Research Article

On Holomorphic Solution for Space- and Time-Fractional Telegraph Equations in Complex Domain

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We consider some classes of space- and time-fractional telegraph equations in complex domain in sense of the Riemann-Liouville fractional operators for time and the Srivastava-Owa fractional operators for space. The existence and uniqueness of holomorphic solution are established. We illustrate our theoretical result by examples.

1. Introduction

Fractional differential equations have emerged as a new branch of applied mathematics which has been used for many mathematical models in science and engineering. In fact, fractional differential equations are considered as an alternative model to nonlinear differential equations. Various types play important roles and tools not only in mathematics but also in physics, control systems, dynamical systems, and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators, Erdelyi-Kober operators, Weyl-Riesz operators, Caputo operators, and Grünwald-Letnikov operators, have appeared during the past three decades with its applications in other fields [1–6]. Recently, the existence of analytic solutions for fractional differential equations in complex domain is posed [7–10].

The study of nonlinear problems is of crucial importance in all areas of mathematics, mechanics, and physics. Some of the most interesting features of physical systems are hidden in their nonlinear behavior and can only be studied with appropriate methods designed to tackle and process nonlinear problems. One of the famous nonlinear problem is the telegraph equations used in signal analysis for transmission and propagation of electrical

signals and also used in modeling reaction diffusion [11, 12]. Numerical solutions for these equations are posed; for example, the Adomian decomposition method (ADM) and homotopy perturbation method (HPM) were used to solve the space- and time-fractional telegraph equations (see [13]), Variational iteration method (VIM) was used to solve the linear and nonlinear telegraph equation (see [14–16]). Note that in [17], the methods (HPM) and (VIM) were compared for linear fractional partial differential equations. Also time-periodic solutions of these equations are studied (see [18]). Recently, Herzallah and Baleanu [19] gave a new model of the abstract fractional-order telegraph equation, and they studied the existence and uniqueness theorems of the strong and mild solutions as well as the continuation of this solution.

In the present paper, we establish the existence and uniqueness of solutions for classes of the space- and time-fractional telegraph equations in complex domain. The fractional derivative for the space is taken in the sense of the Srivastava-Owa operators while the fractional time derivative is taken in the sense of the Riemann-Liouville operators because the Riemann-Liouville fractional derivative could hardly pose the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. One of the most frequently used tools in the theory of fractional calculus is furnished by the Riemann-Liouville operators.

Definition 1.1. The fractional-order (arbitrary) integral of the function f of order $\alpha > 0$ is defined by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau. \quad (1.1)$$

When $a = 0$, one writes $I_a^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $(*)$ denoted the convolution product,

$$\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0, \quad (1.2)$$

and $\phi_\alpha(t) = 0$, $t \leq 0$, and $\phi_\alpha \rightarrow \delta(t)$ as $\alpha \rightarrow 0$ where $\delta(t)$ is the delta function.

Definition 1.2. The fractional-order (arbitrary) derivative of the function f of order $0 < \alpha < 1$ is defined by

$$D_a^\alpha f(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau) d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t). \quad (1.3)$$

From Definitions 1.1 and 1.2, one has

$$\begin{aligned} D^\alpha t^\mu &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \quad \mu > -1, \quad 0 < \alpha \leq 1, \\ I^\alpha t^\mu &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \quad \mu > -1, \quad \alpha > 0. \end{aligned} \quad (1.4)$$

We will use the notation $\partial^\alpha / \partial t^\alpha$ for the fractional-order derivative.

The Srivastava-Owa fractional operators are (see [20]) as follows.

Definition 1.3. The fractional derivative of order α is defined, for a function $f(z)$, by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta, \quad 0 \leq \alpha < 1, \quad (1.5)$$

where the function $f(z)$ is analytic in simply connected region of the complex z -plane $\mathbb{C}$ containing the origin, and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.4. The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta, \quad \alpha > 0, \quad (1.6)$$

where the function $f(z)$ is analytic in simply-connected region of the complex z -plane ($\mathbb{C}$) containing the origin, and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark 1.5. From Definitions 1.3 and 1.4, we have

$$\begin{aligned} D_z^\alpha z^\mu &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \quad \mu > -1, \quad 0 < \alpha < 1, \\ I_z^\alpha z^\mu &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \quad \mu > -1, \quad \alpha > 0. \end{aligned} \quad (1.7)$$

By using the majorant concept, we establish the existence and uniqueness of holomorphic solution.

Definition 1.6. The major are relations, described as follows: if $a(x) = \sum a_i x^i$ and $A(x) = \sum A_i x^i$, then one says that $a(x) \ll A(x)$ if and only if $|a_i| \leq |A_i|$ for each i .

2. Existence of Unique Solution

In the present paper, we consider the fractional differential equations

$$\theta(z) D_z^\alpha u(t, z) + \lambda \frac{\partial^\beta u(t, z)}{\partial t^\beta} = F(t, z, u, u_t), \quad 0 < \alpha \leq 1, \quad 1 < \beta \leq 2, \quad (2.1)$$

subject to the initial condition $u|_{t=0} = 0$ in a neighborhood of $z = 0$, where $t \in J := [0, 1]$, $z \in U := \{z \in \mathbb{C} : |z| < 1\}$, $u(t, z)$ is an unknown function, $\theta(z) := \vartheta z^\alpha$ such that $\vartheta \neq 0$, $z \in U$, $\lambda \neq 0$ is a constant, and F is a holomorphic function on $J \times U \times \mathbb{C} \times \mathbb{C}$. Thus, we can expand it as follows:

$$F(t, z, u, v) := \gamma(t)z + \delta(t)u + \eta(t)v + R_2(t, z, u, v), \quad (2.2)$$

where

$$\gamma(t) := \frac{\partial F}{\partial z}(t, 0, 0, 0), \quad \delta(t) := \frac{\partial F}{\partial u}(t, 0, 0, 0), \quad \eta(t) := \frac{\partial F}{\partial v}(t, 0, 0, 0), \quad (2.3)$$

and the degree of $R_2(t, z, u, v)$ with respect to (t, z, u, v) is greater than or equal to 2.

We need the following result.

Lemma 2.1 (see [21]). *Let $R > 0$, and let $f(x)$ be a holomorphic function on $D_R = \{x \in \mathbb{C} : |x| < R\}$. If for any $r > 0$, $0 < r < R$, $f(x)$ satisfies*

$$\max_{|x| \leq r} |f(x)| \leq \frac{\rho}{(R-r)^\mu} \quad (2.4)$$

for some $\rho > 0$ and $\mu \geq 0$, then one has

$$\max_{|x| \leq r} \left| \frac{\partial f(x)}{\partial x} \right| \leq \frac{(\mu+1)e\rho}{(R-r)^{\mu+1}}. \quad (2.5)$$

Theorem 2.2. *Assume the problem (2.1). If $\alpha \in (0, 1]$, $\vartheta \neq 0$, and*

$$\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} \vartheta - \delta(t) \neq 0, \quad n \in \mathbb{N}, \quad (2.6)$$

then (2.1) has a unique holomorphic solution $u(t, z)$ near $(0, 0) \in J \times U$.

Proof. We realize that (2.1) has a formal solution

$$u(t, z) = \sum_{k=1}^{\infty} u_k(t) z^k, \quad (z \in U). \quad (2.7)$$

Then, substitut the series (2.7) into (2.1) and comparing the coefficients of z^k in two sides of the equation yield

$$\begin{aligned} \left[\frac{\Gamma(2)}{\Gamma(2-\alpha)} \vartheta - \delta(t) \right] u_1(t) + \lambda u_1^{(\beta)}(t) - \eta(t) u_1'(t) &= \gamma(t) \\ \implies u_1^{(\beta)}(t) &= f_1(t, u_1) \end{aligned}$$

$$\begin{aligned}
& \left[\frac{\Gamma(3)}{\Gamma(3-\alpha)} \vartheta - \delta(t) \right] u_2(t) + \lambda u_2^{(\beta)}(t) - \eta(t) u_2'(t) = \phi_1(u_1, u_1') \\
& \implies u_2^{(\beta)}(t) = f_2(t, u_2) \\
& \quad \vdots \\
& \left[\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} \vartheta - \delta(t) \right] u_n(t) + \lambda u_n^{(\beta)}(t) - \eta(t) u_n'(t) + \phi_{n-1}(u_1, \dots, u_{n-1}, u_1', \dots, u_{n-1}') \\
& \implies u_n^{(\beta)}(t) = f_n(t, u_n).
\end{aligned} \tag{2.8}$$

Thus, we obtain the following formula:

$$u_n^{(\beta)}(t) + \frac{[(\Gamma(n+1)/\Gamma(n+1-\alpha)) \vartheta - \delta(t)]}{\lambda} u_n(t) = \Phi(t), \tag{2.9}$$

where $\phi_0(t) = \gamma(t)$ and

$$\Phi(t) := \frac{\phi_{n-1}(u_1, \dots, u_{n-1}, u_1', \dots, u_{n-1}') + \eta(t) u_n'(t)}{\lambda}. \tag{2.10}$$

Hence, from the idea given in [22], solutions to fractional Cauchy problems are obtained by subordinating the solution to the original Cauchy problem, problem (2.9), and consequently problem (2.1) has a unique solution in a neighborhood of $z = 0$.

Now, we proceed to prove that the formal series solution (2.7) is convergent near $(0, 0) \in (J, U)$. We expand the remainder term $R_2(t, z, u, v)$ into Taylor series with respect to (t, u, v) , that is,

$$R_2(t, z, u, v) = \sum_{m+n+p \geq 2} a_{m,n,p}(t) z^m u^n v^p, \tag{2.11}$$

such that

- (i) $a_{m,n,p}(t)$ is holomorphic in J ,
- (ii) $|a_{m,n,p}(t)| \leq A_{m,n,p}$, $A_{m,n,p} > 0$ on J ,
- (iii) $\sum_{m+n+p \geq 2} A_{m,n,p} z^m V^{n+p}$ converges in (z, V) where $V > 0$ satisfies $|u| \leq V$ and $|v| \leq V$.

From (2.9), we observe that

$$\begin{aligned}
 & \left\{ \frac{[(\Gamma(2)/\Gamma(2-\alpha))\mathfrak{D} - \delta(t)]}{\lambda} + \frac{\partial^\beta}{\partial t^\beta} \right\} u_1(t) = \gamma(t) + \eta(t)u'_1 \\
 & \quad \vdots \\
 & \left\{ \frac{[(\Gamma(k)/\Gamma(k-\alpha))\mathfrak{D} - \delta(t)]}{\lambda} + \frac{\partial^\beta}{\partial t^\beta} \right\} u_k(t) \\
 & = \sum_{m+n+p \geq 2} \left[\sum_{m+k_1+\dots+k_n+l_1+\dots+l_p=k} a_{m,n,p}(t) \times u_{k_1} \times \dots \times u_{k_n} \times \frac{\partial u_{l_1}}{\partial t} \times \dots \times \frac{\partial u_{l_p}}{\partial t} \right].
 \end{aligned} \tag{2.12}$$

Without loss of generality, we may assume that there exists a constant $K > 0$ such that

$$|u_1(t)| \leq K, \quad \left| \frac{\partial u_1(t)}{\partial t} \right| \leq K. \tag{2.13}$$

Denoting

$$C := \frac{1}{\|(\Gamma(2)/\Gamma(2-\alpha))\mathfrak{D} - \delta\|}, \tag{2.14}$$

then we pose the following formula:

$$V(z) = Kz + \frac{C}{1-r} \sum_{m+n+p \geq 2} \frac{A_{m,n,p}}{(1-r)^{m+n+p-2}} z^m V^n V^p, \tag{2.15}$$

where r is a parameter with $0 < r < 1$. Since (2.15) is an analytic functional equation in V then in view of the implicit function theorem, (2.15) has a unique holomorphic solution $V(z)$ in a neighborhood of $z = 0$ with $V(0) = 0$. Expanding $V(z)$ into Taylor series in z , we have

$$V(z) = \sum_{k \geq 1} V_k z^k, \tag{2.16}$$

where

$$\begin{aligned}
 V_k &= \frac{C}{1-r} \sum_{m+n+p \geq 2} \left[\sum_{m+k_1+\dots+k_n+l_1+\dots+l_p=k} \frac{A_{m,n,p}}{(1-r)^{m+n+p-2}} \times V_{k_1} \times \dots \right. \\
 &\quad \left. \times V_{k_n} \times 0!(eV_{l_1}) \times \dots \times (p-1)!(eV_{l_p}) \right] \\
 &:= \frac{C_k}{(1-r)^{k-1}}, \quad k \in \mathbb{N} \\
 &> 0,
 \end{aligned} \tag{2.17}$$

with $C_1 = K$.

Next, our aim is to show that the series $\sum_{k \geq 1} V_k z^k$ is a major series for the formal series solution $\sum_{k \geq 1} u_k z^k$ near $z = 0$. For this purpose, we will show that

$$|u_k| \leq V_k \quad \text{on } J_r := [0, r], \quad r < 1, \tag{2.18}$$

$$\left| \frac{\partial u_k}{\partial t} \right| \leq (k-1)!(eV_k) \quad \text{on } J_r. \tag{2.19}$$

Since $(1-r) < 1$ implies

$$\frac{1}{(1-r)^{m+n+p-2}} \geq 1, \quad r < 1, \tag{2.20}$$

then we have

$$\begin{aligned}
 |u_k| &\leq C \sum_{m+n+p \geq 2} \left[\sum_{m+k_1+\dots+k_n+l_1+\dots+l_p=k} A_{m,n,p} \times |u_{k_1}(t)| \times \dots \right. \\
 &\quad \left. \times |u_{k_n}(t)| \times \left| \frac{\partial u_{l_1}(t)}{\partial t} \right| \times \dots \times \left| \frac{\partial u_{l_p}(t)}{\partial t} \right| \right] \\
 &\leq C \sum_{m+n+p \geq 2} \left[\sum_{m+k_1+\dots+k_n+l_1+\dots+l_p=k} A_{m,n,p} \times V_{k_1} \times \dots \right. \\
 &\quad \left. \times V_{k_n} \times 0!(eV_{l_1}) \times \dots \times (p-1)!(eV_{l_p}) \right]
 \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{m+n+p \geq 2} \left[\sum_{m+k_1+\dots+k_n+l_1+\dots+l_p=k} \frac{A_{m,n,p}}{(1-r)^{m+n+p-2}} \times V_{k_1} \times \dots \right. \\
&\quad \left. \times V_{k_n} \times 0!(eV_{l_1}) \times \dots \times (p-1)!(eV_{l_p}) \right] \\
&\leq \frac{C_k}{(1-r)^{k-2}} \leq \frac{C_k}{(1-r)^{k-1}} = V_k.
\end{aligned} \tag{2.21}$$

Hence, we obtain the inequality (2.19). Next, by using Lemma 2.1, we pose that

$$\left| \frac{\partial u_k}{\partial t} \right| \leq (k-1) \frac{eC_k}{(1-r)^{k-1}} \leq (k-1)! \frac{eC_k}{(1-r)^{k-1}} = (k-1)!(eV_k). \tag{2.22}$$

This completes the proof of Theorem 2.2. $\square$

3. Applications

In this section, we pose two examples that demonstrate the performance and efficiency for solving time- and space-fractional telegraph equations.

Example 3.1. Assume the following equation:

$$\begin{aligned}
\frac{z^{0.5}}{1.128} D_z^{0.5} u(t, z) + \frac{\partial^{1.5} u(t, z)}{\partial t^{1.5}} &= u_t(t, z) + 1.5u(t, z), \quad t \in J = [0, 1], \\
u(0, z) &= 0,
\end{aligned} \tag{3.1}$$

where $u(t, z)$ is the unknown function. Since the condition $(\Gamma(n+1)/\Gamma(n+1-\alpha))\vartheta-\delta(t) \neq 0$, $n \in \mathbb{N}$ is satisfied, then in view of Theorem 2.2, the problem (3.1) has a unique holomorphic solution near $(0, 0) \in J \times U$. To verify this, we put

$$u(t, z) = \mu(t)z + v(t, z) \quad \left(v(t, z) = O(z^2) \right) \tag{3.2}$$

as a formal solution. Therefore, $\mu(t)$ satisfies

$$\mu(t)^{(1.5)} = \mu'(t) + 0.5\mu(t). \tag{3.3}$$

The last equation is equivalent to the equation of the form

$$\frac{\partial^\gamma \mu(t)}{\partial t^\gamma} = g(t, \mu(t), \mu'(t)), \quad \gamma \in (1, 2], \tag{3.4}$$

where g is a contraction function whenever $|\mu'(t) - \rho'(t)| \leq a|\mu(t) - \rho(t)|$, $a \in (0, 0.5)$; thus the holomorphic solution $\mu(t)$ exists uniquely and converges in a neighborhood of the origin (the Banach fixed-point theorem).

Example 3.2. Assume the following equation:

$$\begin{aligned} \frac{z^{0.5}}{1.128} D_z^{0.5} u(t, z) + u_{tt} &= u_t(t, z) - u(t, z), \quad t \in J = [0, 1], \\ u(0, z) &= 0, \end{aligned} \quad (3.5)$$

where $u(t, z)$ is the unknown function. The condition of Theorem 2.2 is valid; hence, the problem (3.5) has a unique holomorphic solution near $(0, 0) \in J \times U$. To realize this, (3.5) can be reduced to the following problem:

$$\begin{aligned} \psi''(t) &= \psi'(t) - 2\psi(t), \\ \psi(0) &= 0. \end{aligned} \quad (3.6)$$

Thus, the problem (3.6) has a unique solution.

4. Conclusion

The telegraph equation appears when we look for a mathematical model for the electrical flow in a metallic cable. In virtue of the laws of electricity, we posed generalized fractional partial differential equations (2.1) in a complex domain where the unknown variable is the voltage u . The conventional power series method (tool from classical functional analysis) has been successfully employed for finding the solution of space- and time-fractional telegraph equations. The space- and time-fractional derivatives are considered in the Srivastava-Owa and Riemann-Liouville sense, respectively. The results of the examples show that conventional power series method is a reliable and efficient method for solving space- and time-fractional telegraph equations and also other equations in complex domain.

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