac{l}{\Gamma(\alpha - n + 2)} h(t) \int_0^1 (1-s)^{\alpha-n} ds = \frac{l}{(\alpha - n + 1)\Gamma(\alpha - n + 2)} h(t). \quad (3.10)$$

Hence,

$$0 \leq x_0 \leq \frac{l}{(\alpha - n + 1)\Gamma(\alpha - n + 2)} h. \quad (3.11)$$

Further,

$$\lambda(h)h \leq x_0 + Ah \leq \left[\mu(h) + \frac{l}{(\alpha - n + 1)\Gamma(\alpha - n + 2)} \right] h. \quad (3.12)$$

Hence $x_0 + Ah \in P_h$. Finally, using Theorem 2.5, $v = Av + x_0$ has a unique solution v^* in P_h . That is, v^* is a unique positive solution of the problem (2.3) in P_h . So there are $\mu(v^*)$, $\lambda(v^*)$ with $\mu(v^*) \geq \lambda(v^*) > 0$ such that $\lambda(v^*)t^{\alpha-n+1}(1-t) \leq v^*(t) \leq \mu(v^*)t^{\alpha-n+1}(1-t)$, $t \in [0, 1]$. From Lemma 2.3, $u^* = I_{0+}^{n-2}v^*$ is the solution of the problem (1.1). Evidently, $u^* \in C([0, 1], [0, +\infty))$ is a unique positive solution of the problem (1.1). $\square$

Remark 3.1. Let $f \equiv C > 0$. Then the conditions (H_1) , (H_4) are satisfied and the problem (2.3) has a unique solution $v(t) = \int_0^1 G(t, s)[C + g(t)]ds$, $t \in [0, 1]$. From Lemma 2.4, the unique solution v is a positive solution and satisfies $v \in P_h = P_{t^{\alpha-n+1}(1-t)}$. So $u = I_{0+}^{n-2}v$ is a unique positive solution of the problem (1.1).

To illustrate how our main result can be used in practice we present an example.

Example 3.2. Consider the following problem:

$$\begin{aligned} D_{0+}^{5/2}u(t) + a(t)\left\{u^{1/3}(t) + [u'(t)]^{1/4}\right\} + g(t) &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u'(1) &= 0, \end{aligned} \quad (3.13)$$

where $a, g : [0, 1] \rightarrow [0, +\infty)$ are continuous with $a \not\equiv 0$.

In this example, we have $\alpha = 5/2$. Let $f(t, x, y) = a(t)[x^{1/3} + y^{1/4}]$. Evidently, $f(t, x, y)$ is increasing in x for $t \in [0, 1]$, $y \geq 0$, and increasing in y for $t \in [0, 1]$, $x \geq 0$. Moreover, $f \not\equiv 0$. Set $\varphi(\gamma) = \gamma^{5/12}$, $\gamma \in (0, 1)$. Then

$$f(t, \gamma x, \gamma y) = a(t)\left[\gamma^{1/3}x^{1/3} + \gamma^{1/4}y^{1/4}\right] \geq \varphi(\gamma)f(t, x, y), \quad t \in [0, 1], x, y \geq 0. \quad (3.14)$$

Hence, all the conditions of Theorem 1.2 are satisfied. An application of Theorem 1.2 implies that the problem (3.13) has a unique positive solution.

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