Research Article

Martingale Morrey-Campanato Spaces and Fractional Integrals

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We introduce Morrey-Campanato spaces of martingales and give their basic properties. Our definition of martingale Morrey-Campanato spaces is different from martingale Lipschitz spaces introduced by Weisz, while Campanato spaces contain Lipschitz spaces as special cases. We also give the relation between these definitions. Moreover, we establish the boundedness of fractional integrals as martingale transforms on these spaces. To do this we show the boundedness of the maximal function on martingale Morrey-Campanato spaces.

1. Introduction

The purpose of this paper is to introduce Morrey-Campanato spaces of martingales. The Lebesgue space L^p plays an important role in martingale theory as well as in harmonic analysis. Moreover, in martingale theory, Lorentz spaces, Orlicz spaces, Hardy spaces, Lipschitz spaces, and John-Nirenberg space BMO also have been developed by many authors, see [1–5], and so forth. Recently Kikuchi [6] investigated Banach function spaces of martingales. In this paper we introduce Morrey-Campanato spaces of martingales and give their basic properties. Moreover, we establish the boundedness of fractional integrals as martingale transforms on these spaces. Note that Campanato spaces are not Banach function spaces in general.

We consider a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$, where $\{\mathcal{F}_n\}_{n \geq 0}$ is a nondecreasing sequence of sub- σ -algebras of $\mathcal{F}$. Following Weisz [5], we call $\{\mathcal{F}_n\}_{n \geq 0}$ a stochastic basis. For the sake of simplicity, let $\mathcal{F}_{-1} = \mathcal{F}_0$. We suppose that every σ -algebra $\mathcal{F}_n$ is generated by countable atoms, where $B \in \mathcal{F}_n$ is called an atom (more precisely a $(\mathcal{F}_n, P)$ -atom), if any $A \subset B$ with $A \in \mathcal{F}_n$ satisfies $P(A) = P(B)$ or $P(A) = 0$. Denote by $A(\mathcal{F}_n)$ the

set of all atoms in $\mathcal{F}_n$. The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E and E_n , respectively.

We define Morrey-Campanato spaces as the following: let $p \in [1, \infty)$ and $\lambda \in (-\infty, \infty)$. For $f \in L_1$, let

$$\begin{aligned} \|f\|_{L_{p,\lambda}} &= \sup_{n \geq 0} \sup_{B \in A(\mathcal{F}_n)} \frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f|^p dP \right)^{1/p}, \\ \|f\|_{\mathcal{L}_{p,\lambda}} &= \sup_{n \geq 0} \sup_{B \in A(\mathcal{F}_n)} \frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f - E_n f|^p dP \right)^{1/p}, \end{aligned} \quad (1.1)$$

and let

$$L_{p,\lambda}^0 = \{f \in L_p^0 : \|f\|_{L_{p,\lambda}} < \infty\}, \quad \mathcal{L}_{p,\lambda} = \{f \in L_p^0 : \|f\|_{\mathcal{L}_{p,\lambda}} < \infty\}, \quad (1.2)$$

where L_p^0 is the set of all $f \in L_p$ such that $E_0 f = 0$.

We give basic properties of martingale Morrey-Campanato spaces and compare these spaces with martingale Lipschitz spaces introduced by Weisz [7]. It is well known, in harmonic analysis, that Campanato spaces contain Lipschitz spaces as special cases. Recently, martingale Campanato spaces were introduced in [8] as generalization of martingale Lipschitz spaces. While our definition of martingale Morrey-Campanato spaces is different from the one in [8], we can prove that our martingale Morrey-Campanato spaces contain martingale Lipschitz spaces by Weisz as special cases, under the assumption that every σ -algebra $\mathcal{F}_n$ is generated by countable atoms.

The fractional integrals are very useful tools to analyse function spaces in harmonic analysis. Actually, Hardy and Littlewood [9, 10] and Sobolev [11] investigated the fractional integrals to establish the theory of Lebesgue spaces and Lipschitz spaces. Stein and Weiss [12], Taibleson and Weiss [13], and Krantz [14] also investigated the fractional integrals to establish the theory of Hardy spaces. See also [15]. The L_p - L_q boundedness of the fractional integrals is well known as the Hardy-Littlewood-Sobolev theorem derived from [9–11]. This boundedness has been extended to Morrey-Campanato spaces by Peetre [16] and Adams [17], see also [18]. It is known that Morrey-Campanato spaces contain L_p , BMO, and Lip_α as special cases, see for example [16, 19].

On the other hand, in martingale theory, Watari [20] and Chao and Ombe [21] proved the boundedness of the fractional integrals for L_p (H_p), BMO, and Lipschitz spaces of the dyadic martingale. In this paper, we also establish the boundedness of fractional integrals as martingale transforms on Morrey-Campanato spaces. Our result generalizes and improves the results in [20, 21].

For a martingale $f = (f_n)_{n \geq 0}$ relative to $\{\mathcal{F}_n\}_{n \geq 0}$, denote its martingale difference by $d_n f = f_n - f_{n-1}$ ($n \geq 0$, with convention $d_0 f = 0$). For $\alpha > 0$, we define the fractional integral $I_\alpha f = ((I_\alpha f)_n)_{n \geq 0}$ of f by

$$(I_\alpha f)_n = \sum_{k=0}^n b_{k-1}^\alpha d_k f, \quad (1.3)$$

where b_k is an $\mathcal{F}_k$ -measurable function such that

$$b_k(\omega) = P(B) \quad \text{for a.e. } \omega \in B \text{ with } B \in A(\mathcal{F}_k). \quad (1.4)$$

Then $((I_\alpha f)_n)_{n \geq 0}$ is a martingale for any martingale $f = (f_n)_{n \geq 0}$, since each b_k is bounded, that is, I_α is a martingale transform introduced by Burkholder [22]. This definition of I_α is an extension of the one in [20, 21] which is for dyadic martingales. We can prove the boundedness of fractional integrals I_α as martingale transforms from L_p to L_q , if $1 < p < q < \infty$ and $-1/p + \alpha = -1/q$. That is, if a martingale $f = (f_n)_{n \geq 0}$ is L_p -bounded, then $((I_\alpha f)_n)_{n \geq 0}$ is L_q -bounded and the following inequality holds:

$$\sup_{n \geq 0} \|(I_\alpha f)_n\|_{L_q} \leq C \sup_{n \geq 0} \|f_n\|_{L_p}, \quad (1.5)$$

where C is a positive constant independent of f . Further, we prove the boundedness of fractional integrals I_α as martingale transforms on Morrey-Campanato spaces.

To prove the boundedness of fractional integrals we use a different method from [20, 21]. More precisely, under the assumption that every σ -algebra $\mathcal{F}_n$ is generated by countable atoms, we first prove the boundedness of the maximal function, and then we use the pointwise estimate by the maximal function and its boundedness, namely, Hedberg's method in [23]. We also use the method in [24, 25]. By considering sequences of atoms precisely, we can apply these methods to martingale Morrey-Campanato spaces. From this point of view our assumption seems to be natural to define martingale Morrey-Campanato spaces.

We state notation, definitions, and remarks in the next section and give basic properties of Morrey-Campanato spaces in Section 3. We prove the boundedness of the maximal function and fractional integrals in Sections 4 and 5, respectively.

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p , is dependent on the subscripts. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$ and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

2. Notation, Definitions, and Remarks

Recall that $(\Omega, \mathcal{F}, P)$ is a probability space, and $\{\mathcal{F}_n\}_{n \geq 0}$ a nondecreasing sequence of sub- σ -algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. For the sake of simplicity, let $\mathcal{F}_{-1} = \mathcal{F}_0$. As in Section 1, we always suppose that every σ -algebra $\mathcal{F}_n$ is generated by countable atoms, with denoting by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$. We define the fractional integral I_α as a martingale transform by (1.3).

For a martingale $f = (f_n)_{n \geq 0}$ relative to $\{\mathcal{F}_n\}_{n \geq 0}$, the maximal function f^* of f is defined by

$$f_n^* = \sup_{0 \leq m \leq n} |f_m|, \quad f^* = \sup_{n \geq 0} |f_n|. \quad (2.1)$$

It is known that if $p \in (1, \infty)$, then any L_p -bounded martingale converges in L_p . Moreover, if $f \in L_p$, $p \in [1, \infty)$, then $(f_n)_{n \geq 0}$ with $f_n = E_n f$ is an L_p -bounded martingale and converges to f in L_p (see, e.g., [26]). For this reason a function $f \in L_1$ and the corresponding martingale $(f_n)_{n \geq 0}$ with $f_n = E_n f$ will be denoted by the same symbol f . Note also that $\|f\|_{L_p} = \sup_{n \geq 0} \|E_n f\|_{L_p}$. In this case

$$f_n^* = \sup_{0 \leq m \leq n} |E_m f|, \quad f^* = \sup_{n \geq 0} |E_n f| \quad \text{for } f \in L_1. \quad (2.2)$$

Let $\mathcal{M}$ be the set of all martingales such that $f_0 = 0$. For $p \in [1, \infty]$, let L_p^0 be the set of all $f \in L_p$ such that $E_0 f = 0$. For any $f \in L_p^0$, let $f_n = E_n f$. Then $(f_n)_{n \geq 0}$ is an L_p -bounded martingale in $\mathcal{M}$. For this reason we can regard L_p^0 as a subset of $\mathcal{M}$.

In Section 1 we have introduced Morrey spaces $L_{p,\lambda}$ and Campanato spaces $\mathcal{L}_{p,\lambda}$ as the following.

Definition 2.1. Let $p \in [1, \infty)$ and $\lambda \in (-\infty, \infty)$. For $f \in L_1$, let

$$\begin{aligned} \|f\|_{L_{p,\lambda}} &= \sup_{n \geq 0} \sup_{B \in \mathcal{A}(\mathcal{F}_n)} \frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f|^p dP \right)^{1/p}, \\ \|f\|_{\mathcal{L}_{p,\lambda}} &= \sup_{n \geq 0} \sup_{B \in \mathcal{A}(\mathcal{F}_n)} \frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f - E_n f|^p dP \right)^{1/p}, \end{aligned} \quad (2.3)$$

and define

$$L_{p,\lambda} = \left\{ f \in L_p^0 : \|f\|_{L_{p,\lambda}} < \infty \right\}, \quad \mathcal{L}_{p,\lambda} = \left\{ f \in L_p^0 : \|f\|_{\mathcal{L}_{p,\lambda}} < \infty \right\}. \quad (2.4)$$

Then functionals $\|f\|_{L_{p,\lambda}}$ and $\|f\|_{\mathcal{L}_{p,\lambda}}$ are norms on $L_{p,\lambda}$ and $\mathcal{L}_{p,\lambda}$, respectively. Note that $L_{p,\lambda}$ and $\mathcal{L}_{p,\lambda}$ are not always trivial set $\{0\}$ even if $\lambda > 0$ and $\lambda > 1$, respectively. This property is different from classical Morrey-Campanato spaces on $\mathbb{R}^n$.

The martingale $f = (f_n)_{n \geq 0}$ is said to be $L_{p,\lambda}$ -bounded if $f_n \in L_{p,\lambda}$ ($n \geq 0$) and $\sup_{n \geq 0} \|f_n\|_{L_{p,\lambda}} < \infty$. Similarly, the martingale $f = (f_n)_{n \geq 0}$ is said to be $\mathcal{L}_{p,\lambda}$ -bounded if $f_n \in \mathcal{L}_{p,\lambda}$ ($n \geq 0$) and $\sup_{n \geq 0} \|f_n\|_{\mathcal{L}_{p,\lambda}} < \infty$.

Proposition 2.2. Let $1 \leq p < \infty$. Let $f \in L_1$ and $(f_n)_{n \geq 0}$ be its corresponding martingale with $f_n = E_n f$ ($n \geq 0$).

(i) Assume that $\lambda \in (-\infty, 0]$. If $f \in L_{p,\lambda}$, then $(f_n)_{n \geq 0}$ is $L_{p,\lambda}$ -bounded and

$$\|f\|_{L_{p,\lambda}} \geq \sup_{n \geq 0} \|f_n\|_{L_{p,\lambda}}. \quad (2.5)$$

Conversely, if $(f_n)_{n \geq 0}$ is $L_{p,\lambda}$ -bounded, then $f \in L_{p,\lambda}$ and

$$\|f\|_{L_{p,\lambda}} \leq \sup_{n \geq 0} \|f_n\|_{L_{p,\lambda}}. \quad (2.6)$$

(ii) Assume that $\lambda \in (-\infty, \infty)$. If $f \in \mathcal{L}_{p,\lambda}$, then $(f_n)_{n \geq 0}$ is $\mathcal{L}_{p,\lambda}$ -bounded and

$$\|f\|_{\mathcal{L}_{p,\lambda}} \geq \sup_{n \geq 0} \|f_n\|_{\mathcal{L}_{p,\lambda}}. \quad (2.7)$$

Conversely, if $(f_n)_{n \geq 0}$ is $\mathcal{L}_{p,\lambda}$ -bounded, then $f \in \mathcal{L}_{p,\lambda}$ and

$$\|f\|_{\mathcal{L}_{p,\lambda}} \leq \sup_{n \geq 0} \|f_n\|_{\mathcal{L}_{p,\lambda}}. \quad (2.8)$$

Proof. (i) Let $f \in L_{p,\lambda}$ and $n \geq 0$. Fix any $B \in A(\mathcal{F}_k)$. If $k \leq n$, then

$$\left(\int_B |E_n f|^p dP \right)^{1/p} \leq \left(\int_B E_n [|f|^p] dP \right)^{1/p} = \left(\int_B |f|^p dP \right)^{1/p} \leq P(B)^{\lambda+1/p} \|f\|_{L_{p,\lambda}}. \quad (2.9)$$

If $n < k$, then taking $B_n \in A(\mathcal{F}_n)$ such that $B \subset B_n$, we have

$$E_n f = \frac{1}{P(B_n)} \int_{B_n} f dP \text{ on } B_n. \quad (2.10)$$

Hence

$$\begin{aligned} \left(\frac{1}{P(B)} \int_B |E_n f|^p dP \right)^{1/p} &= \left(\frac{1}{P(B_n)} \int_{B_n} |E_n f|^p dP \right)^{1/p} \\ &\leq \left(\frac{1}{P(B_n)} \int_{B_n} E_n [|f|^p] dP \right)^{1/p} \\ &= \left(\frac{1}{P(B_n)} \int_{B_n} |f|^p dP \right)^{1/p} \\ &\leq P(B_n)^\lambda \|f\|_{L_{p,\lambda}} \leq P(B)^\lambda \|f\|_{L_{p,\lambda}}. \end{aligned} \quad (2.11)$$

Therefore, $f_n = E_n f \in L_{p,\lambda}$ and $\|f_n\|_{L_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}}$ for all $n \geq 0$. This shows that $(f_n)_{n \geq 0}$ is a $L_{p,\lambda}$ -bounded martingale and

$$\sup_{n \geq 0} \|f_n\|_{L_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}}. \quad (2.12)$$

Conversely, from the inequality

$$\int_B |f|^p dP \leq \liminf_{n \rightarrow \infty} \int_B |E_n f|^p dP \quad \forall B \in \bigcup_m A(\mathcal{F}_m), \quad (2.13)$$

it follows that

$$\|f\|_{L_{p,\lambda}} \leq \sup_{n \geq 0} \|E_n f\|_{L_{p,\lambda}}. \quad (2.14)$$

(ii) Let $f \in \mathcal{L}_{p,\lambda}$ and $n \geq 0$. Fix any $B \in A(\mathcal{F}_k)$. If $k \leq n$, then

$$\begin{aligned} \left(\int_B |E_n f - E_k [E_n f]|^p dP \right)^{1/p} &= \left(\int_B |E_n f - E_n [E_k f]|^p dP \right)^{1/p} \\ &\leq \left(\int_B E_n [|f - E_k f|^p] dP \right)^{1/p} \\ &= \left(\int_B |f - E_k f|^p dP \right)^{1/p} \\ &\leq P(B)^{\lambda+1/p} \|f\|_{\mathcal{L}_{p,\lambda}}. \end{aligned} \quad (2.15)$$

If $n < k$, then $E_k[E_n f] = E_n f$. Hence

$$\left(\frac{1}{P(B)} \int_B |E_n f - E_k[E_n f]|^p dP \right)^{1/p} = 0. \quad (2.16)$$

Therefore, $f_n = E_n f \in \mathcal{L}_{p,\lambda}$ and $\|f_n\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{\mathcal{L}_{p,\lambda}}$ for all $n \geq 0$. This shows that $(f_n)_{n \geq 0}$ is a $\mathcal{L}_{p,\lambda}$ -bounded martingale and

$$\sup_{n \geq 0} \|f_n\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{\mathcal{L}_{p,\lambda}}. \quad (2.17)$$

Conversely, from the inequality

$$\int_B |f - f_k|^p dP \leq \liminf_{n \rightarrow \infty} \int_B |E_n[f - f_k]|^p dP \quad \forall B \in A(\mathcal{F}_k), \quad k = 1, 2, \dots, \quad (2.18)$$

it follows that

$$\|f\|_{\mathcal{L}_{p,\lambda}} \leq \sup_{n \geq 0} \|E_n f\|_{\mathcal{L}_{p,\lambda}}. \quad (2.19)$$

□

Remark 2.3. In general, for $f \in L_{p,\lambda}$ (resp., $\mathcal{L}_{p,\lambda}$), $E_n f$ does not always converge to f in $L_{p,\lambda}$ (resp., $\mathcal{L}_{p,\lambda}$). See Remark 3.7.

Remark 2.4. If $\mathcal{F}_0 = \{\emptyset, \Omega\}$, then $L_{p,\lambda} \subset L_p$ and $\|f\|_{L_p} \leq \|f\|_{L_{p,\lambda}}$ for $f \in L_{p,\lambda}$. Therefore, if $(f_n)_{n \geq 0}$ is an $L_{p,\lambda}$ -bounded martingale, then it is an L_p -bounded martingale. If $1 < p < \infty$, from the known result it follows that there exists $f \in L_p^0$ such that $E_n f = f_n$, $n \geq 0$, and $(f_n)_{n \geq 0}$ converges to f in L_p and a.e. Moreover, we can deduce that $f \in L_{p,\lambda}$, since

$$\left(\int_B |f_n(\omega)|^p dP \right)^{1/p} \leq P(B)^{\lambda+1/p} \|f_n\|_{L_{p,\lambda}} \leq P(B)^{\lambda+1/p} \sup_{n \geq 0} \|f_n\|_{L_{p,\lambda}}, \quad B \in \bigcup_m A(\mathcal{F}_m). \quad (2.20)$$

The stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is said to be regular, if there exists a constant $R \geq 2$ such that

$$f_n \leq R f_{n-1} \quad (2.21)$$

holds for all nonnegative martingales $(f_n)_{n \geq 0}$.

Remark 2.5. In general, $L_{p,\lambda} \subset \mathcal{L}_{p,\lambda}$ with $\|f\|_{\mathcal{L}_{p,\lambda}} \leq 2\|f\|_{L_{p,\lambda}}$. Actually, for any $B \in A(\mathcal{F}_n)$,

$$\begin{aligned} \left(\frac{1}{P(B)} \int_B |f - E_n f|^p dP \right)^{1/p} &\leq \left(\frac{1}{P(B)} \int_B |f|^p dP \right)^{1/p} + \left(\frac{1}{P(B)} \int_B |E_n f|^p dP \right)^{1/p} \\ &\leq 2 \left(\frac{1}{P(B)} \int_B |f|^p dP \right)^{1/p}. \end{aligned} \quad (2.22)$$

Moreover, if $\{\mathcal{F}_n\}_{n \geq 0}$ is regular and $\lambda < 0$, then we can prove that $L_{p,\lambda} = \mathcal{L}_{p,\lambda}$ with equivalent norms (Theorem 3.1 and Remark 3.2).

Remark 2.6. By definition, if $0 \geq \lambda \geq \lambda'$, then we have that $L_\infty^0 \subset L_{p,\lambda} \subset L_{p,\lambda'}$ with $\|f\|_{L_{p,\lambda'}} \leq \|f\|_{L_{p,\lambda}} \leq \|f\|_{L_\infty}$ and $L_\infty^0 \subset \mathcal{L}_{p,\lambda} \subset \mathcal{L}_{p,\lambda'}$ with $\|f\|_{\mathcal{L}_{p,\lambda'}} \leq \|f\|_{\mathcal{L}_{p,\lambda}} \leq 2\|f\|_{L_\infty}$. If $\lambda \leq -1/p$, then $L_p^0 \subset L_{p,\lambda} \subset \mathcal{L}_{p,\lambda}$ with $\|f\|_{\mathcal{L}_{p,\lambda}}/2 \leq \|f\|_{L_{p,\lambda}} \leq \|f\|_{L_p}$.

Remark 2.7. By definition and Remark 2.5, if $\mathcal{F}_0 = \{\emptyset, \Omega\}$, then $L_{p,\lambda} \subset \mathcal{L}_{p,\lambda} \subset L_p^0$ with $\|f\|_{L_p} \leq \|f\|_{\mathcal{L}_{p,\lambda}} \leq 2\|f\|_{L_{p,\lambda}}$.

Definition 2.8. Let $\text{BMO} = \mathcal{L}_{1,0}$ and $\text{Lip}_\alpha = \mathcal{L}_{1,\alpha}$ if $\alpha > 0$.

Our definitions of BMO and Lip_α are different from the ones by Weisz [7]. To compare both we give another definition of martingale Morrey and Campanato spaces.

Definition 2.9. Let $p \in [1, \infty)$ and $\lambda \in (-\infty, \infty)$. For $f \in L_1$, let

$$\begin{aligned} \|f\|_{L_{p,\lambda,\mathcal{F}}} &= \sup_{n \geq 0} \sup_{B \in \mathcal{F}_n} \frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f|^p dP \right)^{1/p}, \\ \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} &= \sup_{n \geq 0} \sup_{B \in \mathcal{F}_n} \frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f - E_n f|^p dP \right)^{1/p}, \end{aligned} \quad (2.23)$$

and define

$$L_{p,\lambda,\mathcal{F}} = \left\{ f \in L_p^0 : \|f\|_{L_{p,\lambda,\mathcal{F}}} < \infty \right\}, \quad \mathcal{L}_{p,\lambda,\mathcal{F}} = \left\{ f \in L_p^0 : \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} < \infty \right\}. \quad (2.24)$$

Note that the spaces $L_{p,\lambda,\mathcal{F}}$ and $\mathcal{L}_{p,\lambda,\mathcal{F}}$ can be defined without the assumption that every σ -algebra $\mathcal{F}_n$ is generated by countable atoms.

Remark 2.10. By the definitions we have the relations $L_{p,\lambda,\mathcal{F}} \subset L_{p,\lambda}$ with $\|f\|_{L_{p,\lambda}} \leq \|f\|_{L_{p,\lambda,\mathcal{F}}}$ and $\mathcal{L}_{p,\lambda,\mathcal{F}} \subset \mathcal{L}_{p,\lambda}$ with $\|f\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}}$. If $\lambda \geq 0$, then we can prove that $L_{p,\lambda,\mathcal{F}} = L_{p,\lambda}$ and $\mathcal{L}_{p,\lambda,\mathcal{F}} = \mathcal{L}_{p,\lambda}$ with the same norms, respectively (see Proposition 3.8). If $-1/p < \lambda < 0$, then $L_{p,\lambda,\mathcal{F}} \subsetneq L_{p,\lambda}$ and $\mathcal{L}_{p,\lambda,\mathcal{F}} \subsetneq \mathcal{L}_{p,\lambda}$ in general (see Proposition 3.9).

Remark 2.11. It is known that, if $\{\mathcal{F}_n\}_{n \geq 0}$ is regular and $\lambda \geq 0$, then $\mathcal{L}_{1,\lambda,\mathcal{F}} = \mathcal{L}_{p,\lambda,\mathcal{F}}$ with $\|f\|_{\mathcal{L}_{1,\lambda,\mathcal{F}}} \leq \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \leq C_p \|f\|_{\mathcal{L}_{1,\lambda,\mathcal{F}}}$ for each $p \in [1, \infty)$ (see, e.g., [8]).

We also define weak Morrey spaces.

Definition 2.12. For $p \in [1, \infty)$ and $\lambda \in (-\infty, \infty)$, let

$$\|f\|_{WL_{p,\lambda}} = \sup_{n \geq 0} \sup_{B \in \mathcal{A}(\mathcal{F}_n)} \frac{1}{P(B)^\lambda} \sup_{t > 0} t \left(\frac{P(B \cap \{|f| > t\})}{P(B)} \right)^{1/p}, \quad (2.25)$$

for measurable functions f , and define

$$WL_{p,\lambda} = \left\{ f \in L_1^0 : \|f\|_{WL_{p,\lambda}} < \infty \right\}. \quad (2.26)$$

3. Basic Properties of Morrey and Campanato Spaces

In this section we give basic properties of Morrey and Campanato spaces. The following theorem gives the relation between Morrey and Campanato spaces.

Theorem 3.1. *Let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $(\Omega, \mathcal{F}, P)$ be nonatomic. Let $p \in [1, \infty)$.*

(i) *If $\lambda \leq -1/p$, then $L_{p,\lambda} = \mathcal{L}_{p,\lambda} = L_p^0$ and*

$$\frac{1}{2} \|f\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}} = \|f\|_{L_p} \leq \|f\|_{\mathcal{L}_{p,\lambda}}. \quad (3.1)$$

(ii) *If $-1/p < \lambda < 0$, then $L_\infty^0 \subsetneq L_{p,\lambda} = \mathcal{L}_{p,\lambda} \subsetneq L_p^0$ and*

$$\|f\|_{L_p} \leq \|f\|_{L_{p,\lambda}} \leq \|f\|_{L_\infty}, \quad \frac{1}{2} \|f\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}} \leq C \|f\|_{\mathcal{L}_{p,\lambda}}. \quad (3.2)$$

(iii) *If $\lambda = 0$, then $L_\infty^0 = L_{p,0} \subsetneq \mathcal{L}_{p,0} = \text{BMO}$ and*

$$\|f\|_{L_{p,0}} = \|f\|_{L_\infty}, \quad \|f\|_{\text{BMO}} \leq \|f\|_{\mathcal{L}_{p,0}} \leq C_p \|f\|_{\text{BMO}}. \quad (3.3)$$

(iv) *If $\lambda > 0$, then $\{0\} = L_{p,\lambda} \subsetneq \mathcal{L}_{p,\lambda} = \text{Lip}_\lambda$ and*

$$\|f\|_{\text{Lip}_\lambda} \leq \|f\|_{\mathcal{L}_{p,\lambda}} \leq C_p \|f\|_{\text{Lip}}. \quad (3.4)$$

Remark 3.2. We can prove (i) without the assumption that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular or that $(\Omega, \mathcal{F}, P)$ is nonatomic. In (ii), we can prove that $L_{p,\lambda} = \mathcal{L}_{p,\lambda}$ and $(1/2)\|f\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}} \leq C\|f\|_{\mathcal{L}_{p,\lambda}}$ without the assumption that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ or that $(\Omega, \mathcal{F}, P)$ is nonatomic. To show $L_\infty^0 \subsetneq \mathcal{L}_{p,\lambda}$ in (ii) and (iii), we can replace the condition that $(\Omega, \mathcal{F}, P)$ is nonatomic by a weaker condition as in Proposition 3.6(ii), which follows from the condition that $(\Omega, \mathcal{F}, P)$ is nonatomic. In (iv), we need the condition that $(\Omega, \mathcal{F}, P)$ is nonatomic to show $L_{p,\lambda} = \{0\}$.

To prove the theorem we first prove a lemma and two propositions.

Lemma 3.3. *Let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular. Then every sequence*

$$B_0 \supset B_1 \supset \cdots \supset B_n \supset \cdots, \quad B_n \in A(\mathcal{F}_n), \quad (3.5)$$

has the following property: for each $n \geq 1$,

$$B_n = B_{n-1} \text{ or } \left(1 + \frac{1}{R}\right)P(B_n) \leq P(B_{n-1}) \leq RP(B_n), \quad (3.6)$$

where R is the constant in (2.21).

Remark 3.4. Since $B_n \in A(\mathcal{F}_n)$ is an $(\mathcal{F}_n, P)$ -atom, we always interpret $B_{n-1} \supset B_n$ as the inclusion modulo null sets, that is, $B_{n-1} \supset B_n$ means $P(B_n \setminus B_{n-1}) = 0$. Therefore, $B_n = B_{n-1}$ means $P(B_n \setminus B_{n-1}) = P(B_{n-1} \setminus B_n) = 0$.

Remark 3.5. By the lemma we see that there exists m such that $B_m = B_n$ for all $n \geq m$, if and only if $\lim_{n \rightarrow \infty} P(B_n) > 0$.

Proof of Lemma 3.3. Let $\tilde{B}_{n-1} = \{E_{n-1}[\chi_{B_n}] \geq 1/R\}$. Then $\tilde{B}_{n-1} \in \mathcal{F}_{n-1}$. By the regularity we have $\chi_{B_n} \leq RE_{n-1}[\chi_{B_n}]$. This shows $\tilde{B}_{n-1} \supset B_n$. In this case, $\tilde{B}_{n-1} \supset B_{n-1} \supset B_n$, since $B_{n-1} \in A(\mathcal{F}_{n-1})$. From the definition of $\tilde{B}_{n-1}$ it follows that $\chi_{\tilde{B}_{n-1}} \leq RE_{n-1}[\chi_{B_n}]$. Then we have

$$P(B_{n-1}) \leq P(\tilde{B}_{n-1}) = E[\chi_{\tilde{B}_{n-1}}] \leq E[RE_{n-1}[\chi_{B_n}]] = RE[\chi_{B_n}] = RP(B_n). \quad (3.7)$$

Next we show $B_{n-1} = B_n$ or $(1 + 1/R)P(B_n) \leq P(B_{n-1})$. Suppose that

$$P(B_{n-1}) < \left(1 + \frac{1}{R}\right)P(B_n). \quad (3.8)$$

Then

$$P(B_{n-1} \setminus B_n) = P(B_{n-1}) - P(B_n) < \frac{P(B_n)}{R} \leq \frac{P(B_{n-1})}{R}. \quad (3.9)$$

Therefore,

$$E_{n-1}[\chi_{B_{n-1} \setminus B_n}] = \frac{P(B_{n-1} \setminus B_n)}{P(B_{n-1})} \chi_{B_{n-1}} < \frac{1}{R} \chi_{B_{n-1}}. \quad (3.10)$$

From the regularity and the inequality above it follows that

$$\chi_{B_{n-1} \setminus B_n} \leq RE_{n-1}[\chi_{B_{n-1} \setminus B_n}] < \chi_{B_{n-1}}. \quad (3.11)$$

This means that $B_{n-1} = B_n$. □

Proposition 3.6. Let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular, $1 \leq p < \infty$ and $\lambda > -1/p$.

(i) For a sequence $B_0 \supset B_1 \supset \cdots \supset B_k \supset \cdots$, $B_k \in A(\mathcal{F}_k)$, let $f_0 = 0$ and

$$f_n = \sum_{k=1}^n P(B_k)^\lambda \left(\frac{P(B_{k-1})}{P(B_k)} \chi_{B_k} - \chi_{B_{k-1}} \right), \quad n \geq 1. \quad (3.12)$$

Then $f = (f_n)_{n \geq 0}$ is a martingale in $\mathcal{M}$ and in $\mathcal{L}_{p,\lambda}$.

(ii) Let $0 \geq \lambda > \lambda' > -1/p$. If there exists a sequence $B_0 \supset B_1 \supset \cdots \supset B_k \supset \cdots$, $B_k \in A(\mathcal{F}_k)$ and $\lim_{k \rightarrow \infty} P(B_k) = 0$, then $L_\infty^0 \subsetneq \mathcal{L}_{p,\lambda} \subsetneq \mathcal{L}_{p,\lambda'}$. If $\mathcal{F}_0 = \{\emptyset, \Omega\}$ also, then $L_\infty^0 \subsetneq \mathcal{L}_{p,\lambda} \subsetneq \mathcal{L}_{p,\lambda'} \subsetneq L_p^0$.

Proof. (i) By the definition of the sequence $(f_n)_{n \geq 0}$, we have

$$E_{n-1}[d_n f] = P(B_n)^\lambda \left(\frac{P(B_{n-1})}{P(B_n)} E_{n-1}[\chi_{B_n}] - \chi_{B_{n-1}} \right) = 0, \quad (3.13)$$

for every $n \geq 1$. Hence, we obtain that $(f_n)_{n \geq 0}$ is a martingale.

We next show that the sequence $(f_n)_{n \geq 0}$ converges in L_p . If $\lim_{k \rightarrow \infty} P(B_k) > 0$ then the convergence is clear by Remark 3.5. We assume that $\lim_{k \rightarrow \infty} P(B_k) = 0$. Then, by Lemma 3.3, we can take a sequence of integers $0 = k_0 < k_1 < \dots < k_j < \dots$ that satisfies

$$\left(1 + \frac{1}{R}\right) P(B_{k_j}) \leq P(B_{k_{j-1}}) \leq R P(B_{k_j}), \quad (3.14)$$

and $B_{k_{j-1}} = B_k$ if $k_{j-1} \leq k < k_j$. In this case we can write

$$f_n = \sum_{k_j \leq n} P(B_{k_j})^\lambda \left(\frac{P(B_{k_{j-1}})}{P(B_{k_j})} \chi_{B_{k_j}} - \chi_{B_{k_{j-1}}} \right). \quad (3.15)$$

Using (3.14) and the assumption $\lambda > -1/p$, we have

$$\begin{aligned} & \sum_{k_j > n} P(B_{k_j})^\lambda \left\| \frac{P(B_{k_{j-1}})}{P(B_{k_j})} \chi_{B_{k_j}} - \chi_{B_{k_{j-1}}} \right\|_{L_p} \\ & \leq \sum_{k_j > n} P(B_{k_j})^\lambda \left(R \|\chi_{B_{k_j}}\|_{L_p} + \|\chi_{B_{k_{j-1}}}\|_{L_p} \right) \\ & \leq 2R \sum_{k_j > n} P(B_{k_j})^{\lambda+1/p} \leq 2R \sum_{j=0}^{\infty} \left(1 + \frac{1}{R}\right)^{-(\lambda+1/p)j} P(B_n)^{\lambda+1/p} \\ & \sim P(B_n)^{\lambda+1/p}. \end{aligned} \quad (3.16)$$

Therefore, $(f_n)_{n \geq 0}$ converges in L_p . We denote by f the limit of $(f_n)_{n \geq 0}$.

We can also deduce from (3.16) that

$$\left(\frac{1}{P(B_n)} \int_{B_n} |f - E_n f|^p dP \right)^{1/p} \lesssim P(B_n)^\lambda. \quad (3.17)$$

On the other hand, for $B \in A(\mathcal{F}_n)$, we have

$$(f - E_n f) \chi_B = \begin{cases} f - E_n f, & (B = B_n), \\ 0, & (B \neq B_n). \end{cases} \quad (3.18)$$

Combining (3.17) and (3.18), we have $\|f\|_{\mathcal{L}_{p,\lambda}} \lesssim 1$, that is, we get $f \in \mathcal{L}_{p,\lambda}$.

(ii) First we show $L_\infty^0 \subsetneq \mathcal{L}_{p,\lambda} \subsetneq \mathcal{L}_{p,\lambda'}$. By Remark 2.6 we have $L_\infty^0 \subset \mathcal{L}_{p,\lambda} \subset \mathcal{L}_{p,\lambda'}$. Then we need to show $\mathcal{L}_{p,0} \setminus L_\infty^0 \neq \emptyset$ and $\mathcal{L}_{p,\lambda'} \setminus \mathcal{L}_{p,\lambda} \neq \emptyset$.

We consider f_n in (3.12) for the sequence B_k , $k = 0, 1, \dots$. Then we can write

$$f_n = \sum_{k_j \leq n} \left(\frac{P(B_{k_{j-1}})}{P(B_{k_j})} \chi_{B_{k_j}} - \chi_{B_{k_{j-1}}} \right) \quad (3.19)$$

for $\lambda = 0$ and we have $f = (f_n)_{n \geq 0} \in \mathcal{L}_{p,0}$. On the other hand, for a.e. $\omega \in B_{k_j} \setminus B_{k_{j+1}}$,

$$f = f_{k_{j+1}} = f_{k_j} - \chi_{B_{k_j}} = \sum_{i=0}^j \left(\frac{P(B_{k_{i-1}})}{P(B_{k_i})} - 1 \right) - 1 \geq \frac{1}{R}(j+1) - 1. \quad (3.20)$$

Then $f \notin L_\infty^0$ and $\mathcal{L}_{p,0} \setminus L_\infty^0 \neq \emptyset$.

Next, let

$$g_n = \sum_{k_j \leq n} P(B_{k_j})^{\lambda'} \left(\frac{P(B_{k_{j-1}})}{P(B_{k_j})} \chi_{B_{k_j}} - \chi_{B_{k_{j-1}}} \right). \quad (3.21)$$

Then $g = (g_n)_{n \geq 0} \in \mathcal{L}_{p,\lambda'}$. On the other hand, since

$$|g - E_{k_{j-1}}g| = P(B_{k_j})^{\lambda'} \quad \text{on } B_{k_{j-1}} \setminus B_{k_j}, \quad (3.22)$$

we have

$$\begin{aligned} & \frac{1}{P(B_{k_{j-1}})^\lambda} \left(\frac{1}{P(B_{k_{j-1}})} \int_{B_{k_{j-1}}} |g - E_{k_{j-1}}g|^p dP \right)^{1/p} \\ & \geq \frac{1}{P(B_{k_{j-1}})^\lambda} \left(\frac{1}{P(B_{k_{j-1}})} \int_{B_{k_{j-1}} \setminus B_{k_j}} |g - E_{k_{j-1}}g|^p dP \right)^{1/p} \\ & \geq \frac{P(B_{k_j})^{\lambda'}}{P(B_{k_{j-1}})^\lambda} \left(\frac{P(B_{k_{j-1}} \setminus B_{k_j})}{P(B_{k_{j-1}})} \right)^{1/p} \\ & \geq R^{\lambda'} (1+R)^{-\lambda'-1/p} P(B_{k_{j-1}})^{\lambda'-\lambda} \rightarrow \infty \quad \text{as } j \rightarrow \infty, \end{aligned} \quad (3.23)$$

since $\lambda' < \lambda$. This shows $\mathcal{L}_{p,\lambda'} \setminus \mathcal{L}_{p,\lambda} \neq \emptyset$.

Finally, if $\mathcal{F}_0 = \{\emptyset, \Omega\}$ also, then by Remark 2.7 and $\mathcal{L}_{p,\lambda'} \setminus \mathcal{L}_{p,\lambda} \neq \emptyset$ we have $\mathcal{L}_{p,\lambda} \subsetneq L_p^0$. This shows the conclusion. $\square$

Remark 3.7. In Proposition 3.6, $f = (f_n)_{n \geq 0}$ in (3.12) converges in L_p as in the above proof. Moreover, the limit belongs to both $\mathcal{L}_{p,\lambda}$ and $L_{p,\lambda}$ when $-1/p < \lambda < 0$, since we will show that $L_{p,\lambda} = \mathcal{L}_{p,\lambda}$ in the proof of Theorem 3.1. However, it converges in neither $\mathcal{L}_{p,\lambda}$ nor $L_{p,\lambda}$. Actually, by a similar calculation to $g = (g_n)_{n \geq 0}$ in the above proof, we have

$$\left| (f - f_n) - E_{k_{j-1}}(f - f_n) \right| = \left| f - E_{k_{j-1}}f \right| = P(B_{k_j})^\lambda \quad \text{on } B_{k_{j-1}} \setminus B_{k_j}, \quad (3.24)$$

for $n \leq k_{j-1}$, and then

$$\frac{1}{P(B_{k_{j-1}})^\lambda} \left(\frac{1}{P(B_{k_{j-1}})} \int_{B_{k_{j-1}}} |f - E_{k_{j-1}}f|^p dP \right)^{1/p} \geq R^\lambda (1 + R)^{-\lambda-1/p}. \quad (3.25)$$

By Remark 2.5, we have

$$2\|f - f_n\|_{L_{p,\lambda}} \geq \|f - f_n\|_{\mathcal{L}_{p,\lambda}} \geq R^\lambda (1 + R)^{-\lambda-1/p}. \quad (3.26)$$

This shows that $(f_n)_{n \geq 0}$ converges in neither $\mathcal{L}_{p,\lambda}$ nor $L_{p,\lambda}$.

Proposition 3.8. *Let $1 \leq p < \infty$.*

(i) *If $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\lambda \leq -1/p$, then $L_p^0 = L_{p,\lambda} = L_{p,\lambda,\mathcal{F}} = \mathcal{L}_{p,\lambda} = \mathcal{L}_{p,\lambda,\mathcal{F}}$ with*

$$\|f\|_{L_{p,\lambda,\mathcal{F}}} = \|f\|_{L_{p,\lambda}} = \|f\|_{L_p} \leq \|f\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \leq 2\|f\|_{L_{p,\lambda,\mathcal{F}}}. \quad (3.27)$$

(ii) *If $\lambda \geq 0$, then $L_{p,\lambda} = L_{p,\lambda,\mathcal{F}} \subset \mathcal{L}_{p,\lambda} = \mathcal{L}_{p,\lambda,\mathcal{F}} \subset L_p^0$ with*

$$\|f\|_{L_p} \leq \|f\|_{L_{p,\lambda,\mathcal{F}}} = \|f\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{L_{p,\lambda,\mathcal{F}}} = \|f\|_{L_{p,\lambda}}. \quad (3.28)$$

Proof. (i) Let $\lambda \leq -1/p$. For any $f \in L_p^0$ and any $B \in \mathcal{F}_n$,

$$\begin{aligned} \frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f|^p dP \right)^{1/p} &= P(B)^{-\lambda-1/p} \left(\int_B |f|^p dP \right)^{1/p} \\ &\leq P(\Omega)^{-\lambda-1/p} \left(\int_\Omega |f|^p dP \right)^{1/p} = \|f\|_{L_p} \\ &= P(\Omega)^{-\lambda-1/p} \left(\int_\Omega |f - E_0 f|^p dP \right)^{1/p}. \end{aligned} \quad (3.29)$$

Then by the definition of the norms and the assumption $\mathcal{F}_0 = \{\emptyset, \Omega\}$ we see that $\|f\|_{L_{p,\lambda,\mathcal{F}}} = \|f\|_{L_{p,\lambda}} = \|f\|_{L_p} \leq \|f\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}}$. By the same observation as Remark 2.5 we have $\|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \leq 2\|f\|_{L_{p,\lambda,\mathcal{F}}}$.

(ii) Let $\lambda \geq 0$. By Remark 2.10 we need to show only $L_{p,\lambda} \subset L_{p,\lambda,\mathcal{F}}$ with $\|f\|_{L_{p,\lambda,\mathcal{F}}} \leq \|f\|_{L_{p,\lambda}}$ and $\mathcal{L}_{p,\lambda} \subset \mathcal{L}_{p,\lambda,\mathcal{F}} \subset L_p^0$ with $\|f\|_{L_p} \leq \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \leq \|f\|_{\mathcal{L}_{p,\lambda}}$.

Let $f \in L_{p,\lambda}$. For any $B \in \mathcal{F}_n$, there exists a sequence of atoms $B_\ell \in A(\mathcal{F}_n)$, $\ell = 1, 2, \dots$, such that $B = \cup_\ell B_\ell$ and $P(B) = \sum_\ell P(B_\ell)$. Then

$$\int_B |f|^p dP = \sum_\ell \int_{B_\ell} |f|^p dP \leq \sum_\ell P(B_\ell)^{\lambda p + 1} \|f\|_{L_{p,\lambda}}^p \leq P(B)^{\lambda p + 1} \|f\|_{L_{p,\lambda}}^p, \quad (3.30)$$

since $\lambda p + 1 \geq 1$. Therefore $f \in L_{p,\lambda,\mathcal{F}}$ and $\|f\|_{L_{p,\lambda,\mathcal{F}}} \leq \|f\|_{L_{p,\lambda}}$. Similarly, we have $\|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \leq \|f\|_{\mathcal{L}_{p,\lambda}}$ for $f \in \mathcal{L}_{p,\lambda}$. By the definition of $\mathcal{L}_{p,\lambda,\mathcal{F}}$ norm we have $\|f\|_{L_p} = \|f - E_0 f\|_{L_p} \leq P(\Omega)^{\lambda + 1/p} \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} = \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}}$ for $f \in \mathcal{L}_{p,\lambda,\mathcal{F}}$. $\square$

Proof of Theorem 3.1. (i) We have the conclusion by Proposition 3.8 without the assumption that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular or that $(\Omega, \mathcal{F}, P)$ is nonatomic.

(ii) By Proposition 3.6 we only need to prove $L_{p,\lambda} = \mathcal{L}_{p,\lambda}$ with $(1/2)\|f\|_{\mathcal{L}_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}} \leq C\|f\|_{\mathcal{L}_{p,\lambda}}$. The first norm inequality follows from Remark 2.5. We show the second one. Note that we do not need the assumption that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ or that $(\Omega, \mathcal{F}, P)$ is nonatomic.

Let $f \in \mathcal{L}_{p,\lambda}$. Then, for any $B \in A(\mathcal{F}_n)$,

$$\begin{aligned} \left(\frac{1}{P(B)} \int_B |f|^p dP \right)^{1/p} &\leq \left(\frac{1}{P(B)} \int_B |f - E_n f|^p dP \right)^{1/p} + \left| \frac{1}{P(B)} \int_B f(\omega) dP \right| \\ &\leq P(B)^\lambda \|f\|_{\mathcal{L}_{p,\lambda}} + \left| \frac{1}{P(B)} \int_B f(\omega) dP \right|, \end{aligned} \quad (3.31)$$

since

$$E_n f = \frac{1}{P(B)} \int_B f(\omega) dP \quad \text{on } B. \quad (3.32)$$

If $B \in \mathcal{F}_0$, then $E_n f = 0$ on B . Assume that $B \notin \mathcal{F}_0$. By Lemma 3.3 we can choose $B_{k_j} \in A(\mathcal{F}_{k_j})$, $0 = k_0 < k_1 < \dots < k_m \leq n$, such that $B_{k_0} \supset B_{k_1} \supset B_{k_2} \supset \dots \supset B_{k_m} = B$ and that $(1 + 1/R)P(B_{k_j}) \leq P(B_{k_{j-1}}) \leq RP(B_{k_j})$. Then, since

$$\frac{1}{P(B_{k_0})} \int_{B_{k_0}} f(\omega) dP = 0, \quad (3.33)$$

we have

$$\begin{aligned}
 \frac{1}{P(B)} \int_B f(\omega) dP &= \frac{1}{P(B)} \int_B f(\omega) dP - \frac{1}{P(B_{k_0})} \int_{B_{k_0}} f(\omega) dP \\
 &= \sum_{j=1}^m \left(\frac{1}{P(B_{k_j})} \int_{B_{k_j}} f(\omega) dP - \frac{1}{P(B_{k_{j-1}})} \int_{B_{k_{j-1}}} f(\omega) dP \right) \\
 &= \sum_{j=1}^m \frac{1}{P(B_{k_j})} \int_{B_{k_j}} [f - E_{k_{j-1}} f](\omega) dP.
 \end{aligned} \tag{3.34}$$

By Hölder's inequality and the assumption $\lambda < 0$ we have

$$\begin{aligned}
 \left| \frac{1}{P(B)} \int_B f(\omega) dP \right| &\leq \sum_{j=1}^m \left(\frac{1}{P(B_{k_j})} \int_{B_{k_j}} |f - E_{k_{j-1}} f|^p dP \right)^{1/p} \\
 &\lesssim \sum_{j=1}^m \left(\frac{1}{P(B_{k_{j-1}})} \int_{B_{k_{j-1}}} |f - E_{k_{j-1}} f|^p dP \right)^{1/p} \\
 &\leq \sum_{j=1}^m P(B_{k_{j-1}})^\lambda \|f\|_{\mathcal{L}_{p,\lambda}} \\
 &\leq \sum_{j=1}^m \left(\left(1 + \frac{1}{R}\right)^{m-j+1} P(B_{k_m}) \right)^\lambda \|f\|_{\mathcal{L}_{p,\lambda}} \\
 &\sim P(B)^\lambda \|f\|_{\mathcal{L}_{p,\lambda}}.
 \end{aligned} \tag{3.35}$$

Therefore we have $\|f\|_{L_{p,\lambda}} \lesssim \|f\|_{\mathcal{L}_{p,\lambda}}$.

(iii) Let $\lambda = 0$. By Remark 2.6 and Proposition 3.6 we have $L_\infty^0 \subset L_{p,0}$ with $\|f\|_{L_{p,0}} \leq \|f\|_{L_\infty}$ and $L_\infty^0 \subsetneq \mathcal{L}_{p,0}$.

Next we show $L_{p,0} \subset L_\infty^0$ and $\|f\|_{L_\infty} \leq \|f\|_{L_{p,0}}$.

Let $f \in L_{p,0}$ and $f \neq 0$ a.e. Take a positive number r such that $P(|f| > r) > 0$. For any $\epsilon > 0$, there exists n and $B \in \mathcal{F}_n$ such that

$$P(B \cap \{|f| > r\}) > (1 - \epsilon)P(B), \tag{3.36}$$

because $\mathcal{F}$ is generated by $\cup_n \mathcal{F}_n$. For the above B , we can take a sequence of atoms $B_\ell \in A(\mathcal{F}_n)$, $\ell = 1, 2, \dots$, such that $B = \cup_\ell B_\ell$ and $P(B) = \sum_\ell P(B_\ell)$. Hence, by the pigeonhole principle, there exists $B' \in A(\mathcal{F}_n)$ such that

$$P(B' \cap \{|f| > r\}) > (1 - \epsilon)P(B'). \tag{3.37}$$

Therefore, we have

$$\begin{aligned}
 \|f\|_{L_{p,0}}^p &\geq \frac{1}{P(B')} \int_{B'} |f|^p dP \\
 &\geq \frac{1}{P(B')} \int_{B' \cap \{|f| > r\}} |f|^p dP \\
 &\geq \frac{P(B' \cap \{|f| > r\})}{P(B')} r^p \\
 &\geq (1 - \epsilon) r^p.
 \end{aligned} \tag{3.38}$$

This shows that $P(|f| > r) > 0$ implies $\|f\|_{L_{p,0}} \geq r$. Then we have the conclusion.

By Proposition 3.8 and Remark 2.11 we have that $\text{BMO} (= \mathcal{L}_{1,\lambda}) = \mathcal{L}_{p,\lambda}$ with equivalent norms.

(iv) Let $\lambda > 0$. For $f \in L_{p,\lambda}$, we suppose that there exists $r > 0$ such that $P(|f| > r) > 0$. Then, for any $\epsilon > 0$, we have the same estimate as (3.37). Moreover, we can decompose B' in (3.37) to atoms in $A(\mathcal{F}_{n+1})$ and we have the same estimate as (3.37) for some atom in $A(\mathcal{F}_{n+1})$. Therefore, we can take an atom $\tilde{B}'$ in $A(\mathcal{F}_{n+m})$ for large enough m such that $\tilde{B}'$ satisfies (3.37) and $P(\tilde{B}') < \epsilon$. Hence we have

$$\|f\|_{L_{p,\lambda}}^p \geq \frac{1}{P(\tilde{B}')^{1+\lambda p}} \int_{\tilde{B}'} |f|^p dP \geq \frac{(1-\epsilon)r^p}{P(\tilde{B}')^{\lambda p}} \geq \frac{(1-\epsilon)r^p}{\epsilon^{\lambda p}}, \tag{3.39}$$

that is, $r \leq \epsilon^\lambda / (1 - \epsilon)^{1/p} \|f\|_{L_{p,\lambda}}$. This contradicts that $r > 0$. Then $f = 0$ a.e. and $L_{p,\lambda} = \{0\}$. By Proposition 3.6 we have that $\{0\} \subsetneq \mathcal{L}_{p,\lambda}$.

Finally, by Proposition 3.8 and Remark 2.11 we have that $\mathcal{L}_{1,\lambda} = \mathcal{L}_{p,\lambda}$ with equivalent norms. $\square$

Next we prove that $L_{p,\lambda,\mathcal{F}} \subsetneq L_{p,\lambda}$ and $\mathcal{L}_{p,\lambda,\mathcal{F}} \subsetneq \mathcal{L}_{p,\lambda}$ in general by an example.

Proposition 3.9. *Let $(\Omega, \mathcal{F}, P)$ be as follows:*

$$\Omega = [0, 1), \quad A(\mathcal{F}_n) = \{I_{n,j} = [j2^{-n}, (j+1)2^{-n}) : j = 0, 1, \dots, 2^n - 1\}, \tag{3.40}$$

$$\mathcal{F}_n = \sigma(A(\mathcal{F}_n)), \quad \mathcal{F} = \sigma\left(\bigcup_n \mathcal{F}_n\right), \quad P = \text{the Lebesgue measure}. \tag{3.41}$$

If $-1/p < \lambda < 0$, then $L_{p,\lambda,\mathcal{F}} \subsetneq L_{p,\lambda}$ and $\mathcal{L}_{p,\lambda,\mathcal{F}} \subsetneq \mathcal{L}_{p,\lambda}$.

Proof. We construct f such that

$$f \in L_{p,\lambda} \setminus L_{p,\lambda,\mathcal{F}}, \quad f \in \mathcal{L}_{p,\lambda} \setminus \mathcal{L}_{p,\lambda,\mathcal{F}}. \tag{3.42}$$

Step 1. Denote the characteristic function of $I_{n,j}$ by $\chi_{n,j}$ and let

$$f_n = \sum_{j=0}^{2^n-1} f_{n+m,2^m j}, \quad f_{n,j} = P(I_{n,j})^\lambda (\chi_{n+1,2j} - \chi_{n+1,2j+1}), \quad (3.43)$$

where we choose m such that

$$P(I_{n+m,0})^{p\lambda+1} \leq P(I_{n,0}). \quad (3.44)$$

Note that $I_{n,j} = I_{n+1,2j} \cup I_{n+1,2j+1}$ and $|f_{n,j}| = P(I_{n,0})^\lambda \chi_{n,j}$.

If $k \leq n+m$, then $E_k f_n = 0$, and $|f_n - E_k f_n| = |f_n| = P(I_{n+m,0})^\lambda \sum_{j=1}^{2^n-1} \chi_{n+m,2^m j}$. Hence

$$\begin{aligned} & \left(\frac{1}{P(I_{k,\ell})} \int_{I_{k,\ell}} |f_n - E_k f_n|^p dP \right)^{1/p} \\ &= \left(\frac{1}{P(I_{k,\ell})} \int_{I_{k,\ell}} |f_n|^p dP \right)^{1/p} \\ &= P(I_{n+m,0})^\lambda \left(\frac{\sum_{j: I_{n+m,2^m j} \subset I_{k,\ell}} P(I_{n+m,2^m j})}{P(I_{k,\ell})} \right)^{1/p}. \end{aligned} \quad (3.45)$$

If $k \leq n$, then the number of the elements of $\{j : I_{n+m,2^m j} \subset I_{k,\ell}\}$ is the same as of $\{j : I_{n,j} \subset I_{k,\ell}\}$. Hence

$$\begin{aligned} \left(\frac{1}{P(I_{k,\ell})} \int_{I_{k,\ell}} |f_n - E_k f_n|^p dP \right)^{1/p} &= \left(\frac{1}{P(I_{k,\ell})} \int_{I_{k,\ell}} |f_n|^p dP \right)^{1/p} \\ &\leq P(I_{n+m,0})^\lambda \left(\frac{P(I_{n+m,0})}{P(I_{n,0})} \right)^{1/p} \leq 1 \leq P(I_{k,\ell})^\lambda, \end{aligned} \quad (3.46)$$

where we use (3.44) and $\lambda < 0$ for the last two inequalities. If $n < k \leq n+m$, then the number of the elements of $\{j : I_{n+m,2^m j} \subset I_{k,\ell}\}$ is one at most. Hence

$$\begin{aligned} \left(\frac{1}{P(I_{k,\ell})} \int_{I_{k,\ell}} |f_n - E_k f_n|^p dP \right)^{1/p} &= \left(\frac{1}{P(I_{k,\ell})} \int_{I_{k,\ell}} |f_n|^p dP \right)^{1/p} \\ &\leq P(I_{n+m,0})^\lambda \left(\frac{P(I_{n+m,0})}{P(I_{k,\ell})} \right)^{1/p} \leq P(I_{k,\ell})^\lambda, \end{aligned} \quad (3.47)$$

where we use $-1/q < \lambda < 0$ for the last inequality. In the above, if $I_{k,\ell} = I_{n+m,0}$, then the equality holds.

If $k > n + m$, then $f_n - E_k f_n = 0$ and

$$\left(\frac{1}{P(I_{k,\ell})} \int_{I_{k,\ell}} |f_n|^p dP \right)^{1/p} \leq P(I_{n+m,0})^\lambda \leq P(I_{k,0})^\lambda. \quad (3.48)$$

Therefore,

$$f_n \in L_{p,\lambda} \cap \mathcal{L}_{p,\lambda}, \quad \|f_n\|_{L_{p,\lambda}} = \|f_n\|_{\mathcal{L}_{p,\lambda}} = 1. \quad (3.49)$$

On the other hand, for the set $B = \bigcup_{j=0}^{2^n-1} I_{n+m,2^m j} \in \mathcal{F}_{n+m}$,

$$\begin{aligned} \left(\frac{1}{P(B)} \int_B |f_n - E_{n+m} f_n|^p dP \right)^{1/p} &= \left(\frac{1}{P(B)} \int_B |f_n|^p dP \right)^{1/p} \\ &= P(I_{n+m,0})^\lambda \\ &= (2^{-n} P(B))^\lambda. \end{aligned} \quad (3.50)$$

Therefore, $\|f_n\|_{L_{p,\lambda,\mathcal{F}}}, \|f_n\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \geq 2^{-n\lambda} \rightarrow \infty$ as $n \rightarrow \infty$.

Step 2. Let f_n be as in Step 1. If $k < n$, then, by the same observation as Step 1 we also have that

$$f_n \chi_{k,\ell} \in L_{p,\lambda} \cap \mathcal{L}_{p,\lambda}, \quad \|f_n \chi_{k,\ell}\|_{L_{p,\lambda}} = \|f_n \chi_{k,\ell}\|_{\mathcal{L}_{p,\lambda}} = 1. \quad (3.51)$$

Moreover, if $k < n$, then, for the set $B = (\bigcup_{j=0}^{2^n-1} I_{n+m,2^m j}) \cap I_{k,\ell} \in \mathcal{F}_{n+m}$,

$$\begin{aligned} \left(\frac{1}{P(B)} \int_B |f_n \chi_{k,\ell} - E_{n+m} [f_n \chi_{k,\ell}]|^p dP \right)^{1/p} &= \left(\frac{1}{P(B)} \int_B |f_n \chi_{k,\ell}|^p dP \right)^{1/p} \\ &= P(I_{n+m,0})^\lambda \\ &= (2^{-n+k} P(B))^\lambda. \end{aligned} \quad (3.52)$$

Therefore, $\|f_n \chi_{k,\ell}\|_{L_{p,\lambda,\mathcal{F}}}, \|f_n \chi_{k,\ell}\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \geq 2^{(-n+k)\lambda}$.

Step 3. Let f_n be as in Step 1 and let

$$f = \sum_{n=1}^{\infty} 2^{n\lambda/2} f_{2n} \chi_{n,1}. \quad (3.53)$$

Note that $\sum_{n=1}^{\infty} \chi_{n,1} \equiv 1$. Then

$$\|f\|_{L_{p,\lambda}}, \|f\|_{\mathcal{L}_{p,\lambda}} \leq \sum_{n=1}^{\infty} 2^{n\lambda/2} = \frac{2^{\lambda/2}}{1 - 2^{\lambda/2}}. \quad (3.54)$$

On the other hand, by the same observation as Step 2, we have that

$$\|f\|_{L_{p,\lambda,\mathcal{F}}}, \|f\|_{\mathcal{L}_{p,\lambda,\mathcal{F}}} \geq 2^{n\lambda/2} \times 2^{(-2n+n)\lambda} = 2^{-n\lambda/2}, \quad (3.55)$$

for all n . This shows (3.42). $\square$

At the end of this section we prove the relation of $\|f\|_{L_{1,\lambda}}$ and $\|f\|_{WL_p}$. Recall that

$$\|f\|_{WL_p} = \sup_{t>0} tP(|f| > t)^{1/p}. \quad (3.56)$$

The following is well known for classical Morrey spaces on $\mathbb{R}^n$. We give a proof for convenience, though it is the same as the proof for classical Morrey spaces on $\mathbb{R}^n$.

Proposition 3.10. *If $1 < q < p < \infty$ and $-1/p = \lambda$, then*

$$\|f\|_{L_{1,\lambda}} \leq \|f\|_{L_{q,\lambda}} \leq C\|f\|_{WL_p}. \quad (3.57)$$

Proof. The first inequality followed from Hölder's inequality. To show the second inequality, we assume that $\|f\|_{WL_p} = 1$ and prove that $\|f\|_{L_{q,\lambda}} \leq C$. From $\|f\|_{WL_p} = 1$, we have $P(|f| > t) \leq t^{-p}$. For any atom $B \in A(\mathcal{F}_n)$, let $\eta = P(B)^\lambda = P(B)^{-1/p}$ and

$$f = f^\eta + f_\eta, \quad f^\eta(\omega) = \begin{cases} f(\omega), & |f(\omega)| > \eta, \\ 0, & |f(\omega)| \leq \eta. \end{cases} \quad (3.58)$$

Then

$$\begin{aligned} \frac{1}{P(B)^{q\lambda+1}} \int_B |f^\eta(\omega)|^q dP &\leq \frac{1}{P(B)^{q\lambda+1}} \int_0^\infty P(|f^\eta| > t) (qt^{q-1}) dt \\ &\leq \frac{1}{P(B)^{q\lambda+1}} \left(\int_0^\eta P(|f^\eta| > \eta) (qt^{q-1}) dt + \int_\eta^\infty t^{-p} (qt^{q-1}) dt \right) \\ &\leq \frac{1}{P(B)^{q\lambda+1}} \left(1 + \frac{q}{p-q} \right) \eta^{q-p} = \frac{p}{p-q}. \end{aligned} \quad (3.59)$$

Since $|f_\eta(\omega)| \leq \eta$ we have

$$\frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B |f_\eta(\omega)|^q dP \right)^{1/q} \leq \frac{\eta}{P(B)^\lambda} = 1. \quad (3.60)$$

We get the conclusion. $\square$

4. Maximal Function

It is known as Doob's inequality that (see for example [5, Pages 20-21])

$$\|f^*\|_{L_p} \leq \frac{p}{p-1} \|f\|_{L_p}, \quad f \in L_p \ (p > 1), \quad (4.1)$$

$$\|f^*\|_{WL_1} \leq \|f\|_{L_1}, \quad f \in L_1. \quad (4.2)$$

In this section we extend (4.1) and (4.2) to Morrey norms. Note that we do not need the regularity of the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$.

Theorem 4.1. *Let $1 \leq p < \infty$ and $-1/p \leq \lambda < 0$. Then, for $f \in L_1$,*

$$\begin{aligned} \|f^*\|_{L_{p,\lambda}} &\leq C_p \|f\|_{L_{p,\lambda}}, \quad \text{if } p > 1, \\ \|f^*\|_{WL_{1,\lambda}} &\leq C_1 \|f\|_{L_{1,\lambda}}, \quad \text{if } p = 1. \end{aligned} \quad (4.3)$$

Proof. Case 1 ($p > 1$). For any $B \in A(\mathcal{F}_m)$ and $m \geq 0$, let $f = g + h$ and $g = f\chi_B$. Then, using (4.1), we have

$$\int_B (g^*)^p dP \leq \int_\Omega (g^*)^p dP \lesssim \int_\Omega |g|^p dP = \int_B |f|^p dP. \quad (4.4)$$

Hence

$$\frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B (g^*)^p dP \right)^{1/p} \lesssim \|f\|_{L_{p,\lambda}}. \quad (4.5)$$

Next, take $B_n \in A(\mathcal{F}_n)$, $n = 0, 1, \dots, m$, such that $B = B_m \subset B_{m-1} \subset \dots \subset B_0$. Then, for a.e. $\omega \in B$,

$$E_n h(\omega) = \begin{cases} 0, & (n \geq m), \\ \frac{1}{P(B_n)} \int_{B_n} h dP, & (n < m). \end{cases} \quad (4.6)$$

If $n < m$, then

$$|E_n h(\omega)| \leq \left(\frac{1}{P(B_n)} \int_{B_n} |h|^p dP \right)^{1/p} \leq P(B_n)^\lambda \|f\|_{L_{p,\lambda}} \leq P(B)^\lambda \|f\|_{L_{p,\lambda}}, \quad (4.7)$$

since $\lambda < 0$. Hence

$$h^* \leq P(B)^\lambda \|f\|_{L_{p,\lambda}} \quad \text{on } B. \quad (4.8)$$

Then

$$\frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B (h^*)^p dP \right)^{1/p} \leq \|f\|_{L_{p,\lambda}}. \quad (4.9)$$

By (4.5), (4.9), and the inequality $f^* \leq g^* + h^*$, we have

$$\frac{1}{P(B)^\lambda} \left(\frac{1}{P(B)} \int_B (f^*)^p dP \right)^{1/p} \lesssim \|f\|_{L_{p,\lambda}}, \quad (4.10)$$

which shows the conclusion.

Case 2 ($p = 1$). Let g and h be as in Case 1. Using (4.2), we have, for all $t > 0$,

$$tP(B \cap \{g^* > t\}) \leq tP(g^* > t) \leq \int_\Omega |g| dP = \int_B |f| dP \quad (4.11)$$

and then

$$\frac{1}{P(B)^\lambda} \frac{tP(B \cap \{g^* > t\})}{P(B)} \leq \|f\|_{L_{1,\lambda}}. \quad (4.12)$$

We also have (4.8) for the case $p = 1$. Then

$$\frac{1}{P(B)^\lambda} \frac{tP(B \cap \{h^* > t\})}{P(B)} \leq \|f\|_{L_{1,\lambda}}. \quad (4.13)$$

Therefore we have the conclusion. $\square$

5. Fractional Integrals

In this section we establish the boundedness of the fractional integrals. To do this we first prove norm inequalities for functions, and then we get the boundedness of I_α as a martingale transform.

For normed spaces M_1 and M_2 of functions, we denote by $B(M_1, M_2)$ the set of all bounded martingale transforms from M_1 to M_2 , that is, $I_\alpha \in B(M_1, M_2)$ means that

$$\sup_{n \geq 0} \|(I_\alpha f)_n\|_{M_2} \leq C \sup_{n \geq 0} \|f_n\|_{M_1}, \quad (5.1)$$

for all M_1 -bounded martingales $f = (f_n)_{n \geq 0}$.

We state our results in Section 5.1. To prove Theorems 5.1 and 5.5 we show the pointwise estimate for $I_\alpha f$ with the assumptions $\|f\|_{L_p} < \infty$ and $\|f\|_{L_{p,\lambda}} < \infty$, respectively. To avoid repetition we prove first Theorem 5.5 in Section 5.2 and then Theorem 5.1 in Section 5.3. The proof of Theorem 5.8 is in Section 5.4.

5.1. Boundedness of Fractional Integrals

The following is for L_p - L_q boundedness.

Theorem 5.1. Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular. Let $1 \leq p < q < \infty$, $-1/p + \alpha = -1/q$. Then, for $f \in L_1$,

$$\|(I_\alpha f)^*\|_{L_q} \leq C \|f\|_{L_p}, \quad \text{if } p > 1, \quad (5.2)$$

$$\|(I_\alpha f)^*\|_{WL_q} \leq C \|f\|_{L_1}, \quad \text{if } p = 1. \quad (5.3)$$

Remark 5.2. Let $f = (f_n)_{n \geq 0}$ be an L_1 -bounded martingale. For each L_1 -function f_m , $m \geq 0$, consider the corresponding martingale $f^m = (E_n[f_m])_{n \geq 0} = (f_{\min(n,m)})_{n \geq 0}$. Then $(I_\alpha f^m)_n = (I_\alpha f)_n$ for $m \geq n$. Therefore, from (5.3) it follows that, for $m \geq n$,

$$\|(I_\alpha f)_n\|_{WL_q} = \|(I_\alpha f^m)_n\|_{WL_q} \lesssim \|f_m\|_{L_1} \leq \sup_{n \geq 0} \|f_n\|_{L_1}. \quad (5.4)$$

This shows that

$$\sup_{n \geq 0} \|(I_\alpha f)_n\|_{WL_q} \leq C \sup_{n \geq 0} \|f_n\|_{L_1}. \quad (5.5)$$

Remark 5.3. Let a martingale $f = (f_n)_{n \geq 0}$ be L_1 -bounded. Since $WL_q \subset L_{q_1}$ and $\|(I_\alpha f)_n\|_{L_{q_1}} \leq \|(I_\alpha f)_n\|_{WL_q}$ for $1 < q_1 < q$, from Remark 5.2 it follows that the martingale $((I_\alpha f)_n)_{n \geq 0}$ is L_{q_1} -bounded and that it converges in L_{q_1} . Denote this limit by $I_\alpha f$. Then, $I_\alpha f = \sum_{k=0}^{\infty} b_{k-1}^\alpha d_k f$ and $E_n[I_\alpha f] = (I_\alpha f)_n$.

If $p > 1$, then the same observation as in Remark 5.2 with (5.2) shows that

$$\sup_{n \geq 0} \|(I_\alpha f)_n\|_{L_q} \leq C \sup_{n \geq 0} \|f_n\|_{L_p}. \quad (5.6)$$

Hence we have the following corollary.

Corollary 5.4. Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular. Let $1 \leq p < q < \infty$, $-1/p + \alpha = -1/q$. Then

$$\begin{aligned} I_\alpha &\in B(L_p, L_q), & \text{if } p > 1, \\ I_\alpha &\in B(L_1, WL_q), & \text{if } p = 1. \end{aligned} \quad (5.7)$$

For Morrey norms, one has the following.

Theorem 5.5. Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular. Let $1 \leq p < \infty$, $-1/p \leq \lambda < 0$, and $\mu = \lambda + \alpha < 0$. Then, for $f \in L_1$,

$$\|(I_\alpha f)^*\|_{L_{q,\mu}} \leq C\|f\|_{L_{p,\lambda}}, \quad \text{if } p > 1, \ 1 \leq q \leq (\lambda/\mu)p, \quad (5.8)$$

$$\|(I_\alpha f)^*\|_{L_{q,\mu}} \leq C\|f\|_{L_{1,\lambda}}, \quad \text{if } p = 1, \ 1 \leq q < (\lambda/\mu)p, \quad (5.9)$$

$$\|(I_\alpha f)^*\|_{WL_{q,\mu}} \leq C\|f\|_{L_{1,\lambda}}, \quad \text{if } p = 1, \ q = (\lambda/\mu)p. \quad (5.10)$$

Note that Theorem 5.1 is not a corollary of Theorem 5.5, since $\|f\|_{L_p} \not\lesssim \|f\|_{L_{p,-1/p}} \leq \|f\|_{L_p}$ in general.

Remark 5.6. In order to prove (5.8) it suffices to prove it in the case where $q = (\lambda/\mu)p$, since $\|f\|_{L_{q_1,\lambda}} \leq \|f\|_{L_{q,\lambda}}$ for $q_1 \leq q$ by Hölder's inequality. The inequality (5.9) follows from (5.10), since $\|f\|_{L_{q_1,\lambda}} \lesssim \|f\|_{WL_{q,\lambda}}$ for $q_1 < q$.

By the same observation as in Remark 5.2 we have the following.

Corollary 5.7. Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular. Let $1 \leq p < \infty$, $-1/p \leq \lambda < 0$, and $\mu = \lambda + \alpha < 0$. Then

$$\begin{aligned} I_\alpha &\in B(L_{p,\lambda}, L_{q,\mu}), \quad \text{if } p > 1, \ 1 \leq q \leq (\lambda/\mu)p, \\ I_\alpha &\in B(L_{1,\lambda}, L_{q,\mu}), \quad \text{if } p = 1, \ 1 \leq q < (\lambda/\mu)p, \\ I_\alpha &\in B(L_{1,\lambda}, WL_{q,\mu}), \quad \text{if } p = 1, \ q = (\lambda/\mu)p. \end{aligned} \quad (5.11)$$

For Campanato spaces, one has the following.

Theorem 5.8. Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular. Let $1 \leq p < \infty$, $\lambda \geq -1/p$, $1 \leq q < \infty$, and $\mu = \lambda + \alpha \geq 0$. Then

$$\|I_\alpha f\|_{\mathcal{L}_{q,\mu}} \leq C\|f\|_{\mathcal{L}_{p,\lambda}}, \quad f \in L_1, \quad (5.12)$$

and

$$I_\alpha \in B(\mathcal{L}_{p,\lambda}, \mathcal{L}_{q,\mu}), \quad (5.13)$$

where $I_\alpha f$ in (5.12) denotes the limit function described in Remark 5.3, for $f = (E_n f)_{n \geq 0}$, $f \in L_1$.

Remark 5.9. If $\mu < 0$, then, by Theorem 3.1 (ii), the inequalities (5.8) and (5.9) in Theorem 5.5 hold with $L_{p,\lambda}$ and $L_{q,\mu}$ replaced by $\mathcal{L}_{p,\lambda}$ and $\mathcal{L}_{q,\mu}$, respectively.

If $1 < p < \infty$ and $-1/p = \lambda$, then $\|f\|_{\mathcal{L}_{1,\lambda}} \sim \|f\|_{L_{1,\lambda}} \lesssim \|f\|_{WL_p} \lesssim \|f\|_{L_p}$ by Remark 2.5 and Proposition 3.10. Since $L_p \subsetneq L_{1,\lambda}$ in general, the following is an improvement of the results in [21].

Corollary 5.10. Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular. Then, for $f \in L_1$,

$$\begin{aligned} \|I_\alpha f\|_{\text{BMO}} &\leq C \|f\|_{L_{1,-1/p}}, \quad \text{if } -\frac{1}{p} + \alpha = 0, \quad 1 < p < \infty, \\ \|I_\alpha f\|_{\text{Lip}_\beta} &\leq C \|f\|_{L_{1,-1/p}}, \quad \text{if } -\frac{1}{p} + \alpha = \beta > 0, \quad 1 < p < \infty, \\ \|I_\alpha f\|_{\text{Lip}_\alpha} &\leq C \|f\|_{\text{BMO}}, \end{aligned} \quad (5.14)$$

$$\begin{aligned} \|I_\alpha f\|_{\text{Lip}_\gamma} &\leq C \|f\|_{\text{Lip}_\beta}, \quad \text{if } \beta + \alpha = \gamma, \quad \beta > 0, \\ I_\alpha &\in B(L_{1,-1/p}, \text{BMO}), \quad \text{if } -\frac{1}{p} + \alpha = 0, \quad 1 < p < \infty, \\ I_\alpha &\in B(L_{1,-1/p}, \text{Lip}_\alpha), \quad \text{if } -\frac{1}{p} + \alpha = \beta > 0, \quad 1 < p < \infty, \end{aligned} \quad (5.15)$$

$$\begin{aligned} I_\alpha &\in B(\text{BMO}, \text{Lip}_\alpha), \\ I_\alpha &\in B(\text{Lip}_\beta, \text{Lip}_\gamma), \quad \text{if } \beta + \alpha = \gamma, \quad \beta > 0. \end{aligned}$$

5.2. Proof of Theorem 5.5

By Remark 5.6 we only need to prove (5.8) and (5.10) in the case $q = (\lambda/\mu)p$. So we always assume that $q = (\lambda/\mu)p$ in this proof.

First we show the pointwise estimate

$$(I_\alpha f)^*(\omega) \leq C f^*(\omega)^{\mu/\lambda} \|f\|_{L_{p,\lambda}}^{-\alpha/\lambda}, \quad (5.16)$$

where C is a positive constant independent of ω . To do this we prove that, for any $m \geq 1$ and any $B_m \in A(\mathcal{F}_m)$,

$$|(I_\alpha f)_m(\omega)| \leq C f^*(\omega)^{\mu/\lambda} \|f\|_{L_{p,\lambda}}^{-\alpha/\lambda}, \quad \omega \in B_m. \quad (5.17)$$

Take $B_k \in A(\mathcal{F}_k)$, $0 \leq k < m$, such that $B_m \subset B_{m-1} \subset \dots \subset B_0$ and let

$$K = \{k : 0 < k \leq m, B_k \neq B_{k-1}\} = \{k_1, k_2, \dots, k_\ell\}, \quad (5.18)$$

where $0 = k_0 < k_1 < k_2 < \dots < k_\ell$. Then, by Lemma 3.3,

$$\left(1 + \frac{1}{R}\right) b_{k_j} \leq b_{k_{j-1}} \leq R b_{k_j} \quad \text{on } B_m. \quad (5.19)$$

Hence, we can write

$$(I_\alpha f)_m = \sum_{0 < k_j \leq m} b_{k_{j-1}}^\alpha d_{k_j} f = \sum_{j=1}^{\ell} b_{k_{j-1}}^\alpha d_{k_j} f \quad \text{on } B_m, \quad (5.20)$$

since $b_k = b_{k-1}$ and $d_k f = 0$ for $k \notin K$. Note that for $\omega \in B_m$,

$$\begin{aligned} |d_{k_j} f(\omega)| &= |f_{k_j}(\omega) - f_{k_{j-1}}(\omega)| \leq |f_{k_j}(\omega)| + |f_{k_{j-1}}(\omega)| \\ &= \left| \frac{1}{P(B_{k_j})} \int_{B_{k_j}} f dP \right| + \left| \frac{1}{P(B_{k_{j-1}})} \int_{B_{k_{j-1}}} f dP \right| \\ &\leq \left(\frac{1}{P(B_{k_j})} \int_{B_{k_j}} |f|^p dP \right)^{1/p} + \left(\frac{1}{P(B_{k_{j-1}})} \int_{B_{k_{j-1}}} |f|^p dP \right)^{1/p} \\ &\leq \left(P(B_{k_j})^\lambda + P(B_{k_{j-1}})^\lambda \right) \|f\|_{L_{p,\lambda}} \\ &\leq \left(1 + \left(\frac{1}{R} \right)^\lambda \right) b_{k_{j-1}}(\omega)^\lambda \|f\|_{L_{p,\lambda}}. \end{aligned} \quad (5.21)$$

Then, for $\omega \in B_m$ and for $0 < n \leq m$,

$$\begin{aligned} \left| \sum_{k=0}^n b_{k-1}(\omega)^\alpha d_k f(\omega) \right| &\leq \left(1 + \left(\frac{1}{R} \right)^\lambda \right) \sum_{0 < k_j \leq n} b_{k_{j-1}}(\omega)^\mu \|f\|_{L_{p,\lambda}} \\ &\leq \frac{1 + (1/R)^\lambda}{1 - (1 + 1/R)^\mu} b_n(\omega)^\mu \|f\|_{L_{p,\lambda}}. \end{aligned} \quad (5.22)$$

On the other hand, for $0 \leq n < m$, letting $j(n) = \min\{j : n < k_j\}$, we have, for $\omega \in B_m$,

$$\begin{aligned} \sum_{k=n+1}^m b_{k-1}(\omega)^\alpha d_k f(\omega) &= \sum_{j=j(n)}^{\ell} b_{k_{j-1}}(\omega)^\alpha d_{k_j} f(\omega) \\ &= \sum_{j=j(n)}^{\ell} b_{k_{j-1}}(\omega)^\alpha f_{k_j}(\omega) - \sum_{j=j(n)}^{\ell} b_{k_{j-1}}(\omega)^\alpha f_{k_{j-1}}(\omega) \\ &= b_{k_{\ell-1}}(\omega)^\alpha f_{k_\ell}(\omega) + \sum_{j=j(n)}^{\ell-1} (b_{k_{j-1}}(\omega)^\alpha - b_{k_j}(\omega)^\alpha) f_{k_j}(\omega) \\ &\quad - b_{k_{j(n)-1}}(\omega)^\alpha f_{k_{j(n)-1}}(\omega), \end{aligned} \quad (5.23)$$

$$\begin{aligned}
\left| \sum_{k=n+1}^m b_{k-1}(\omega)^\alpha d_k f(\omega) \right| &\leq b_{k_{\ell-1}}(\omega)^\alpha f^*(\omega) \\
&+ \sum_{j=j(n)}^{\ell-1} \left| b_{k_{j-1}}(\omega)^\alpha - b_{k_j}(\omega)^\alpha \right| f^*(\omega) \\
&+ b_{k_{j(n)-1}}(\omega)^\alpha f^*(\omega) \\
&\leq 2b_{k_{j(n)-1}}(\omega)^\alpha f^*(\omega) = 2b_n(\omega)^\alpha f^*(\omega).
\end{aligned} \tag{5.24}$$

Here, let

$$\Omega_1 = \left\{ \omega \in \Omega : \left(\frac{f^*(\omega)}{\|f\|_{L_{p,\lambda}}} \right)^{1/\lambda} \leq b_0(\omega) \right\}, \quad \Omega_2 = \Omega \setminus \Omega_1. \tag{5.25}$$

If $\omega \in \Omega_1 \cap B_m$ and

$$\left(\frac{f^*(\omega)}{\|f\|_{L_{p,\lambda}}} \right)^{1/\lambda} \leq b_m(\omega), \tag{5.26}$$

then, by (5.22) we have

$$|(I_\alpha f)_m(\omega)| \lesssim b_m(\omega)^\mu \|f\|_{L_{p,\lambda}} \leq \left(\frac{f^*(\omega)}{\|f\|_{L_{p,\lambda}}} \right)^{\mu/\lambda} \|f\|_{L_{p,\lambda}} = f^*(\omega)^{\mu/\lambda} \|f\|_{L_{p,\lambda}}^{-\alpha/\lambda}, \tag{5.27}$$

since $\mu < 0$ and $1 - \mu/\lambda = -\alpha/\lambda$. If $\omega \in \Omega_1 \cap B_m$ and

$$b_m(\omega) < \left(\frac{f^*(\omega)}{\|f\|_{L_{p,\lambda}}} \right)^{1/\lambda}, \tag{5.28}$$

choosing n such that

$$R^{-1}b_n(\omega) < \left(\frac{f^*(\omega)}{\|f\|_{L_{p,\lambda}}} \right)^{1/\lambda} \leq b_n(\omega), \tag{5.29}$$

we have by (5.22) and (5.24)

$$\begin{aligned}
|(I_\alpha f)_m(\omega)| &\lesssim b_n(\omega)^\mu \|f\|_{L_{p,\lambda}} + b_n(\omega)^\alpha f^*(\omega) \\
&\lesssim \left(\frac{f^*(\omega)}{\|f\|_{L_{p,\lambda}}} \right)^{\mu/\lambda} \|f\|_{L_{p,\lambda}} + \left(\frac{f^*(\omega)}{\|f\|_{L_{p,\lambda}}} \right)^{\alpha/\lambda} f^*(\omega) \\
&\lesssim f^*(\omega)^{\mu/\lambda} \|f\|_{L_{p,\lambda}}^{-\alpha/\lambda}.
\end{aligned} \tag{5.30}$$

If $\omega \in \Omega_2 \cap B_m$, then by (5.24) we have

$$\begin{aligned}
|(I_\alpha f)_m(\omega)| &= \left| \sum_{k=1}^m b_{k-1}(\omega)^\alpha d_k f(\omega) \right| \lesssim b_0(\omega)^\alpha f^*(\omega) \\
&\leq \left(\frac{f^*(\omega)}{\|f\|_{L_{p,\lambda}}} \right)^{\alpha/\lambda} f^*(\omega) = f^*(\omega)^{\mu/\lambda} \|f\|_{L_{p,\lambda}}^{-\alpha/\lambda}.
\end{aligned} \tag{5.31}$$

Therefore, we have (5.17).

Next, applying the boundedness of the maximal function (Theorem 4.1), we show

$$\left(\frac{1}{P(B)} \int_B [(I_\alpha f(\omega))^*]^q dP \right)^{1/q} \lesssim P(B)^\mu \|f\|_{L_{p,\lambda}}, \quad \text{if } p > 1, \tag{5.32}$$

$$\sup_{t>0} t \left(\frac{P(B \cap \{(I_\alpha f)^* > t\})}{P(B)} \right)^{1/q} \lesssim P(B)^\mu \|f\|_{L_{1,\lambda}}, \quad \text{if } p = 1, \tag{5.33}$$

for any n and any $B \in \mathcal{A}(\mathcal{F}_n)$. To show (5.32), using (5.16) and the relations $q\mu/\lambda = p$ and $-\alpha/\lambda = 1 - p/q$, we have

$$\begin{aligned}
\left(\frac{1}{P(B)} \int_B [(I_\alpha f(\omega))^*]^q dP \right)^{1/q} &\lesssim \left(\frac{1}{P(B)} \int_B f^*(\omega)^{q\mu/\lambda} dP \right)^{1/q} \|f\|_{L_{p,\lambda}}^{-\alpha/\lambda} \\
&= \left(\frac{1}{P(B)} \int_B f^*(\omega)^p dP \right)^{(1/p)(p/q)} \|f\|_{L_{p,\lambda}}^{1-p/q}.
\end{aligned} \tag{5.34}$$

Applying the boundedness of the maximal function, we have

$$\left(\frac{1}{P(B)} \int_B f^*(\omega)^p dP \right)^{(1/p)(p/q)} \leq \left(P(B)^\lambda \|f^*\|_{L_{p,\lambda}} \right)^{p/q} \lesssim P(B)^\mu \|f\|_{L_{p,\lambda}}^{p/q}. \tag{5.35}$$

Then we have (5.32). To show (5.33), using (5.16) and the relations $p = 1$, $\mu/\lambda = 1/q$ and $-\alpha/\lambda = 1 - 1/q$, we have

$$\begin{aligned}
 \sup_{t>0} t \left(\frac{P(B \cap \{(I_\alpha f)^* > t\})}{P(B)} \right)^{1/q} &\lesssim \sup_{t>0} t \left(\frac{P(B \cap \{(f^*)^{\mu/\lambda} \|f\|_{L_{1,\lambda}}^{-\alpha/\lambda} > t\})}{P(B)} \right)^{1/q} \\
 &= \sup_{t>0} t \left(\frac{P(B \cap \{(f^*)^{1/q} \|f\|_{L_{1,\lambda}}^{1-1/q} > t\})}{P(B)} \right)^{1/q} \\
 &= \sup_{t>0} t \left(\frac{P(B \cap \{(f^*)^{1/q} > t\})}{P(B)} \right)^{1/q} \|f\|_{L_{1,\lambda}}^{1-1/q} \\
 &= \left(\sup_{t>0} t \frac{P(B \cap \{(f^*) > t\})}{P(B)} \right)^{1/q} \|f\|_{L_{1,\lambda}}^{1-1/q}.
 \end{aligned} \tag{5.36}$$

Applying the boundedness of the maximal function, we have

$$\left(\sup_{t>0} t \frac{P(B \cap \{(f^*) > t\})}{P(B)} \right)^{1/q} \leq (P(B)^\lambda \|f^*\|_{WL_{1,\lambda}})^{1/q} \lesssim P(B)^\mu \|f\|_{L_{p,\lambda}}^{1/q}. \tag{5.37}$$

Then we have (5.33). The proof is complete.

5.3. Proof of Theorem 5.1

Let $\lambda = -1/p$ and $\mu = -1/q$. Then $\mu/\lambda = p/q$ and $-\alpha/\lambda = (1/p - 1/q)p = 1 - p/q$. In this case the pointwise estimate (5.16) implies

$$(I_\alpha f)^*(\omega) \leq C f^*(\omega)^{p/q} \|f\|_{L_{p,-1/p}}^{1-p/q} \leq C f^*(\omega)^{p/q} \|f\|_{L_p}^{1-p/q}. \tag{5.38}$$

Applying the boundedness of the maximal function on L_p , we get the conclusion.

5.4. Proof of Theorem 5.8

Note that, for $f \in L_1$, $\|f\|_{\mathcal{L}_{p,\lambda}} = \|f - E_0 f\|_{\mathcal{L}_{p,\lambda}}$ and $I_\alpha f = I_\alpha(f - E_0 f)$. Then it is sufficient to prove (5.12) only for $f \in L_1^0$. Therefore, (5.12) follows from (5.13) and Proposition 2.2(ii).

By Hölder's inequality we have $\|f\|_{\mathcal{L}_{1,\lambda}} \leq \|f\|_{\mathcal{L}_{p,\lambda}}$ for $p \geq 1$. Furthermore, by Theorem 3.1(iii) and (iv) we have $\|I_\alpha f\|_{\mathcal{L}_{q,\mu}} \sim \|I_\alpha f\|_{\mathcal{L}_{1,\mu}}$ for $q \geq 1$. So we only need to prove that

$$\sup_{m \geq 0} \|(I_\alpha f)_m\|_{\mathcal{L}_{1,\mu}} \leq C \sup_{m \geq 0} \|f_m\|_{\mathcal{L}_{1,\lambda}}, \tag{5.39}$$

for all $\mathcal{L}_{1,\lambda}$ -bounded martingales $f = (f_m)_{m \geq 0}$.

To prove (5.39) we show that, for any n , any $m > n$ and any $B_n \in A(\mathcal{F}_n)$,

$$\int_{B_n} |(I_\alpha f)_m - (I_\alpha f)_n| dP \lesssim P(B_n)^{1+\mu} \|f_m\|_{\mathcal{L}_{1,\lambda}}. \quad (5.40)$$

Note that

$$(I_\alpha f)_m - (I_\alpha f)_n = \sum_{k=n+1}^m b_{k-1}^\alpha d_k f = \sum_{k=n+1}^{m-1} (b_{k-1}^\alpha - b_k^\alpha)(f_k - f_n) + b_{m-1}^\alpha(f_m - f_n). \quad (5.41)$$

Since b_m is nonincreasing with respect to m , we have

$$\int_{B_n} b_{m-1}^\alpha |f_m - f_n| dP \leq P(B_n)^\alpha \int_{B_n} |f_m - f_n| dP \lesssim P(B_n)^{1+\mu} \|f_m\|_{\mathcal{L}_{1,\lambda}}. \quad (5.42)$$

Therefore, to prove (5.40), we only need to show that

$$\sum_{k=n+1}^m \int_{B_n} (b_{k-1}^\alpha - b_k^\alpha) |f_k - f_n| dP \leq P(B_n)^{1+\mu} \|f_m\|_{\mathcal{L}_{1,\lambda}} \quad \forall m > n. \quad (5.43)$$

Using the inequality $|f_k - f_n| \leq E_k |f_m - f_n|$ for $n < k \leq m$, we have

$$\int_{B_n} (b_{k-1}^\alpha - b_k^\alpha) |f_k - f_n| dP \leq \int_{B_n} (b_{k-1}^\alpha - b_k^\alpha) E_k |f_m - f_n| dP = \int_{B_n} (b_{k-1}^\alpha - b_k^\alpha) |f_m - f_n| dP. \quad (5.44)$$

Therefore, we have

$$\begin{aligned} \sum_{k=n+1}^m \int_{B_n} (b_{k-1}^\alpha - b_k^\alpha) |f_k - f_n| dP &\leq \sum_{k=n+1}^m \int_{B_n} (b_{k-1}^\alpha - b_k^\alpha) |f_m - f_n| dP \\ &= \int_{B_n} (b_n^\alpha - b_m^\alpha) |f_m - f_n| dP \\ &\leq P(B_n)^\alpha \int_{B_n} |f_m - f_n| dP \\ &\leq P(B_n)^{1+\mu} \|f_m\|_{\mathcal{L}_{1,\lambda}}. \end{aligned} \quad (5.45)$$

We have obtained (5.43). The proof of Theorem 5.8 is completed.

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