Research Article

A Weak Comparison Principle for Reaction-Diffusion Systems

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We prove a weak comparison principle for a reaction-diffusion system without uniqueness of solutions. We apply the abstract results to the Lotka-Volterra system with diffusion, a generalized logistic equation, and to a model of fractional-order chemical autocatalysis with decay. Moreover, in the case of the Lotka-Volterra system a weak maximum principle is given, and a suitable estimate in the space of essentially bounded functions L^∞ is proved for at least one solution of the problem.

1. Introduction

Comparison results for parabolic equations and ordinary differential equations are well known in the literature (see, e.g., [1–4] among many others). One of the important applications of such kind of results is the theory of monotone dynamical systems, which leads to a more precise characterization of ω -limit sets and attractors. In the last years, several authors have been working in this direction (see, e.g., [4–8] for the deterministic case, and [9–12] for the stochastic case). In all these papers, it is considered the classical situation where the initial-value problem possesses a unique solution.

However, the situation is more complicated when we consider a differential equation for which uniqueness of the Cauchy problem fails (or just it is not known to hold). Let us consider an abstract parabolic problem:

$$\begin{aligned}\frac{du}{dt} &= A(t, u(t)), \quad \tau \leq t \leq T, \\ u(\tau) &= u_\tau,\end{aligned}\tag{1.1}$$

for which we can prove that for every initial data in the phase space X (with a partial order $\leq$) there exists at least one solution.

If we try to compare solutions of (1.1) for two ordered initial data $u_\tau^1 \leq u_\tau^2$, then we can consider a strong comparison principle and a weak one.

The strong version would imply the existence of a solution u_1 with $u_1(\tau) = u_\tau^1$ such that

$$u_1(t) \leq u_2(t) \quad \text{for } t \in [\tau, T], \quad (1.2)$$

for any solution u_2 with $u_2(\tau) = u_\tau^2$, and, viceversa, the existence of a solution u_2 with $u_2(\tau) = u_\tau^2$ such that (1.2) is satisfied for any solution u_1 with $u_1(\tau) = u_\tau^1$. This kind of result is established in [13] for a delayed ordinary differential equations, defining then a multivalued order-preserving dynamical system.

The weak version of the comparison principle says that if $u_\tau^1 \leq u_\tau^2$, then there exist two solutions u_1, u_2 of (1.1) such that $u_1(\tau) = u_\tau^1$, $u_2(\tau) = u_\tau^2$, and (1.2) hold.

There is in fact an intermediate version of the comparison principle, which says that if we fix a solution u_1 of (1.1) with $u_1(\tau) = u_\tau^1$, then there exists a solution u_2 with $u_2(\tau) = u_\tau^2$ such that (1.2) is satisfied (and vice versa). This is proved in [14] for a differential inclusion generated by a subdifferential map.

In this paper, we establish a weak comparison principle for a reaction-diffusion system in which the nonlinear term satisfies suitable dissipative and growth conditions, ensuring existence of solutions but not uniqueness. This principle is applied to several well-known models in physics and biology. Namely, a weak comparison of solutions is proved for the Lotka-Volterra system, the generalized logistic equation and for a model of fractional-order chemical autocatalysis with decay. Moreover, in the case of the Lotka-Volterra system a weak maximum principle is given, and a suitable estimate in the space of essentially bounded functions L^∞ is proved for at least one solution of the problem.

We note that in the papers [15, 16] the existence of a global attractor is proved for such kind of reaction-diffusion systems. In the near future, we will apply these results to obtain theorems concerning the structure of the global attractor.

2. Comparison Results for Reaction-Diffusion Systems

We shall denote by $|\cdot|$ and $(\cdot, \cdot)$ the norm and scalar product in the space $\mathbb{R}^m$, $m \geq 1$. Let $d > 0$ be an integer and $\Omega \subset \mathbb{R}^N$ be a bounded open subset with smooth boundary. Consider the problem:

$$\begin{aligned} \frac{\partial u}{\partial t} - a \Delta u + f(t, u) &= h(t, x), \quad (t, x) \in (\tau, T) \times \Omega, \\ u|_{x \in \partial \Omega} &= 0, \\ u|_{t=\tau} &= u_\tau(x), \end{aligned} \quad (2.1)$$

where $\tau, T \in \mathbb{R}$, $T > \tau$, $x \in \Omega$, $u = (u^1(t, x), \dots, u^d(t, x))$, $f = (f^1, \dots, f^d)$, a is a real $d \times d$ matrix with a positive symmetric part $(a + a^t)/2 \geq \beta I$, $\beta > 0$, $h \in L^2(\tau, T; (L^2(\Omega))^d)$.

Moreover, $f = (f^1(t, u), \dots, f^d(t, u))$ is jointly continuous on $[\tau, T] \times \mathbb{R}^d$ and satisfies the following conditions:

$$\begin{aligned} \sum_{i=1}^d |f^i(t, u)|^{p_i/(p_i-1)} &\leq C_1 \left(1 + \sum_{i=1}^d |u^i|^{p_i} \right), \\ (f(t, u), u) &\geq \alpha \sum_{i=1}^d |u^i|^{p_i} - C_2, \end{aligned} \quad (2.2)$$

where $p_i \geq 2$, $\alpha, C_1, C_2 > 0$.

Let $H = (L^2(\Omega))^d$, $V = (H_0^1(\Omega))^d$, and let V' be the dual space of V . By $\|\cdot\|$, $\|\cdot\|_V$ we denote the norm by H and V , respectively. For $p = (p_1, \dots, p_d)$, we define the spaces

$$\begin{aligned} L^p(\Omega) &= L^{p_1}(\Omega) \times \dots \times L^{p_d}(\Omega), \\ L^p(\tau, T; L^p(\Omega)) &= L^{p_1}(\tau, T; L^{p_1}(\Omega)) \times \dots \times L^{p_d}(\tau, T; L^{p_d}(\Omega)). \end{aligned} \quad (2.3)$$

We take $q = (q_1, \dots, q_d)$, where $(1/p_i) + (1/q_i) = 1$.

We say that the function $u(\cdot)$ is a weak solution of (2.1) if $u \in L^p(\tau, T; L^p(\Omega)) \cap L^2(\tau, T; V) \cap C([\tau, T]; H)$, $du/dt \in L^2(\tau, T; V') + L^q(\tau, T; L^q(\Omega))$, $u(\tau) = u_\tau$, and

$$\int_\tau^T \left\langle \frac{du}{dt}, \xi \right\rangle dt + \int_\tau^T \int_\Omega (\nabla(au), \nabla \xi) dx dt + \int_\tau^T \int_\Omega (f(t, u), \xi) dx dt = \int_\tau^T \int_\Omega (h, \xi) dx dt, \quad (2.4)$$

for all $\xi \in L^p(\tau, T; L^p(\Omega)) \cap L^2(\tau, T; V)$, where $\langle \cdot, \cdot \rangle$ denotes pairing in the space $V' + L^q(\Omega)$, and $(\nabla u, \nabla v) = \sum_{i=1}^d (\nabla u^i, \nabla v^i)$.

Under conditions (2.2), it is known [17, page 284] that for any $u_\tau \in H$ there exists at least one weak solution $u = u(t, x)$ of (2.1), and also that the function $t \mapsto \|u(t)\|^2$ is absolutely continuous on $[\tau, T]$ and $(d/dt)\|u(t)\|^2 = 2\langle du/dt, u \rangle$ for a.a. $t \in (\tau, T)$.

Denote $r = (r_1, \dots, r_d)$, $r_i = \max\{1; N(1/q_i - 1/2)\}$. Any weak solution satisfies $(du/dt) \in L^q(\tau, T; H^{-r}(\Omega))$ and

$$L^q(0, T; H^{-r}(\Omega)) = L^{q_1}(0, T; H^{-r_1}(\Omega)) \times \dots \times L^{q_d}(0, T; H^{-r_d}(\Omega)). \quad (2.5)$$

If, additionally, we assume that that $f(t, u)$ is continuously differentiable with respect to u for any $t \in [\tau, T]$, $u \in \mathbb{R}^d$, and

$$(f_u(t, u)w, w) \geq -C_3(t)|w|^2, \quad \forall w, u \in \mathbb{R}^d, \quad (2.6)$$

where $C_3(\cdot) \in L^1(\tau, T)$, $C_3(t) \geq 0$, the weak solution of (2.1) is unique. Here, f_u denotes the Jacobian matrix of f .

We consider also the following assumption: there exists $R_0 > 0$ such that

$$f^i(t, u) \geq f^i(t, v), \quad (2.7)$$

for any $t \in [\tau, T]$ and any $u, v \in \mathbb{R}^d$ such that $u^i = v^i$ and $u^j \leq v^j$ if $j \neq i$, and $|u|, |v| \leq R_0$, which means that the systems is cooperative in the ball with radius R_0 centered at 0.

Consider the two problems:

$$\begin{aligned} \frac{\partial u}{\partial t} - a\Delta u + f_1(t, u) &= h_1(t, x), \quad (t, x) \in (\tau, T) \times \Omega, \\ u|_{x \in \partial\Omega} &= 0, \\ u|_{t=\tau} &= u_\tau(x), \end{aligned} \tag{2.8}$$

$$\begin{aligned} \frac{\partial u}{\partial t} - a\Delta u + f_2(t, u) &= h_2(t, x), \quad (t, x) \in (\tau, T) \times \Omega, \\ u|_{x \in \partial\Omega} &= 0, \\ u|_{t=\tau} &= u_\tau(x), \end{aligned} \tag{2.9}$$

where f_j are jointly continuous on $[\tau, T] \times \mathbb{R}^d$. Among conditions (2.2) and (2.6)-(2.7), we shall consider the following:

$$h_1(t, x) \leq h_2(t, x), \quad \text{for a.a. } (t, x), f_1^i(t, u) \geq f_2^i(t, u), \quad \forall t, u. \tag{2.10}$$

Lemma 2.1. *If f_j satisfy (2.2) and (2.10), then the constants p_i have to be the same for f_1 and f_2 .*

Proof. Denote by p_i^j, α^j, C_1^j , and C_2^j the constants corresponding to f_j in (2.2). By contradiction let, for example, $p_1^2 > p_1^1$. Take the sequence $u_n = (u_n^1, 0, \dots, 0)$, where $u_n^1 \rightarrow +\infty$ as $n \rightarrow \infty$. Then by (2.2), (2.10), and Young's inequality, we have

$$\begin{aligned} \alpha^2 |u_n^1|^{p_1^2} - C_2^2 &\leq (f_2(t, u_n), u_n) \leq (f_1(t, u_n), u_n) \\ &= f_1^1(t, u_n) u_n^1 \leq \left(C_1^1 \left(1 + |u_n^1|^{p_1^1} \right) \right)^{(p_1^1-1)/p_1^1} u_n^1 \leq C \left(1 + |u_n^1|^{p_1^1} \right). \end{aligned} \tag{2.11}$$

But $p_1^2 > p_1^1$ implies the existence of n such that $\alpha^2 |u_n^1|^{p_1^2} - C_2^2 > C(1 + |u_n^1|^{p_1^1})$, which is a contradiction. Hence, $p_1^2 \leq p_1^1$.

Conversely, let $p_1^2 < p_1^1$. Then we take $u_n = (u_n^1, 0, \dots, 0)$ with $u_n^1 \rightarrow -\infty$ as $n \rightarrow \infty$ so that

$$\begin{aligned} \alpha^1 |u_n^1|^{p_1^1} - C_1^1 &\leq (f_1(t, u_n), u_n) \leq (f_2(t, u_n), u_n) \\ &= f_2^1(t, u_n) u_n^1 \leq \left(C_1^2 \left(1 + |u_n^1|^{p_1^2} \right) \right)^{(p_1^2-1)/p_1^2} u_n^1 \leq C \left(1 + |u_n^1|^{p_1^2} \right). \end{aligned} \tag{2.12}$$

As before, we obtain a contradiction, so $p_1^2 = p_1^1$.

Repeating similar arguments for the other p_i^j , we obtain that $p_i^1 = p_i^2$ for $i = 1, \dots, d$. $\square$

We recall [15] that under conditions (2.2) any solution $u(\cdot)$ of (2.8) satisfies the inequality:

$$\|u(t)\|^2 + 2\beta \int_s^t \|\nabla u(\tau)\|^2 d\tau + \alpha \sum_{i=1}^d \int_s^t \|u^i(r)\|_{L^{p_i}(\Omega)}^{p_i} dr \leq \|u(s)\|^2 + C \int_s^t (\|h_1(r)\|^2 + 1) dr, \quad (2.13)$$

for some constant $C > 0$. Of course, the same is valid for any solution of (2.9). From (2.13), for any $T > \tau$ we obtain

$$\|u(t)\|^2 \leq \|u_\tau\|^2 + C \int_\tau^t (\|h_1(r)\|^2 + 1) dr = K^2(\|u_\tau\|, \tau, T) \quad \forall \tau \leq t \leq T. \quad (2.14)$$

We shall denote by $u_1(t)$ the solution of (2.8) corresponding to the initial data u_τ^1 and by $u_2(t)$ the solution of (2.9) corresponding to the initial data u_τ^2 . Also, we take $u^+ = \max\{u, 0\}$ for $u \in \mathbb{R}$.

We obtain the following comparison result.

Theorem 2.2. Assume that f_j, h_j satisfy (2.2), (2.6), and (2.10). If $u_\tau^1 \leq u_\tau^2$ and we suppose that f_2 satisfies (2.7) with $R_0^2 \geq 2 \max\{K^2(\|u_\tau^1\|, \tau, T), K^2(\|u_\tau^2\|, \tau, T)\}$, where $K(\|u_\tau^j\|, \tau, T)$ is taken from (2.14), we have $u_1(t) \leq u_2(t)$, for all $t \in [\tau, T]$.

Remark 2.3. The results remain valid if, instead, f_1 satisfies (2.7) with $R_0^2 \geq K^2(\|u_\tau^1\|, T)$.

Remark 2.4. If f_2 satisfies (2.7) for an arbitrary $R_0 > 0$ (i.e., in the whole space $\mathbb{R}^d$), then the result is true for any initial data $u_\tau^1 \leq u_\tau^2$.

Proof. Let $g_2(t, u) = f_2(t, u) + C_3(t)u$. The function $g_2(t, \cdot)$ satisfies (2.6) with $C_3 \equiv 0$. For any $u_1, u_2 \in \mathbb{R}^d$ define $v_2(u_1, u_2)$ by

$$v_2^i = \begin{cases} u_2^i, & \text{if } u_1^i \geq u_2^i, \\ u_1^i, & \text{if } u_1^i < u_2^i. \end{cases} \quad (2.15)$$

Note that $u_1 - v_2 = (u_1 - u_2)^+$ and $v_2 - u_2 = -(u_2 - u_1)^+$, so $v_2^i \leq u_2^i$ for all i . For the function $(u_1 - u_2)^+$, we can obtain by (2.10) and the mean value theorem that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(u_1 - u_2)^+\|^2 + a \|(u_1 - u_2)^+\|_V^2 \\ & \leq - \int_\Omega (f_1(t, u_1) - f_2(t, u_2), (u_1 - u_2)^+) dx \\ & \leq - \int_\Omega (f_2(t, u_1) - f_2(t, v_2), (u_1 - u_2)^+) dx - \int_\Omega (f_2(t, v_2) - f_2(t, u_2), (u_1 - u_2)^+) dx \end{aligned}$$

$$\begin{aligned}
&= - \int_{\Omega} (g_{2u}(t, v(t, x, u_1, u_2))(u_1 - v_2), (u_1 - u_2)^+) dx + C_3(t) \|(u_1 - u_2)^+\|^2 \\
&\quad - \int_{\Omega} (f_2(t, v_2) - f_2(t, u_2), (u_1 - u_2)^+) dx \\
&= - \int_{\Omega} (g_{2u}(t, v(t, x, u_1, u_2))(u_1 - u_2)^+, (u_1 - u_2)^+) dx + C_3(t) \|(u_1 - u_2)^+\|^2 \\
&\quad - \int_{\Omega} (f_2(t, v_2) - f_2(t, u_2), (u_1 - u_2)^+) dx \\
&\leq C_3(t) \|(u_1 - u_2)^+\|^2 - \int_{\Omega} (f_2(t, v_2) - f_2(t, u_2), (u_1 - u_2)^+) dx.
\end{aligned} \tag{2.16}$$

For all t , we have

$$(f_2(t, v_2) - f_2(t, u_2), (u_1 - u_2)^+) = \sum_{i \in J} (f_2^i(t, v_2) - f_2^i(t, u_2)) (u_1^i - u_2^i)^+, \tag{2.17}$$

where $u_1^i - u_2^i > 0$, for $i \in J$, and $u_1^i - u_2^i \leq 0$ if $i \notin J$. For any $i \in J$, we have that $v_2^i = u_2^i$, and then by $v_2 \leq u_2$, $|v_2|^2 \leq |u_1|^2 + |u_2|^2$, (2.14), and (2.7), we get

$$f_2^i(t, v_2) - f_2^i(t, u_2) \geq 0. \tag{2.18}$$

By Gronwall's lemma, we get

$$\|(u_1(t) - u_2(t))^+\|^2 \leq \left\| (u_1^1 - u_2^1)^+ \right\|^2 e^{\int_{\tau}^t 2C_3(s) ds} = 0. \tag{2.19}$$

Thus $\|(u_1 - u_2)^+\| = 0$, which means that $u_1^i(x, t) - u_2^i(x, t) \leq 0$, for a.a. $x \in \Omega$, and all $i \in \{1, \dots, d\}$, $t \in [\tau, T]$. $\square$

Remark 2.5. In the scalar case, that is, $d = 1$, condition (2.7) is trivially satisfied.

When condition (2.6) fails to be true, we will obtain a weak comparison principle. Define a sequence of smooth functions $\varphi_k : \mathbb{R}_+ \rightarrow [0, 1]$ satisfying

$$\varphi_k(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq k, \\ 0 \leq \varphi_k(s) \leq 1, & \text{if } k \leq s \leq k+1, \\ 0, & \text{if } s \geq k+1. \end{cases} \tag{2.20}$$

For every $k \geq 1$ we put $f_k^i(t, u) = \varphi_k(|u|)f^i(t, u) + (1 - \varphi_k(|u|))g^i(u)$, where $g^i(u) = |u^i|^{p_i-2}u^i$. Then $f_k \in \mathbb{C}([\tau, T] \times \mathbb{R}^d; \mathbb{R}^d)$ and for any $A > 0$,

$$\sup_{t \in [\tau, T]} \sup_{|u| \leq A} |f_k(t, u) - f(t, u)| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \quad (2.21)$$

Let $\rho_\epsilon : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be a mollifier, that is, $\rho_\epsilon \in \mathbb{C}_0^\infty(\mathbb{R}^d; \mathbb{R})$, $\text{supp } \rho_\epsilon \subset B_\epsilon = \{x \in \mathbb{R}^d : |x| < \epsilon\}$, $\int_{\mathbb{R}^d} \rho_\epsilon(s) ds = 1$ and $\rho_\epsilon(s) \geq 0$ for all $s \in \mathbb{R}^d$. We define the functions

$$f_k^\epsilon(t, u) = \int_{\mathbb{R}^d} \rho_\epsilon(s) f_k(t, u - s) ds. \quad (2.22)$$

Since for any $k \geq 1$ f_k is uniformly continuous on $[0, T] \times [-k-1, k+1]$, there exist $\epsilon_k \in (0, 1)$ such that for all u satisfying $|u| \leq k$, and for all s for which $|u - s| < \epsilon_k$ we have

$$\sup_{t \in [\tau, T]} |f_k(t, u) - f_k(t, s)| \leq \frac{1}{k}. \quad (2.23)$$

We put $f^k(t, u) = f_k^{\epsilon_k}(t, u)$. Then $f^k(t, \cdot) \in \mathbb{C}^\infty(\mathbb{R}^d; \mathbb{R}^d)$, for all $t \in [\tau, T]$, $k \geq 1$.

For further arguments we need the following technical result [16, Lemma 2].

Lemma 2.6. *Let f satisfy (2.2). For all $k \geq 1$, the following statements hold:*

$$\sup_{t \in [\tau, T]} \sup_{|u| \leq A} |f^k(t, u) - f(t, u)| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty, \quad \forall A > 0, \quad (2.24)$$

$$\sum_{i=1}^d |f^{ki}(t, u)|^{p_i/(p_i-1)} \leq D_1 \left(1 + \sum_{i=1}^d |u^i|^{p_i} \right), \quad (f^k(t, u), u) \geq \gamma \sum_{i=1}^d |u^i|^{p_i} - D_2, \quad (2.25)$$

$$(f_u^k(t, u)w, w) \geq -D_3(k)|w|^2, \quad \forall u, w, \quad (2.26)$$

where $D_3(k)$ is a nonnegative number, and the positive constants $D_1, D_2 \geq C_2, \gamma$ do not depend on k .

Consider first the scalar case.

Theorem 2.7. *Let $d = 1$. Assume that f_j, h_j satisfy (2.2) and (2.10). If $u_\tau^1 \leq u_\tau^2$, there exist two solutions u_1, u_2 (of (2.8) and (2.9), resp.) such that $u_1(t) \leq u_2(t)$ for all $t \in [\tau, T]$.*

Proof. For the functions f_j we take the approximations f_j^k (defined in Lemma 2.6), which satisfy (2.24)–(2.26), and consider the problems

$$\begin{aligned} \frac{\partial u}{\partial t} - a\Delta u + f_j^k(t, u) &= h_j(t, x), \quad (t, x) \in (\tau, T) \times \Omega, \\ u|_{x \in \partial\Omega} &= 0, \\ u|_{t=\tau} &= u_\tau, \end{aligned} \quad (2.27)$$

for $j = 1, 2$. Problem (2.27) has a unique solution for any initial data $u_\tau \in H$. In view of Lemma 2.1, the constant p is the same for f^1 and f^2 . We note that

$$\begin{aligned} f_{1k}(t, u) &= \varphi_k(|u|)f_1(t, u) + (1 - \varphi_k(|u|))|u|^{p-2}u \\ &\geq \varphi_k(|u|)f_2(t, u) + (1 - \varphi_k(|u|))|u|^{p-2}u = f_{2k}(t, u). \end{aligned} \quad (2.28)$$

Then, it is clear that $f_1^k(t, u) \geq f_2^k(t, u)$ for every (t, u) .

By Theorem 2.2 we know that as $u_\tau^1 \leq u_\tau^2$, we have $u_1^k(t) \leq u_2^k(t)$, for all $t \in [\tau, T]$, for the corresponding solutions of (2.27).

In view of Lemma 2.6, one can obtain in a standard way that (2.13) is satisfied for the solutions of (2.27) with a constant C not depending on k and replacing α by γ . Hence, the sequences $u_j^k(\cdot)$ are bounded in $L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^p(\tau, T; L^p(\Omega))$. It follows from (2.25) that $f_{jk}(\cdot, u_j^k(\cdot))$ are bounded in $L^q(\tau, T; L^q(\Omega))$ and also that $\{(du^k/dt)(\cdot)\}$ is bounded in $L^q(\tau, T; H^{-r}(\Omega))$, where $r_i = \max\{1; (1/2 - 1/p_i)N\}$. By the Compactness Lemma [18], we have that for some functions $u_j = u_j(t, x)$, $j = 1, 2$:

$$u_j^k \rightharpoonup u_j \quad \text{weakly star in } L^\infty(\tau, T; H), \quad (2.29)$$

$$u_j^k \rightharpoonup u_j \quad \text{in } L^2(\tau, T; H), \quad u_j^k(t) \rightharpoonup u_j(t) \quad \text{in } H \text{ for a.a. } t \in (\tau, T), \quad (2.30)$$

$$u_j^k(t, x) \rightharpoonup u_j(t, x) \quad \text{for a.a. } (t, x) \in (\tau, T) \times \Omega, \quad (2.31)$$

$$u_j^k \rightharpoonup u_j \quad \text{weakly in } L^2(\tau, T; V), \quad (2.32)$$

$$\frac{du_j^k}{dt} \rightharpoonup \frac{du_j}{dt} \quad \text{weakly in } L^q(\tau, T; H^{-r}(\Omega)), \quad (2.33)$$

$$u_j^k \rightharpoonup u_j \quad \text{weakly in } L^p(\tau, T; L^p(\Omega)). \quad (2.34)$$

Also, arguing as in [19, page 3037] we obtain

$$u_j^k(t) \rightharpoonup u_j(t) \quad \text{weakly in } H \quad \forall t \in [\tau, T]. \quad (2.35)$$

Moreover, by (2.24) and (2.31) we have $f_{jk}(t, u_j^k(t, x)) \rightarrow f_j(t, u(t, x))$ for a.a. $(t, x) \in (\tau, T) \times \Omega$ and then the boundedness of $f_{jk}(\cdot, u_j^k(\cdot))$ in $L^q(\tau, T; L^q(\Omega))$ implies that $f_{jk}(\cdot, u_j^k(\cdot))$ converges to $f(\cdot, u(\cdot))$ weakly in $L^q(\tau, T; L^q(\Omega))$ [18]. It follows that $u_1(\cdot)$, $u_2(\cdot)$ are weak solutions of (2.8) and (2.9), respectively, with $u_1(\tau) = u_\tau^1$, $u_2(\tau) = u_\tau^2$.

Moreover, one can prove that

$$u_j^k(t) \rightarrow u_j(t) \quad \text{strongly in } H \quad \forall t \in [\tau, T]. \quad (2.36)$$

Indeed, we define the functions $J_{jk}(t) = \|u_j^k(t)\|^2 - C \int_\tau^t (\|h_j(s)\|^2 + 1) ds$, $J_j(t) = \|u_j(t)\|^2 - C \int_\tau^t (\|h_j(s)\|^2 + 1) ds$, which are nonincreasing in view of (2.13). Also, from (2.30) we have $J_{jk}(t) \rightarrow J_j(t)$ for a.a. $t \in (\tau, T)$. Then one can prove that $\limsup_{k \rightarrow \infty} J_{jk}(t) \leq J_j(t)$ for all

$t \in [\tau, T]$ (see [15, page 623] for the details). Hence, $\limsup_{k \rightarrow \infty} \|u_j^k(t)\| \leq \|u_j(t)\|$. Together with (2.35) this implies (2.36) (see again [15, page 623] for more details).

Hence, passing to the limit we obtain

$$u_1(t) \leq u_2(t), \quad \forall t \in [\tau, T]. \quad (2.37)$$

□

Further, let us prove the general case for an arbitrary $d \in \mathbb{N}$.

Theorem 2.8. *Assume that f_j, h_j satisfy (2.2) and (2.10). Also, suppose that either f_1 or f_2 satisfies (2.7) for an arbitrary $R_0 > 0$. If $u_\tau^1 \leq u_\tau^2$, there exist two solutions u_1, u_2 (of (2.8) and (2.9), resp.) such that $u_1(t) \leq u_2(t)$, for all $t \in [\tau, T]$.*

Proof. Let f_1 be the function which satisfies (2.7). We take the approximations f_1^k, f_2^k (defined in Lemma 2.6), which satisfy (2.24)–(2.26). Then, we consider problems (2.27).

In view of Lemma 2.1, the constants p_i are the same for f^1 and f^2 . We note that

$$\begin{aligned} f_{1k}^i(t, u) &= \varphi_k(|u|)f_1^i(t, u) + (1 - \varphi_k(|u|))|u^i|^{p_i-2}u^i \\ &\geq \varphi_k(|u|)f_2^i(t, u) + (1 - \varphi_k(|u|))|u^i|^{p_i-2}u^i = f_{2k}^i(t, u). \end{aligned} \quad (2.38)$$

Then, it is clear that $f_1^k(t, u) \geq f_2^k(t, u)$ for every (t, u) .

Using Lemma 2.6 it is standard to obtain estimate (2.14) with a constant C not depending on k . Hence, the solutions $u_j^k(\cdot)$ of (2.27) satisfy

$$\|u_j^k(t)\|^2 \leq \|u_\tau^j\|^2 + C \int_\tau^t (\|h_j(r)\|^2 + 1) dr = K^2(\|u_\tau^j\|, \tau, T). \quad (2.39)$$

We note that

$$f_{1k}(t, u) = f_1(t, u), \quad (2.40)$$

if $|u| \leq k$, since in such a case $\varphi_k(|u|) = 1$. Hence, if $k^2 \geq 2 \max\{K^2(\|u_\tau^1\|, \tau, T), K^2(\|u_\tau^2\|, \tau, T)\}$, the functions f_{1k} satisfy condition (2.7) with $R_0 = k$. Therefore, for any $t \in [\tau, T]$ and any $u, v \in \mathbb{R}^d$ such that $u^i = v^i$ and $u^j \leq v^j$ if $j \neq i$, and $|u|, |v| \leq k - 1$, we have

$$f_1^k(t, u) = \int_{\mathbb{R}^d} \rho_{\varepsilon_k}(s) f_{1k}(t, u - s) ds \geq \int_{\mathbb{R}^d} \rho_{\varepsilon_k}(s) f_{1k}(t, v - s) ds = f_2^k(t, u). \quad (2.41)$$

Thus, if $(k - 1)^2 \geq 2 \max\{K^2(\|u_\tau^1\|, T), K^2(\|u_\tau^2\|, T)\}$, the functions f_1^k satisfy condition (2.7) with $R_0 = k - 1$.

By Theorem 2.2 we know that as $u_\tau^1 \leq u_\tau^2$, we have $u_1^k(t) \leq u_2^k(t)$, for all $t \in [\tau, T]$, $k \geq 1 + (2 \max\{K^2(\|u_\tau^1\|, T), K^2(\|u_\tau^2\|, T)\})^{1/2}$, for the corresponding solutions of (2.27).

Repeating the same proof of Theorem 2.7, we obtain that the sequences u_1^k, u_2^k converge (up to a subsequence) in the sense of (2.29)–(2.36) to the solutions u_1, u_2 of problems (2.8) and (2.9), respectively. Also, it holds

$$u_1(t) \leq u_2(t), \quad \forall t \in [\tau, T]. \quad (2.42)$$

□

In the applications we need to generalize this theorem to the case where the constant α can be negative. We shall do this when f_1, f_2 have sublinear growth (i.e., $p_i = 2$ for all i). Consider for (2.1) the following conditions:

$$\begin{aligned} |f(t, u)| &\leq C_1(1 + |u|), \\ (f(t, u), u) &\geq \alpha|u|^2 - C_2, \end{aligned} \quad (2.43)$$

where $\alpha \in \mathbb{R}$, and $C_1, C_2 > 0$.

Let f_1, f_2 satisfy (2.43) with constants α^j, C_1^j, C_2^j , $j = 1, 2$. Then if $\min\{\alpha^1, \alpha^2\} \leq 0$, we make in (2.1) the change of variable $v = e^{-\beta t}u$, where $\beta > -\min\{\alpha^1, \alpha^2\}$. Hence, multiplying (2.8) and (2.9) by $e^{-\beta t}$ we have

$$\begin{aligned} \frac{\partial v}{\partial t} - a\Delta v + e^{-\beta t}f_1(t, e^{\beta t}v) + \beta v &= e^{-\beta t}h_1(t, x), \quad (t, x) \in (\tau, T) \times \Omega, \\ v|_{x \in \partial\Omega} &= 0, \end{aligned} \quad (2.44)$$

$$v|_{t=\tau} = e^{-\beta\tau}u_\tau(x),$$

$$\begin{aligned} \frac{\partial v}{\partial t} - a\Delta v + e^{-\beta t}f_2(t, e^{\beta t}v) + \beta v &= e^{-\beta t}h_2(t, x), \quad (t, x) \in (\tau, T) \times \Omega, \\ v|_{x \in \partial\Omega} &= 0, \end{aligned} \quad (2.45)$$

$$v|_{t=\tau} = e^{-\beta\tau}u_\tau(x).$$

It is easy to check that if $v(t)$ is a weak solution of (2.44), then $u(t) = e^{\beta t}v(t)$ is a weak solution of (2.8) (and the same is true, of course, for (2.45) and (2.9)). Conversely, if $u(t)$ is a weak solution of (2.8), then $v(t) = e^{-\beta t}u(t)$ is a weak solution of (2.44).

The functions $\tilde{f}_j(t, v) = e^{-\beta t}f_j(t, e^{\beta t}v) + \beta v$ satisfy (2.2) with $p_i = 2$ for all i . Indeed,

$$\begin{aligned} |\tilde{f}_j(t, v)| &\leq e^{-\beta t}C_1^j(1 + e^{\beta t}|v|) + \beta|v| \leq \tilde{C}_1^j(1 + |v|), \\ (\tilde{f}_j(t, v), v) &\geq e^{-2\beta t}(f_j(t, e^{\beta t}v), e^{\beta t}v) + \beta|v|^2 \\ &\geq (\alpha^j + \beta)|v|^2 - C_2^j, \end{aligned} \quad (2.46)$$

where $\alpha^j + \beta > 0$.

Then, we obtain the following.

Theorem 2.9. Assume that f_j, h_j satisfy (2.43) and (2.10). Also, suppose that either f_1 or f_2 satisfies (2.7) for an arbitrary $R_0 > 0$. If $u_\tau^1 \leq u_\tau^2$, there exist two solutions u_1, u_2 (of (2.8) and (2.9), resp.) such that $u_1(t) \leq u_2(t)$, for all $t \in [\tau, T]$.

Proof. We consider problems (2.44) and (2.45). In view of (2.46) $\tilde{f}_j(t, v) = e^{-\beta t} f_j(t, e^{\beta t} v) + \beta v$ satisfy (2.2). Also, defining $\tilde{h}_j(t, x) = e^{-\beta t} h_j(t, x)$ it is clear that (2.10) holds. Finally, if, for example, f_1 satisfies (2.7) for any $R_0 > 0$, then it is obvious that for $\tilde{f}_1$ is true as well.

Hence, by Theorem 2.8 there exist two solutions v_1, v_2 (of (2.44) and (2.45), resp.), with $v_j(\tau) = e^{-\beta \tau} u_\tau^j$ such that $v_1(t) \leq v_2(t)$ for all $t \in [\tau, T]$. Thus

$$u_1(t) = e^{\beta t} v_1(t) \leq e^{\beta t} v_2(t) = u_2(t), \quad \text{for } t \in [\tau, T], \quad (2.47)$$

and u_1, u_2 are solutions of (2.8) and (2.9), respectively such that $u_j(\tau) = u_\tau^j$. $\square$

Remark 2.10. If f_j satisfy (2.6), then the solutions u_1, u_2 given in Theorem 2.9 are unique for the corresponding initial data.

3. Comparison for Positive Solutions

Denote $\mathbb{R}_+^d = \{u \in \mathbb{R}^d : u^i \geq 0\}$. Let us consider the previous results in the case where the solutions have to be positive. Consider now the following conditions.

The matrix a is diagonal, (3.1)

$$h^i(t, x) - f^i(t, u^1, \dots, u^{i-1}, 0, u^{i+1}, \dots, u^d) \geq 0, \quad (3.2)$$

for all i , a.e. $(t, x) \in (\tau, T) \times \Omega$ and $u^j \geq 0$ if $j \neq i$. Obviously, in the scalar case these conditions just mean that

$$h(t, x) - f(t, 0) \geq 0, \quad (3.3)$$

for a.e. $(t, x) \in (\tau, T) \times \Omega$.

It is well known (see [16, Lemma 5] for a detailed proof) that if we assume conditions (2.2) only for $u \in \mathbb{R}_+^d$, and also (2.6) and (3.1)-(3.2), then for any $u_\tau \geq 0$ there exists a unique weak solution $u(\cdot)$ of (2.1). Moreover, $u(\cdot)$ is such that $u(t) \geq 0$ for all $t \in [\tau, T]$.

On the other hand, if we assume these conditions except (2.6), then there exists at least one weak solution $u(\cdot)$ of (2.1) such that $u(t) \geq 0$ for all $t \in [\tau, T]$ [16, Theorem 4]. Moreover, we can prove the following.

Lemma 3.1. Assume conditions (2.2), (2.6) only for $u \in \mathbb{R}_+^d$, and also (3.1)-(3.2). Then there exists a weak solution $u(\cdot)$ of (2.1), which is unique in the class of solutions satisfying $u(t) \geq 0$ for all $t \in [\tau, T]$.

Proof. Let u_1, u_2 be two solutions with $u_i(\tau) = u_\tau, i = 1, 2$ such that $u_i(t) \geq 0$ for all t . Denote $w(t) = u_1(t) - u_2(t)$. Then in a standard way by the mean value theorem, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w(t)\|^2 &\leq - \int_{\Omega} (f(t, u_1(t, x)) - f(t, u_2(t, x)), w(t, x)) dx \\ &= - \int_{\Omega} (f_u(t, v(t, x, u_1, u_2)) w(t, x), w(t, x)) dx \\ &\leq C_3(t) \|w(t)\|^2, \end{aligned} \quad (3.4)$$

where $v(t, x, u_1, u_2) \in L(u_1(t, x), u_2(t, x)) = \{\alpha u_1(t, x) + (1 - \alpha)u_2(t, x) : \alpha \in [0, 1]\}$ so that $v(t, x, u_1, u_2) \geq 0$. The uniqueness follows from Gronwall's lemma $\square$

We prove also a result, which is similar to Lemma 2.1. Denote by p_i^j, α^j, C_1^j , and C_2^j the constants corresponding to f_j in (2.2) for problems (2.8) and (2.9), respectively. Arguing as in the proof of Lemma 2.1 we obtain the following lemma.

Lemma 3.2. *If f_j satisfy (2.2) and (2.10) for $u \in \mathbb{R}_+^d$, then $p_i^1 \geq p_i^2$ for all i .*

Theorem 3.3. *Let f_j, h_j satisfy (2.6) and (3.1)-(3.2). Assume that f_j, h_j satisfy (2.2) and (2.10) for $u \in \mathbb{R}_+^d$. If $0 \leq u_\tau^1 \leq u_\tau^2$ and one supposes that f_2 satisfies (2.7) for $u \in \mathbb{R}_+^d$ with $R_0^2 \geq 2 \max\{K^2(\|u_\tau^1\|, \tau, T), K^2(\|u_\tau^2\|, \tau, T)\}$, where $K(\|u_\tau^j\|, \tau, T)$ is taken from (2.14), one has $0 \leq u_1(t) \leq u_2(t)$ for all $t \in [\tau, T]$, where $u_1(\cdot), u_2(\cdot)$ are the solutions corresponding to u_τ^1 and u_τ^2 , respectively.*

Proof. As the solutions $u_1(\cdot), u_2(\cdot)$ corresponding to u_τ^1 and u_τ^2 are nonnegative, repeating exactly the same steps of the proof of Theorem 2.2 we obtain the desired result. $\square$

Remark 3.4. The results remain valid if, instead, f_1 satisfies (2.7) with the same R_0 .

Remark 3.5. If f_2 satisfies (2.7) for an arbitrary $R_0 > 0$ (i.e., in the whole space $\mathbb{R}^d$), then the result is true for any initial data $0 \leq u_\tau^1 \leq u_\tau^2$.

We shall need also the following modification of Theorem 3.3.

Theorem 3.6. *Let f_j, h_j satisfy (2.6) and (3.1)-(3.2). Assume that f_j, h_j satisfy (2.2) and (2.10) for $u \in \mathbb{R}_+^d$. Let $0 \leq u_\tau^1 \leq u_\tau^2$. One supposes that f_2 satisfies*

$$f_2^i(t, u) \geq f_2^i(t, v) - \varepsilon, \quad (3.5)$$

for any $t \in [\tau, T]$ and any $u, v \in \mathbb{R}_+^d$ such that $u^i = v^i$ and $u^j \leq v^j$ if $j \neq i$, and $|u|, |v| \leq R_0$ with $R_0^2 \geq 2 \max\{K^2(\|u_\tau^1\|, \tau, T), K^2(\|u_\tau^2\|, \tau, T)\}$, where $K(\|u_\tau^j\|, \tau, T)$ is taken from (2.14).

Then there exists a constant $C(\tau, T)$ such that

$$\|(u_1(t) - u_2(t))^+\| \leq C(\tau, T)\varepsilon, \quad \forall t \in [\tau, T], \quad (3.6)$$

where $u_1(\cdot), u_2(\cdot)$ are the solutions corresponding to u_τ^1 and u_τ^2 , respectively.

Proof. Arguing as in the proof of Theorem 2.2 we obtain the inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(u_1 - u_2)^+\|^2 + a \|(u_1 - u_2)^+\|_V^2 \\ \leq C_3(t) \|(u_1 - u_2)^+\|^2 - \int_{\Omega} (f_2(t, v_2) - f_2(t, u_2), (u_1 - u_2)^+) dx, \end{aligned} \quad (3.7)$$

where v_2 is defined in (2.15).

Using (2.17), $v_2 \leq u_2$, $|v_2|^2 \leq |u_1|^2 + |u_2|^2$, (2.14), and (3.5), we get

$$f_2^i(t, v_2) - f_2^i(t, u_2) \geq -\varepsilon. \quad (3.8)$$

Thus

$$\begin{aligned} \frac{d}{dt} \|(u_1 - u_2)^+\|^2 &\leq 2C_3(t) \|(u_1 - u_2)^+\|^2 + 2\varepsilon \int_{\Omega} \sum_{i \in J} (u_1^i - u_2^i)^+ dx \\ &\leq (2C_3(t) + 1) \|(u_1 - u_2)^+\|^2 + K\varepsilon^2, \end{aligned} \quad (3.9)$$

for some constant $K > 0$. By Gronwall's lemma, we get

$$\begin{aligned} \|(u_1(t) - u_2(t))^+\|^2 &\leq \left\| (u_{\tau}^1 - u_{\tau}^2)^+ \right\|^2 e^{\int_{\tau}^t (2C_3(s)+1)ds} + K\varepsilon^2 \int_{\tau}^t e^{\int_r^t (2C_3(s)+1)ds} dr \\ &\leq C^2(\tau, T) \varepsilon^2. \end{aligned} \quad (3.10)$$

□

Let us consider now the multivalued case. We will obtain first some auxiliary statements.

We shall define suitable approximations. For any $n \geq 1$ we put $f_n^i(t, u) = \varphi_n(|u|) f^i(t, u) + (1 - \varphi_n(|u|)) g^i(t, u)$, where $g^i(t, u) = |u^i|^{p_i-2} u^i + f^i(t, 0, \dots, 0)$, and φ_n was defined in (2.20). Then $f_n \in C([\tau, T] \times \mathbb{R}^d; \mathbb{R}^d)$ and for any $A > 0$,

$$\sup_{t \in [\tau, T]} \sup_{|u| \leq A} |f_n(t, u) - f(t, u)| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (3.11)$$

We will check first that f_n satisfy conditions (2.2) for $u \in \mathbb{R}_+^d$, where the constants do not depend on n .

Lemma 3.7. *Let f satisfy (2.2) for $u \in \mathbb{R}_+^d$. For all $n \geq 1$ one has*

$$\sum_{i=1}^d |f_n^i(t, u)|^{p_i/(p_i-1)} \leq D_1 \left(1 + \sum_{i=1}^d |u^i|^{p_i} \right), \quad (f_n(t, u), u) \geq \gamma \sum_{i=1}^d |u^i|^{p_i} - D_2, \quad (3.12)$$

for $u \in \mathbb{R}_+^d$, where the positive constants D_1 , D_2 , and γ do not depend on n .

If $|u| > n + 1$, then for any $w \in \mathbb{R}^d$ one has

$$(f_{nu}(t, u)w, w) \geq 0. \quad (3.13)$$

Moreover, if f, h satisfy (3.2), then f_n, h also satisfies this condition.

Proof. In view of (2.2) we get

$$\begin{aligned} (f_n(t, u), u) &= \varphi_n(|u|)(f(t, u), u) + (1 - \varphi_n(|u|))(g(u), u) \\ &\geq \varphi_n(|u|) \left(\alpha \sum_{i=1}^d |u^i|^{p_i} - C_2 \right) + (1 - \varphi_n(|u|)) \sum_{i=1}^d |u^i|^{p_i} \\ &\quad + (1 - \varphi_n(|u|)) \sum_{i=1}^d f^i(t, 0, \dots, 0) u^i \\ &\geq \varphi_n(|u|) \alpha \sum_{i=1}^d |u^i|^{p_i} - C_2 + (1 - \varphi_n(|u|)) \frac{1}{2} \sum_{i=1}^d |u^i|^{p_i} \\ &\quad - K_1 (1 - \varphi_n(|u|)) \sum_{i=1}^d |f^i(t, 0, \dots, 0)|^{p_i/(p_i-1)} \\ &\geq \tilde{\alpha} \sum_{i=1}^d |u^i|^{p_i} - C_2 - K_1 C_1, \end{aligned} \quad (3.14)$$

where $\tilde{\alpha} = \min\{1/2, \alpha\}$, for some constant $K_1 > 0$. Also,

$$\begin{aligned} \sum_{i=1}^d |f_n^i(t, u)|^{p_i/(p_i-1)} &\leq K_2 \left(\sum_{i=1}^d |f^i(t, u)|^{p_i/(p_i-1)} + \sum_{i=1}^d |g^i(u)|^{p_i/(p_i-1)} \right) \\ &\leq K_3 \left(C_1 \left(1 + \sum_{i=1}^d |u^i|^{p_i} \right) + \sum_{i=1}^d |u^i|^{p_i} + \sum_{i=1}^d |f^i(t, 0, \dots, 0)|^{p_i/(p_i-1)} \right) \\ &\leq K_4 \left(\sum_{i=1}^d |u^i|^{p_i} + 1 \right), \end{aligned} \quad (3.15)$$

for some constant $K_4 > 0$. Thus, for $D_1 = K_4$, $D_2 = C_2 + K_1 C_1$, $\tilde{\alpha} = \min\{1/2, \alpha\}$ we have (3.12) for the functions f_n .

Moreover, if $|u| > n + 1$, then for any $w \in \mathbb{R}^d$,

$$(f_{nu}(t, u)w, w) = (g_u(t, u)w, w) = \sum_{i=1}^d (p_i - 1) |u^i|^{p_i-2} w_i^2 \geq 0. \quad (3.16)$$

Finally, if (3.2) is satisfied, then

$$h^i(t, x) - f_n^i(t, u) = \varphi_n(|u|) \left(h^i(t, x) - f^i(t, u) \right) + (1 - \varphi_n(|u|)) \left(h^i(t, x) - f^i(t, 0, \dots, 0) \right) \geq 0, \quad (3.17)$$

for all i , a.e. $(t, x) \in (\tau, T) \times \Omega$ and u such that $u^i = 0$ and $u^j \geq 0$ if $j \neq i$. $\square$

Let $2 \leq q_i \leq p_i$, $i = 1, \dots, d$. We define also the following approximations $\tilde{f}_n^i(t, u) = \varphi_n(|u|) f^i(t, u) + (1 - \varphi_n(|u|)) \tilde{g}^i(t, u)$, where $\tilde{g}^i(t, u) = |u^i|^{p_i-2} u^i + |u^i|^{q_i-2} u^i + f^i(t, 0, \dots, 0)$. Then (3.11) holds. We check that $\tilde{f}_n$ satisfy conditions (2.2) for $u \in \mathbb{R}_+^d$, where the constants do not depend on n .

Lemma 3.8. *Let f satisfy (2.2) for $u \in \mathbb{R}_+^d$. For all $n \geq 1$ one has*

$$\sum_{i=1}^d \left| \tilde{f}_n^i(t, u) \right|^{p_i/(p_i-1)} \leq D_1 \left(1 + \sum_{i=1}^d |u^i|^{p_i} \right), \quad (\tilde{f}_n(t, u), u) \geq \gamma \sum_{i=1}^d |u^i|^{p_i} - D_2, \quad (3.18)$$

for $u \in \mathbb{R}_+^d$, where the positive constants D_1 , D_2 , and γ do not depend on n .

If $|u| > n + 1$, then for any $w \in \mathbb{R}^d$ one has

$$(\tilde{f}_{nu}(t, u)w, w) \geq 0. \quad (3.19)$$

Moreover, if f, h satisfy (3.2), then $\tilde{f}_n, h$ also satisfy this condition.

Proof. In view of (3.12), we have

$$\begin{aligned} (\tilde{f}_n(t, u), u) &= (f_n(t, u), u) + (1 - \varphi_n(|u|)) \sum_{i=1}^d |u^i|^{q_i} \geq \gamma \sum_{i=1}^d |u^i|^{p_i} - D_2, \\ \sum_{i=1}^d \left| \tilde{f}_n^i(t, u) \right|^{p_i/(p_i-1)} &\leq K_1 \left(\sum_{i=1}^d |f_n^i(t, u)|^{p_i/(p_i-1)} + \sum_{i=1}^d |u^i|^{p_i(q_i-1)/(p_i-1)} \right) \\ &\leq K_1 \left(D_1 \left(1 + \sum_{i=1}^d |u^i|^{p_i} \right) + \sum_{i=1}^d |u^i|^{q_i} \right) \leq K_2 \left(1 + \sum_{i=1}^d |u^i|^{p_i} \right), \end{aligned} \quad (3.20)$$

where we have used that $p_i \geq q_i$ implies $p_i/(p_i-1) \leq q_i/(q_i-1)$. Finally, (3.19) and condition (3.2) are proved in the same way as in Lemma 3.7. $\square$

For every $n \geq 1$ consider the sequence $f_n^\varepsilon(t, u)$ defined by $f_n^\varepsilon(t, u) = \int_{\mathbb{R}^d} \rho_\varepsilon(s) b_n(t, u - s) ds$, where either $b_n = f_n$ or $b_n = \tilde{f}_n$, defined before. Since any b_n are uniformly continuous on $[\tau, T] \times [-k-1, k+1]$, for any $k \geq 1$, there exist $\varepsilon_{k,n} \in (0, 1)$ such that for all u satisfying $|u| \leq k$, and for all s for which $|u - s| < \varepsilon_{k,n}$ we have

$$\sup_{t \in [\tau, T]} |b_n(t, u) - b_n(t, s)| \leq \frac{1}{k}. \quad (3.21)$$

We put $f_n^k(t, u) = f_n^{\varepsilon_{k,n}}(t, u)$. Then, $f_n^k(t, \cdot) \in \mathbb{C}^\infty(\mathbb{R}^d; \mathbb{R}^d)$ for all $t \in [\tau, T]$, $k, n \geq 1$. Since for any compact subset $A \subset \mathbb{R}^d$ and any n we have $f_n^k \rightarrow b_n$ uniformly on $[\tau, T] \times A$, we obtain the existence of a sequence $\delta_{nk} \in (0, 1)$ such that $\delta_{nk} \rightarrow 0$, as $k \rightarrow \infty$, and $|f_n^{ki}(t, u) - b_n^i(t, u)| \leq \delta_{nk}$, for any i, n and any u satisfying $|u| \leq n + 2$. We define the function $F_n^k = (F_n^{k1}, \dots, F_n^{kd})$ given by

$$F_n^{ki}(t, u) = f_n^{ki}(t, u) - p\delta_{nk}, \quad (3.22)$$

where $p \in \mathbb{N}$.

Lemma 3.9. *Let f satisfy (2.2) for $u \in \mathbb{R}_+^d$. For all $n, k \geq 1$ we have*

$$\sum_{i=1}^d |F_n^{ki}(t, u)|^{p_i/(p_i-1)} \leq D_3 \left(1 + \sum_{i=1}^d |u^i|^{p_i} \right), \quad (F_n^k(t, u), u) \geq \nu \sum_{i=1}^d |u^i|^{p_i} - D_4, \quad (3.23)$$

for $u \in \mathbb{R}_+^d$, where the positive constants D_3 , D_4 , and ν do not depend neither on n nor k .

Moreover, if f, h satisfy (3.2), then F_n^k, h also satisfy this condition if $|u| \leq n + 2$.

Proof. Since f_n satisfy (3.12) and $\tilde{f}_n$ satisfies (3.18), we have

$$\begin{aligned} \sum_{i=1}^d |F_n^{ki}(t, u)|^{p_i/(p_i-1)} &\leq R_1 \left(\sum_{i=1}^d |f_n^{ki}(t, u)|^{p_i/(p_i-1)} + 1 \right) \\ &\leq R_1 \left(\sum_{i=1}^d \left(\left(\int_{\mathbb{R}^d} \rho_{\varepsilon_k}(s) ds \right)^{1/(p_i-1)} \int_{\mathbb{R}^d} \rho_{\varepsilon_k}(s) |b_n^i(t, u-s)|^{p_i/(p_i-1)} ds \right) + 1 \right) \\ &\leq R_2 \left(\sum_{i=1}^d \int_{\mathbb{R}^d} \rho_{\varepsilon_k}(s) (1 + |u^i - s^i|^{p_i}) ds + 1 \right) \\ &\leq R_3 \left(\sum_{i=1}^d \int_{\mathbb{R}^d} \rho_{\varepsilon_k}(s) (|u^i|^{p_i} + \varepsilon_k^{p_i}) ds + 1 \right) \\ &\leq R_4 \left(1 + \sum_{i=1}^d |u^i|^{p_i} \right), \end{aligned} \quad (3.24)$$

for some constant $R_4 > 0$.

On the other hand,

$$\begin{aligned}
(F_n^k(t, u), u) &= \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)(b_n(t, u-s), u-s)ds + \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s)(b_n(t, u-s), s)ds - p\delta_{nk} \sum_{i=1}^n u^i \\
&\geq \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \left(\gamma \sum_{i=1}^d |u^i - s^i|^{p_i} - D_2 \right) ds \\
&\quad - \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \sum_{i=1}^d \left(\frac{\gamma}{2D_1} |b_n^i(t, u-s)|^{p_i/(p_i-1)} + R_5 |s^i|^{p_i} \right) ds \\
&\quad - p\delta_{nk} \sum_{i=1}^n u^i \geq \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \left(\gamma \sum_{i=1}^d |u^i - s^i|^{p_i} - D_2 \right) ds \\
&\quad - \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \left(\frac{\gamma}{2} \sum_{i=1}^d |u^i - s^i|^{p_i} + R_6 \right) ds \\
&\quad - p\delta_{nk} \sum_{i=1}^n u^i \geq \frac{\gamma}{2} \int_{\mathbb{R}^d} \rho_{\epsilon_k}(s) \sum_{i=1}^d |u^i - s^i|^{p_i} ds - R_7 - p\delta_{nk} \sum_{i=1}^n u^i \geq \nu \sum_{i=1}^d |u^i|^{p_i} - R_8,
\end{aligned} \tag{3.25}$$

for some constants $\nu, R_8 > 0$, where in the last inequality we have used that for some $D > 0$,

$$|u^i|^{p_i} = |u^i - s^i + s^i|^{p_i} \leq D(|u^i - s^i|^{p_i} + |s^i|^{p_i}) \leq D(|u^i - s^i|^{p_i} + \epsilon_k^{p_i}). \tag{3.26}$$

Hence, (3.23) holds.

In view of Lemmas 3.7 and 3.8 the functions b_n, h satisfy (3.2). Hence, $|f_n^{ki}(t, u) - b_n^i(t, u)| \leq \delta_{nk}$, for any i, n and any u satisfying $|u| \leq n+2$, implies that

$$\begin{aligned}
h^i(t, x) - F_n^{ki}(t, u) &= h^i(t, x) - f_n^{ki}(t, u) + p\delta_{nk} \\
&\geq h^i(t, x) - b_n^i(t, u) \geq 0,
\end{aligned} \tag{3.27}$$

for u such that $u_i = 0, u_j \geq 0, j \neq i$, and $|u| \leq n+2$. □

Define a smooth function $\phi_n : \mathbb{R}_+ \rightarrow [0, 1]$ satisfying

$$\phi_n(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq n+1+\bar{\gamma}, \\ 0 \leq \phi_n(s) \leq 1, & \text{if } n+1+\bar{\gamma} \leq s \leq n+2, \\ 0, & \text{if } s \geq n+2, \end{cases} \tag{3.28}$$

where $0 < \bar{\gamma} < 1$ is fixed. Let $l_n^k(t, u)$ be given by

$$l_n^k(t, u) = \phi_n(|u|)F_n^k(t, u) + (1 - \phi_n(|u|))b_n(t, u). \tag{3.29}$$

Since for any compact subset $A \subset \mathbb{R}^d$ and any n we have $f_n^k \rightarrow b_n$ uniformly on $[\tau, T] \times A$ as $k \rightarrow \infty$, it is clear that

$$\sup_{t \in [\tau, T]} \sup_{|u| \leq A} |l_n^k(t, u) - b_n(t, u)| \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (3.30)$$

Lemma 3.10. *Let f satisfy (2.2) for $u \in \mathbb{R}_+^d$. For all $n, k \geq 1$ we have*

$$\sum_{i=1}^d |l_n^{ki}(t, u)|^{p_i/(p_i-1)} \leq D_5 \left(1 + \sum_{i=1}^d |u^i|^{p_i} \right), \quad (l_n^k(t, u), u) \geq \lambda \sum_{i=1}^d |u^i|^{p_i} - D_6, \quad (3.31)$$

for $u \in \mathbb{R}_+^d$, where the positive constants D_5, D_6 , and λ do not depend neither on n nor k . Also,

$$(l_{nu}^k(t, u)w, w) \geq -D_7(k, n)|w|^2, \quad \forall u, w, \quad (3.32)$$

where $D_7(k, n)$ is a nonnegative number.

Moreover, if f, h satisfy (3.2), then l_n^k, h also satisfies this condition.

Proof. The inequalities given in (3.31) are an easy consequence of (3.12), (3.18), and (3.23).

On the other hand, if u is such that $u_i = 0, u_j \geq 0, j \neq i$, then in view of Lemmas 3.7, 3.8, and 3.9 we have

$$\phi_n(|u|)(h^i(t, x) - F_n^{ki}(t, u)) \geq 0, \quad (1 - \phi_n(|u|))(h^i(t, x) - b_n^i(t, u)) \geq 0, \quad (3.33)$$

as $\phi_n(|u|) = 0$, for $|u| \geq n + 2$. Hence, $h^i(t, x) - l_n^{ki}(t, u) \geq 0$ so that condition (3.2) holds.

It is also clear that $l_n^k(t, u)$ is continuously differentiable with respect to u for any t and u . We obtain the existence of $D_7(k, n)$ such that (3.32) holds. Indeed, if $|u| \leq n + 1 + \bar{\gamma}$, then $l_n^k(t, u) = F_n^k(t, u)$ so that

$$\begin{aligned} |(l_{nu}^k(t, u)w, w)| &= |(F_{nu}^k(t, u)w, w)| = |(f_{nu}^k(t, u)w, w)| \\ &\leq |w|^2 \int_{\mathbb{R}^d} |\nabla \rho_{\epsilon_k}(u - s)| |b_n(s, u)| ds \leq R_1(k, n)|w|^2. \end{aligned} \quad (3.34)$$

If $|u| \geq n + 2$, then $l_n^k(t, u) = b_n(t, u)$ so that by (3.13), (3.19) we have $(l_{nu}^k(t, u)w, w) = (b_{nu}(t, u)w, w) \geq 0$. Finally, if $n + 1 + \bar{\gamma} < |u| < n + 2$, we have

$$\begin{aligned} (l_{nu}^k(t, u)w, w) &= \phi_n(|u|)(F_{nu}^k(t, u)w, w) + (1 - \phi_n(|u|))(b_{nu}(t, u)w, w) \\ &\quad + \sum_{i,j=1}^d \frac{\partial}{\partial u_j} \phi_n(|u|) F_n^{ki}(t, u) w_i w_j - \sum_{i,j=1}^d \frac{\partial}{\partial u_j} \phi_n(|u|) b_n^i(t, u) w_i w_j \\ &\geq -R_2(k, n)|w|^2, \end{aligned} \quad (3.35)$$

where we have used similar arguments as in (3.34), (3.13), and (3.19) and also that

$$\left| \frac{\partial}{\partial u_j} \phi_n(|u|) \right| \leq R_3(n), \quad |b_n^i(t, u)| \leq R_4(n), \quad |F_n^{ki}(t, u)| \leq R_5(k, n), \quad (3.36)$$

for any u satisfying $n + 1 + \bar{\gamma} < |u| < n + 2$, $t \in [\tau, T]$ and any i, j . $\square$

Now we are ready to obtain the weak comparison principle for positive solutions.

Theorem 3.11. *Let f_j, h_j satisfy (3.1)-(3.2). Assume that f_j, h_j satisfy (2.2) and (2.10) for $u \in \mathbb{R}_+^d$. One supposes that either f_1 or f_2 satisfies (2.7) for $u \in \mathbb{R}_+^d$ and for an arbitrary $R_0 > 0$. If $0 \leq u_\tau^1 \leq u_\tau^2$, there exist two solutions u_1, u_2 of (2.8) and (2.9), respectively, with $u_1(\tau) = u_\tau^1, u_2(\tau) = u_\tau^2$ such that $0 \leq u_1(t) \leq u_2(t)$ for all $t \in [\tau, T]$.*

Proof. For f_j let us consider the approximations $\tilde{f}_{1n}, f_{2n}$ defined before with $q_i = p_i^2 \leq p_i^1$, where the last inequality follows from Lemma 3.2. Then by Lemmas 3.7 and 3.8 we have that $\tilde{f}_{1n}, f_{2n}$ satisfy (2.2) for $u \in \mathbb{R}_+^d$ with constants not depending on n . Also, (3.13), (3.19) hold and (3.2) is satisfied in both cases. Moreover, by (2.10) for $u \in \mathbb{R}_+^d$, we have

$$\begin{aligned} \tilde{f}_{1n}^i(t, u) &= \varphi_n(|u|) f_1^i(t, u) + (1 - \varphi_n(|u|)) \left(|u^i|^{p_i^1-1} + |u^i|^{p_i^2-1} + f_1^i(t, 0, \dots, 0) \right) \\ &\geq \varphi_n(|u|) f_2^i(t, u) + (1 - \varphi_n(|u|)) \left(|u^i|^{p_i^2-1} + f_2^i(t, 0, \dots, 0) \right) = f_{2n}^i(t, u), \end{aligned} \quad (3.37)$$

if $u \in \mathbb{R}_+^d$.

As explained before, we can choose a sequence $\delta_{nk} \in (0, 1)$ such that $\delta_{nk} \rightarrow 0$, as $k \rightarrow \infty$, and $|f_{1n}^{ki}(t, u) - \tilde{f}_{1n}^i(t, u)| \leq \delta_{nk}$, $|f_{2n}^{ki}(t, u) - f_{2n}^i(t, u)| \leq \delta_{nk}$, for any i, n and any u satisfying $|u| \leq n + 2$. Further, we consider the functions

$$\begin{aligned} F_{1n}^{ki}(t, u) &= f_{1n}^{ki}(t, u) - \delta_{nk}, \\ F_{2n}^{ki}(t, u) &= f_{2n}^{ki}(t, u) - 3\delta_{nk}. \end{aligned} \quad (3.38)$$

By Lemma 3.9 we know that F_{jn}^k satisfy (2.2) for $u \in \mathbb{R}_+^d$ with constants not depending neither on k nor n , and condition (3.2) for $|u| \leq n + 2$, as well. Moreover, by (3.37) we have

$$\begin{aligned} F_{2n}^{ki}(t, u) &= f_{2n}^{ki}(t, u) - 3\delta_{nk} \\ &\leq f_{2n}^i(t, u) - 2\delta_{nk} \leq f_{1n}^i(t, u) - 2\delta_{nk} \\ &\leq f_{1n}^{ki}(t, u) - \delta_{nk} = F_{1n}^{ki}(t, u), \end{aligned} \quad (3.39)$$

if $u \in \mathbb{R}_+^d$ and $|u| \leq n + 2$.

Suppose, for example, that f_2 satisfies condition (2.7) for $u \in \mathbb{R}_+^d$. For any $t \in [\tau, T]$ and any $u, v \in \mathbb{R}_+^d$ such that $u^i = v^i$ and $u^j \leq v^j$ if $j \neq i$, $|u|, |v| \leq n$, we have

$$\begin{aligned} F_{2n}^{ki}(t, u) &= f_{2n}^{ki}(t, u) - 3\delta_{nk} \geq f_{2n}(t, u) - 4\delta_{nk} \\ &= f_2^i(t, u) - 4\delta_{nk} \geq f_2^i(t, v) - 4\delta_{nk} \\ &= f_{2n}^i(t, v) - 4\delta_{nk} \geq f_{2n}^{ki}(t, v) - 5\delta_{nk} = F_{2n}^{ki}(t, v) - 2\delta_{nk}. \end{aligned} \quad (3.40)$$

Hence, (3.5) is satisfied with $R_0 = n$ and $\varepsilon = 2\delta_{nk}$.

Now, we will define the following functions:

$$\begin{aligned} l_{1n}^k(t, u) &= \phi_n(|u|)F_{1n}^k(t, u) + (1 - \phi_n(|u|))\tilde{f}_{1n}^k(t, u), \\ l_{2n}^k(t, u) &= \phi_n(|u|)F_{2n}^k(t, u) + (1 - \phi_n(|u|))f_{2n}(t, u). \end{aligned} \quad (3.41)$$

By Lemma 3.10, these functions satisfy (2.2) for $u \in \mathbb{R}_+^d$ with constants not depending neither on k nor n , inequality (3.32), and condition (3.2).

In view of (3.37), (3.39), and $\phi_n(|u|) = 0$, if $|u| \geq n + 2$, we obtain

$$\phi_n(|u|)F_{1n}^{ki}(t, u) \geq \phi_n(|u|)F_{2n}^{ki}(t, u), \quad (1 - \phi_n(|u|))\tilde{f}_{1n}^i(t, u) \geq (1 - \phi_n(|u|))f_{2n}^i(t, u) \quad (3.42)$$

and then if $u \in \mathbb{R}_+^d$,

$$l_{1n}^k(t, u) \geq l_{2n}^k(t, u). \quad (3.43)$$

On the other hand, since $l_{2n}^k(t, u) = F_{2n}^k(t, u)$ if $|u| \leq n + 1 + \bar{\gamma}$, (3.40) implies

$$l_{2n}^k(t, u) \geq l_{2n}^k(t, v) - 2\delta_{nk} \quad (3.44)$$

for any $t \in [\tau, T]$ and any $u, v \in \mathbb{R}_+^d$ such that $u^i = v^i$ and $u^j \leq v^j$ if $j \neq i$, $|u|, |v| \leq n$. Thus, (3.5) is satisfied with $R_0 = n$ and $\varepsilon = 2\delta_{nk}$.

We consider now the following problems:

$$\begin{aligned} \frac{\partial u}{\partial t} - a\Delta u + l_{jn}^k(t, u) &= h_j(t, x), \quad (t, x) \in (\tau, T) \times \Omega, \\ u|_{x \in \partial\Omega} &= 0, \\ u|_{t=\tau} &= u_\tau, \end{aligned} \quad (3.45)$$

$$\begin{aligned} \frac{\partial u}{\partial t} - a\Delta u + b_{jn}(t, u) &= h_j(t, x), \quad (t, x) \in (\tau, T) \times \Omega, \\ u|_{x \in \partial\Omega} &= 0, \\ u|_{t=\tau} &= u_\tau, \end{aligned} \quad (3.46)$$

where $b_{1n} = \tilde{f}_{1n}$ and $b_{2n} = f_{2n}$.

In view of Lemma 3.10 and [16, Lemma 5], problem (3.45) has a unique weak solution $u_n^k(\cdot)$ such that $u_n^k(t) \geq 0$ for all $t \in [\tau, T]$. Let $u_{jn}^k(\cdot)$ be the solutions of (3.45) corresponding to the initial data u_τ^j , where $j = 1, 2$. Using Lemma 3.10 it is standard to obtain estimate (2.14) with a constant D not depending neither on n nor k . Then, the solutions $u_{jn}^k(\cdot)$ of (3.45) satisfy

$$\|u_{jn}^k(t)\|^2 \leq \|u_\tau^j\|^2 + D \int_\tau^T (\|h_j(r)\|^2 + 1) dr = K^2(\|u_\tau^j\|, \tau, T). \quad (3.47)$$

Thus, since l_{2n}^k satisfy condition (3.5) with $R_0^2 = n^2 \geq 2 \max\{K^2(\|u_\tau^1\|, \tau, T), K^2(\|u_\tau^2\|, \tau, T)\}$, by Theorem 3.6 we know that as $u_\tau^1 \leq u_\tau^2$, we have

$$\left\| \left(u_{1n}^k(t) - u_{2n}^k(t) \right)^+ \right\| \leq 2C(\tau, T) \delta_{nk} \quad (3.48)$$

for all $t \in [\tau, T]$, all k and $n \geq (2 \max\{K^2(\|u_\tau^1\|, \tau, T), K^2(\|u_\tau^2\|, \tau, T)\})^{1/2}$.

Arguing as in the proof of Theorem 2.7 we obtain that for $j = 1, 2$ the sequence u_{jn}^k converges (up to a subsequence) in the sense of (2.29)–(2.36) to a solution u_{jn} of problem (3.46) with initial data u_τ^j . In particular, as $k \rightarrow \infty$ we have

$$u_{jn}^k(t) \rightarrow u_{jn}(t) \text{ in } H \quad \forall t \in [\tau, T]. \quad (3.49)$$

Fix $n \geq (2 \max\{K^2(\|u_\tau^1\|, \tau, T), K^2(\|u_\tau^2\|, \tau, T)\})^{1/2}$, $i \in \{1, \dots, d\}$ and take any $t \in [\tau, T]$. Denote by Ω_{nk}^{it} the set

$$\Omega_{nk}^{it} = \left\{ x \in \Omega : u_{1n}^{ki}(t, x) - u_{2n}^{ki}(t, x) \geq 0 \right\}. \quad (3.50)$$

By (3.48) as $k \rightarrow \infty$ we have

$$\int_{\Omega_{nk}^{it}} \left(u_{1n}^{ki}(t, x) - u_{2n}^{ki}(t, x) \right)^2 dx = \int_{\Omega} \left(\left(u_{1n}^{ki}(t) - u_{2n}^{ki}(t) \right)^+ \right)^2 dx \leq 4C^2(\tau, T) \delta_{nk}^2 \rightarrow 0. \quad (3.51)$$

Hence, (3.49) implies

$$\int_{\Omega \setminus \Omega_{nk}^{it}} \left(u_{1n}^{ki}(t, x) - u_{2n}^{ki}(t, x) \right)^2 dx \rightarrow \int_{\Omega} \left(u_{1n}^i(t, x) - u_{2n}^i(t, x) \right)^2 dx \quad \text{as } k \rightarrow \infty. \quad (3.52)$$

Define the sequence

$$w_n^{ki}(t, x) = \begin{cases} u_{1n}^{ki}(t, x) - u_{2n}^{ki}(t, x), & \text{if } x \in \Omega \setminus \Omega_{nk}^{it}, \\ 0, & \text{if } x \in \Omega_{nk}^{it}. \end{cases} \quad (3.53)$$

Then, it is clear that $\|w_n^{ki}(t)\|_{L^2(\Omega)} \rightarrow \|u_{1n}^{ki}(t) - u_{2n}^{ki}(t)\|_{L^2(\Omega)}$ as $k \rightarrow \infty$. We note also that $w_n^{ki}(t) \rightarrow u_{1n}^i(t) - u_{2n}^i(t)$ weakly in $L^2(\Omega)$, as for any $\xi \in L^2(\Omega)$, (3.51) gives

$$\begin{aligned} \int_{\Omega} w_n^{ki}(t, x) \xi(x) dx &= \int_{\Omega} (u_{1n}^{ki}(t, x) - u_{2n}^{ki}(t, x)) \xi(x) dx \\ &\quad - \int_{\Omega_{nk}^t} (u_{1n}^{ki}(t, x) - u_{2n}^{ki}(t, x)) \xi(x) dx \rightarrow 0. \end{aligned} \quad (3.54)$$

Therefore, $w_n^{ki}(t) \rightarrow u_{1n}^i(t) - u_{2n}^i(t)$ strongly in $L^2(\Omega)$ so that $w_n^{ki}(t, x) \rightarrow u_{1n}^i(t, x) - u_{2n}^i(t, x)$ for a.a. $x \in \Omega$. Since $w_n^{ki}(t, x) \leq 0$, for a.a. Ω , we obtain

$$0 \leq u_{1n}(t) \leq u_{2n}(t) \quad \forall t \in [\tau, T]. \quad (3.55)$$

Arguing again as in the proof of Theorem 2.8 we obtain that the sequences u_{1n}, u_{2n} converge (up to a subsequence) in the sense of (2.29)–(2.36) to the solutions u_1, u_2 of problems (2.8) and (2.9), respectively. Also, it holds

$$0 \leq u_1(t) \leq u_2(t), \quad \forall t \in [\tau, T]. \quad (3.56)$$

□

As in the previous section, we shall generalize this theorem to the case where the constant α can be negative. We note that if f_j, h_j satisfy (3.2), then $\tilde{f}_j(t, v) = e^{-\beta t} f_j(t, e^{\beta t} v) + \beta v$, $\tilde{h}_j = e^{-\beta t} h_j$ also satisfy (3.2). Arguing as in Theorem 2.9, we obtain the following.

Theorem 3.12. *Let f_j, h_j satisfy (3.1)–(3.2). Assume that f_j, h_j satisfy (2.43) and (2.10) for $u \in \mathbb{R}_+^d$. Also, suppose that either f_1 or f_2 satisfies (2.7) for $u \in \mathbb{R}_+^d$ for an arbitrary $R_0 > 0$. If $u_\tau^1 \leq u_\tau^2$, there exist two solutions u_1, u_2 (of (2.8) and (2.9), resp.), with $u_1(\tau) = u_\tau^1$, $u_2(\tau) = u_\tau^2$ such that $0 \leq u_1(t) \leq u_2(t)$, for all $t \in [\tau, T]$.*

But in this case we can consider another interesting situation, as the values p_i^1 and p_i^2 are not necessarily equal.

Let f_1 satisfy (2.2) for $u \in \mathbb{R}_+^d$, whereas f_2 satisfy (2.43) for $u \in \mathbb{R}_+^d$, with constants α^j, C_1^j, C_2^j , $j = 1, 2$. Then if $\alpha^2 < 0$, we make in (2.1) the change of variable $v = e^{-\beta t} u$, where $\beta > -\alpha^2$. We obtain problems (2.44)–(2.45).

If $v(t)$ is a weak solution of (2.44), then $u(t) = e^{\beta t} v(t)$ is a weak solution of (2.8) (and the same is true, of course, for (2.45) and (2.9)).

The function $\tilde{f}_1(t, v) = e^{-\beta t} f_1(t, e^{\beta t} v) + \beta v$ satisfies (2.2) for $v \in \mathbb{R}_+^d$ with the same p_i^1 as f_1 . Indeed, as $e^{\beta t(p_i^1-2)} \geq 1$, we get

$$\begin{aligned} \sum_{i=1}^d |\tilde{f}_1^i(t, v)|^{p_i^1/(p_i^1-1)} &\leq K_1 \left(\sum_{i=1}^d |f_1^i(t, e^{\beta t} v)|^{p_i^1/(p_i^1-1)} + \sum_{i=1}^d |v^i|^{p_i^1/(p_i^1-1)} \right) \\ &\leq K_2 \left(1 + \sum_{i=1}^d |v^i|^{p_i^1} \right), \end{aligned} \quad (3.57)$$

$$(\tilde{f}_1(t, v), v) \geq e^{-2\beta t} \left(\alpha^1 \sum_{i=1}^d e^{\beta t p_i} |v^i|^{p_i^1} - C_2^1 \right) + \beta |v|^2 \geq \alpha^1 \sum_{i=1}^d |v^i|^{p_i^1} - \tilde{C}_2^1.$$

Then, we obtain the following.

Theorem 3.13. Let f_j, h_j satisfy (3.1)-(3.2). Let f_1 satisfy (2.2) for $u \in \mathbb{R}_+^d$, whereas f_2 satisfy (2.43) for $u \in \mathbb{R}_+^d$, with constants α^j, C_1^j, C_2^j , $j = 1, 2$. Assume that f_j, h_j satisfy (2.10). Also, suppose that either f_1 or f_2 satisfies (2.7) for $u \in \mathbb{R}_+^d$ for an arbitrary $R_0 > 0$. If $u_\tau^1 \leq u_\tau^2$, there exist two solutions u_1, u_2 (of (2.8) and (2.9), resp.), with $u_1(\tau) = u_\tau^1$, $u_2(\tau) = u_\tau^2$ such that $0 \leq u_1(t) \leq u_2(t)$ for all $t \in [\tau, T]$.

Proof. We consider problems (2.44) and (2.45). In view of (2.46) and (3.57), $\tilde{f}_2(t, v) = e^{-\beta t} f_2(t, e^{\beta t} v) + \beta v$ satisfy (2.2) for $v \in \mathbb{R}_+^d$ with $p_i^2 = 2$, and $\tilde{f}_1$ satisfies (2.2) for $v \in \mathbb{R}_+^d$ with the same p_i^1 as f_1 . Also, defining $\tilde{h}_j(t, x) = e^{-\beta t} h_j(t, x)$ it is clear that (2.10) and (3.2) hold. Finally, if, for example, f_1 satisfies (2.7) for $u \in \mathbb{R}_+^d$ for any $R_0 > 0$, then it is obvious that for $\tilde{f}_1$ is true as well.

Hence, by Theorem 3.11 there exist two solutions v_1, v_2 (of (2.44) and (2.45), resp.), with $v_j(\tau) = e^{-\beta \tau} u_\tau^j$ such that $0 \leq v_1(t) \leq v_2(t)$ for all $t \in [\tau, T]$. Thus,

$$0 \leq u_1(t) = e^{\beta t} v_1(t) \leq e^{\beta t} v_2(t) = u_2(t), \quad \text{for } t \in [\tau, T], \quad (3.58)$$

and u_1, u_2 are solutions of (2.8) and (2.9), respectively such that $u_j(\tau) = u_\tau^j$. $\square$

Remark 3.14. If f_j satisfy (2.6), then the solutions u_1, u_2 given in Theorems 3.12 and 3.13 are unique for the corresponding initial data.

Remark 3.15. All the result proved so far are true if instead of Dirichlet boundary conditions we consider Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{in } \partial\Omega, \quad (3.59)$$

where ν is the unit outward normal. In such a case, the space V will be $(H^1(\Omega))^d$. The proofs remain the same.

4. Applications

We shall apply now the previous results to some model of physical and biological interest.

4.1. The Lotka-Volterra System

We also study the Lotka-Volterra system with diffusion:

$$\begin{aligned}\frac{\partial u^1}{\partial t} &= D_1 \Delta u^1 + u^1 \left(a_1(t) - u^1 - a_{12}(t)u^2 - a_{13}(t)u^3 \right), \\ \frac{\partial u^2}{\partial t} &= D_2 \Delta u^2 + u^2 \left(a_2(t) - u^2 - a_{21}(t)u^1 - a_{23}(t)u^3 \right), \\ \frac{\partial u^3}{\partial t} &= D_3 \Delta u^3 + u^3 \left(a_3(t) - u^3 - a_{31}(t)u^1 - a_{32}(t)u^2 \right),\end{aligned}\tag{4.1}$$

with either Dirichlet or Neumann boundary conditions, where $u^i = u^i(x, t) \geq 0$ and the functions $a_i(t)$, $a_{ij}(t)$ are positive and continuous. Also, D_i are positive constants and $\Omega \subset \mathbb{R}^3$. The initial data u_τ belongs to $(L^2(\Omega))^3$.

In this case, the functions f_1, h_1 are given by

$$f_1(t, u) = \begin{pmatrix} -u^1(a_1(t) - u^1 - a_{12}(t)u^2 - a_{13}(t)u^3) \\ -u^2(a_2(t) - u^2 - a_{21}(t)u^1 - a_{23}(t)u^3) \\ -u^3(a_3(t) - u^3 - a_{31}(t)u^1 - a_{32}(t)u^2) \end{pmatrix}, \quad h_1(t) \equiv 0.\tag{4.2}$$

Uniqueness of the Cauchy problem for this system has been proved only if we consider solutions confined in an invariant region (e.g., in a parallelepiped $\mathfrak{D} = \{(u^1, u^2, u^3) : 0 \leq u^i \leq k^i\}$ when the parameters do not depend on t) (see [20, 21]). However, in the general case for initial data just in $(L^2(\Omega))^3$ it is an open problem so far.

System (4.1) satisfies conditions (2.2) with $p_1 = p_2 = p_3 = 3$ for $u \in \mathbb{R}_+^3$ [16, page 263]. Also, it is clear that (3.1)-(3.2) hold.

We shall compare with the following system:

$$\begin{aligned}\frac{\partial u^1}{\partial t} &= D_1 \Delta u^1 + u^1 \left(a_1(t) - u^1 \right), \\ \frac{\partial u^2}{\partial t} &= D_2 \Delta u^2 + u^2 \left(a_2(t) - u^2 \right), \\ \frac{\partial u^3}{\partial t} &= D_3 \Delta u^3 + u^3 \left(a_3(t) - u^3 \right),\end{aligned}\tag{4.3}$$

which is a system of three uncoupled logistic equations. The functions f_2, h_2 are given by

$$f_2(t, u) = \begin{pmatrix} -u^1(a_1(t) - u^1) \\ -u^2(a_2(t) - u^2) \\ -u^3(a_3(t) - u^3) \end{pmatrix}, \quad h_2(t) \equiv 0. \quad (4.4)$$

It is easy to see that system (4.3) satisfies conditions (2.2) with $p_1 = p_2 = p_3 = 3$ for $u \in \mathbb{R}^3$. Also, it is clear that (3.1)-(3.2) hold, and that condition (2.7) is trivially satisfied. Moreover,

$$f_1^i(t, u) \geq f_2^i(t, u), \quad (4.5)$$

for all $t, u \in \mathbb{R}_+^3$ and i so that (2.10) holds for $u \in \mathbb{R}_+^3$.

Also, we have

$$\begin{aligned} (f_{2u}(t, u)w, w) &\geq -a_1(t)w_1^2 - a_2(t)w_2^2 - a_3(t)w_3^2 \\ &\quad + 2(u^1w_1^2 + u^2w_2^2 + u^3w_3^2) \geq -C|w|^2, \end{aligned} \quad (4.6)$$

for all $u \in \mathbb{R}_+^3$, $t \in [\tau, T]$ and $w \in \mathbb{R}^3$, where $C > 0$.

Thus, by Theorem 3.11, Lemma 3.1, and Remark 3.15, we obtain the following theorem.

Theorem 4.1. *If $0 \leq u_\tau^1 \leq u_\tau^2$, there exists a solution u_1 for (4.1) with $u_1(\tau) = u_\tau^1$ such that $0 \leq u_1(t) \leq u_2(t)$, for all $t \in [\tau, T]$, where u_2 is the unique solution with $u_2(\tau) = u_\tau^2$ of (4.3) in the class of solutions satisfying $u_2(t) \geq 0$ for all t .*

This theorem says that there exists at least one solution for the Lotka-Volterra system which is dominated by the unique nonnegative solution of the uncoupled logistic system (4.3).

Further, we shall compare with the uncoupled linear system

$$\begin{aligned} \frac{\partial u^1}{\partial t} &= D_1 \Delta u^1 + u^1 a_1(t), \\ \frac{\partial u^2}{\partial t} &= D_2 \Delta u^2 + u^2 a_2(t), \\ \frac{\partial u^3}{\partial t} &= D_3 \Delta u^3 + u^3 a_3(t). \end{aligned} \quad (4.7)$$

Hence,

$$f_2(t, u) = \begin{pmatrix} -u^1 a_1(t) \\ -u^2 a_2(t) \\ -u^3 a_3(t) \end{pmatrix}, \quad h_2(t) \equiv 0. \quad (4.8)$$

Obviously, system (4.7) satisfies conditions (2.43) for $u \in \mathbb{R}^3$. Also, it is clear that (3.1)-(3.2) and condition (2.7) are trivially satisfied. Moreover,

$$f_1^i(t, u) \geq f_2^i(t, u), \quad (4.9)$$

for all $t, u \in \mathbb{R}_+^3$ and i so that (2.10) holds for $u \in \mathbb{R}_+^3$. Also, we have

$$(f_{2u}(t, u)w, w) \geq -a_1(t)w_1^2 - a_2(t)w_2^2 - a_3(t)w_3^2 \geq -C|w|^2, \quad (4.10)$$

for all $u \in \mathbb{R}^3, t \in [\tau, T]$ and $w \in \mathbb{R}^3$, where $C > 0$.

Thus, by Theorem 3.13 and Remark 3.15, we have the following.

Theorem 4.2. *If $0 \leq u_\tau^1 \leq u_\tau^2$, there exist a solution u_1 for (4.1) with $u_1(\tau) = u_\tau^1$ such that $0 \leq u_1(t) \leq u_2(t)$ for all $t \in [\tau, T]$, where u_2 is the unique solution of (4.7) with $u_2(\tau) = u_\tau^2$.*

Let us consider now the autonomous case, that is, $a_i(t) \equiv a_i > 0$ and $\tau = 0$, with Diriclet boundary conditions.

By the changes of variable $v^i(t) = e^{-a_i t} u^i(t)$, system (4.7) becomes

$$\begin{aligned} \frac{\partial v^i}{\partial t} &= D_i \Delta v^i, \quad i = 1, 2, 3, \\ v^i \Big|_{x \in \partial \Omega} &= 0, \end{aligned} \quad (4.11)$$

with initial data $v_0 = u_0$.

The operator $A = -D_i \Delta$ in the space $L^2(\Omega)$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ is sectorial [22]. Moreover, since the eigenvalues of A are $0 < D_i \lambda_1 \leq D_i \lambda_2 \leq \dots$, we have that the minimum eigenvalue is strictly greater than 0. Denote by e^{-At} the analytic semigroup generated by the operator A . Then, $v(t) = e^{-At} v_0$ is the unique solution of (4.11) with $v(0) = v_0$.

It is well known [22] that the operator A generates a scale of interpolation spaces $X^\alpha = D(A^\alpha)$ with the norm $\|v\|_\alpha = \|A^\alpha v\|_{L^2(\Omega)}$, $\alpha \geq 0$, where $X^\alpha \subset H^{2\alpha}(\Omega)$ with continuous embedding. Take $0 < \delta < D_1 \lambda_1$. Then, by Theorem 1.4.3 in [22] we obtain

$$\|A^\alpha e^{-At} v_0^i\|_{L^2(\Omega)} \leq C_\alpha t^{-\alpha} e^{-\delta t} \|v_0^i\|_{L^2(\Omega)} \quad \text{for } t > 0. \quad (4.12)$$

Since this is true for every $\delta < D_i \lambda_1$, we obtain that

$$\begin{aligned} \|A^\alpha e^{-At} v_0^i\|_{L^2(\Omega)} &\leq C_\alpha t^{-\alpha} e^{-D_i \lambda_1 t} \|v_0^i\|_{L^2(\Omega)}, \\ \|e^{-At} v_0^i\|_{H^{2\alpha}(\Omega)} &\leq \tilde{C}_\alpha t^{-\alpha} e^{-D_i \lambda_1 t} \|v_0^i\|_{L^2(\Omega)} \quad \text{for } t > 0. \end{aligned} \quad (4.13)$$

We note that $H^{2\alpha}(\Omega) \subset L^\infty(\Omega)$ with continuous embedding if $\alpha > 3/4$. Since the constants C_α are bounded for α in compact sets, we obtain the existence of C such that

$$\|e^{-At} v_0^i\|_{L^\infty(\Omega)} \leq C t^{-3/4} e^{-D \lambda_1 t} \|v_0^i\|_{L^2(\Omega)} \quad \text{for } t > 0, \quad (4.14)$$

where $D = \min\{D_1, D_2, D_3\}$. Then the unique solution of (4.7) with $u(0) = u_0$ satisfies

$$\|u(t)\|_{(L^\infty(\Omega))^3} \leq C t^{-3/4} e^{(a-D\lambda_1)t} \|u_0\| \quad \text{for } t > 0, \quad (4.15)$$

where $a = \max\{a_1, a_2, a_3\}$.

Joining (4.15) and Theorem 4.2, we obtain the following result.

Theorem 4.3. *There exist at least one solution $u(t)$ for the autonomous system (4.1) with $u(0) = u_0$ and Dirichlet boundary conditions such that*

$$\|u(t)\|_{(L^\infty(\Omega))^3} \leq C t^{-3/4} e^{(a-D\lambda_1)t} \|u_0\| \quad \text{for } t \in (0, T], \quad (4.16)$$

where $C > 0$ and $D = \min\{D_1, D_2, D_3\}$, $a = \max\{a_1, a_2, a_3\}$.

We shall obtain also a weak maximum principle for the autonomous Lotka-Volterra system with Dirichlet boundary conditions.

By the maximum principle for the heat equation it is well known (see [23]) that for any $t \geq 0$ the unique solution of (4.11) satisfies

$$0 \leq v^i(t, x) \leq \sup_{\Omega} v_0^i \quad \text{for a.a. } x \in \Omega. \quad (4.17)$$

Hence, the unique solution of (4.7) satisfies

$$0 \leq u^i(t, x) \leq e^{a_i t} \sup_{\Omega} v_0^i \quad \text{for a.a. } x \in \Omega. \quad (4.18)$$

By (4.18) and Theorem 4.2, we obtain the following weak maximum principle.

Theorem 4.4. *There exist at least one solution $u(t)$ for the autonomous system (4.1) with $u(0) = u_0$ and Dirichlet boundary conditions such that for any $t \in [0, T]$,*

$$0 \leq u^i(t, x) \leq e^{a_i t} \sup_{\Omega} u_0^i \quad \text{for a.a. } x \in \Omega, \quad i = 1, 2, 3. \quad (4.19)$$

In particular, if $u_0 \in (L^\infty(\Omega))^3$, then

$$\|u(t)\|_{(L^\infty(\Omega))^3} \leq e^{at} \|u_0\|_{(L^\infty(\Omega))^3} \quad \text{for } t \in [0, T], \quad (4.20)$$

where $a = \max\{a_1, a_2, a_3\}$.

4.2. A Model of Fractional-Order Chemical Autocatalysis with Decay

Consider the following scalar problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + (1 - u)u^m - ku^r, \\ \frac{\partial u}{\partial x}(0, t) &= \frac{\partial u}{\partial x}(a, t) = 0, \\ u|_{t=0} &= u_0(x), \end{aligned} \quad (4.21)$$

where $u \geq 0$, $N = 1$, $d = 1$, $\Omega = (0, L)$, and $k > 0$, $0 < m$, $r < 1$. The initial data u_0 belongs to $L^2(0, a)$. This equation models an isothermal chemical autocatalysis (see [24]). In [24], the authors study the travelling waves of the equation in the case where $\Omega = (0, +\infty)$ with Neumann boundary conditions at $x = 0$. The variable u is nonnegative, since it represents a chemical concentration.

The functions f, h are given by $f(u) = (u - 1)u^m + ku^r$, $h \equiv 0$. Clearly, conditions (2.2) hold (with $p = m + 2$) for $u \geq 0$. In this case (3.1)-(3.2) and (2.7) are trivially satisfied.

We take $f_1 = f_2 = f$ and apply Theorem 3.11 and Remark 3.15 to obtain the following.

Theorem 4.5. *If $0 \leq u_\tau^1 \leq u_\tau^2$, there exist solutions u_1, u_2 for (4.21) with $u_1(0) = u_0^1, u_2(0) = u_0^2$ such that $0 \leq u_1(t) \leq u_2(t)$ for all $t \in [0, T]$.*

4.3. A Generalized Logistic Equation

Consider the following scalar problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + (1 - u^q)u^r, \\ \frac{\partial u}{\partial x}(0, t) &= \frac{\partial u}{\partial x}(a, t) = 0, \\ u|_{t=0} &= u_0(x), \end{aligned} \quad (4.22)$$

where $u \geq 0$, $N = 1$, $\Omega = (0, L)$, $r, q > 0$, and $r + q \geq 1$. The initial data u_0 belongs to $L^2(0, a)$.

This kind of nonlinearities for the logistic equation (instead of the classical $(1 - u)u$) has been considered in [25, Chapter 11].

The functions f, h are given by $f(u) = (u^q - 1)u^r$, $h \equiv 0$. Clearly, conditions (2.2) hold (with $p = r + q + 1$) for $u \geq 0$. In this case, (3.1)-(3.2) and (2.7) are trivially satisfied.

We take $f_1 = f_2 = f$ and apply Theorem 3.11 and Remark 3.15 to obtain the following.

Theorem 4.6. *If $0 \leq u_\tau^1 \leq u_\tau^2$, there exist two solutions u_1, u_2 for (4.22) with $u_1(0) = u_0^1$, $u_2(0) = u_0^2$ such that $0 \leq u_1(t) \leq u_2(t)$ for all $t \in [0, T]$.*

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