Research Article

Best Constants between Equivalent Norms in Lorentz Sequence Spaces

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We find the best constants in inequalities relating the standard norm, the dual norm, and the norm $\|x\|_{(p,s)} := \inf\{\sum_k \|x^{(k)}\|_{p,s}\}$, where the infimum is taken over all finite representations $x = \sum_k x^{(k)}$ in the classical Lorentz sequence spaces. A crucial point in this analysis is the concept of level sequence, which we introduce and discuss. As an application, we derive the best constant in the triangle inequality for such spaces.

1. Introduction

The study of Lorentz spaces goes back to the work of Lorentz [1, 2] (see also [3, 4] for more recent results concerning functional properties of Lorentz spaces). They play an important role in the theory of Banach function spaces, in particular in the interpolation theory of linear operators.

Let $1 < p < \infty$ and $1 \leq s \leq \infty$. For a sequence $x = (x_n)_n$, the decreasing rearrangement $x^* = (x_n^*)_n$ is obtained by rearranging $(|x_n|)_n$ in decreasing order. We recall the definition of Lorentz sequence spaces

$$\ell^{p,s} = \left\{ x : \|x\|_{p,s} := \left(\sum_{n=1}^{\infty} (x_n^*)^s n^{s/p-1} \right)^{1/s} < \infty \right\}, \quad (1.1)$$

with the usual modification if $s = \infty$. Lorentz proved that $\|\cdot\|_{p,s}$ is a norm if and only if $1 \leq s \leq p < \infty$ and that the space $\ell^{p,s}$ is always normable (i.e., there exists a norm which is equivalent to $\|\cdot\|_{p,s}$) when $1 < p < s \leq \infty$ (for the remaining cases, it is known that $\ell^{p,s}$ cannot be endowed with an equivalent norm).

The study of normability for $p < s$ was carried out by means of the *maximal norm*:

$$\|x\|_{p,s}^* = \|(x_n)_n\|_{p,s}^* = \left(\sum_{n=1}^{\infty} n^{s/p-1} \left(\frac{1}{n} \sum_{i=1}^n x_i^* \right)^s \right)^{1/s} \quad (1.2)$$

defined in terms of the discrete Hardy operator. It is well known that $\|\cdot\|_{p,s}^*$ is always a norm. Moreover, one can prove that $\|\cdot\|_{p,s}^*$ is equivalent to $\|\cdot\|_{p,s}$ with the following estimates:

$$\|x\|_{p,s} \leq \|x\|_{p,s}^* \leq p' \|x\|_{p,s}, \quad (1.3)$$

see, for example, [5, 6].

As a consequence of the fact that $\|\cdot\|_{p,s}$ is equivalent to a norm, it is easy to see that it is a quasinorm satisfying the triangle inequality uniformly in the numbers of terms; that is, there exists a constant $C_{ps} > 0$ such that, for every finite collection $\{x^{(k)}\}_{k=1,2,\dots,N} \subset \ell^{p,s}$, it yields that

$$\left\| \sum_{k=1}^N x^{(k)} \right\|_{p,s} \leq C_{ps} \sum_{k=1}^N \|x^{(k)}\|_{p,s}. \quad (1.4)$$

Also the converse result holds, namely that if (1.4) holds, then $\|\cdot\|_{p,s}$ is equivalent with a norm and an alternative equivalent norm is given by means of the following *decomposition norm*:

$$\|x\|_{(p,s)} := \inf \left\{ \sum_{k=1}^N \|x^{(k)}\|_{p,s} : x = \sum_{k=1}^N x^{(k)} \right\}. \quad (1.5)$$

One can prove that $\|\cdot\|_{(p,s)}$ is always a norm and that $\|\cdot\|_{(p,s)}$ is equivalent with $\|\cdot\|_{p,s}$, $1 \leq p, s \leq \infty$. If $1 \leq s \leq p$, then $\|\cdot\|_{(p,s)} = \|\cdot\|_{p,s}$ (see, e.g., [7, 8]). We also remark that the best constant C_{ps} in the inequality

$$\|x\|_{p,s} \leq C_{ps} \|x\|_{(p,s)} \quad (1.6)$$

is the same as the optimal one in (1.4).

It is natural to consider also the following norm defined in terms of Köthe duality, which we will call the *dual norm*:

$$\|x\|'_{p,s} = \sup \left\{ \sum_{n=1}^{\infty} x_n y_n : \|y\|_{p',s'} = 1 \right\}. \quad (1.7)$$

The dual norm $\|\cdot\|'_{p,s}$ is a norm indeed, equivalent to $\|\cdot\|_{p,s}$ and, moreover, if $1 \leq s \leq p$, then

$$\|x\|'_{p,s} = \|x\|_{p,s} = \|x\|_{(p,s)}. \quad (1.8)$$

In the recent papers [9, 10], the authors considered estimates between the dual norm, decomposition norm, and the norm which defines the Lorentz spaces over nonatomic resonant measure spaces. The main reason for these consideration was that the technique based on the norm defined in terms of the maximal function (or Hardy operator) does not give the best constant in the triangle inequality with n -terms. In [9], the case of the classical Lorentz spaces $L^{p,s}$ was considered, while in [10] similar results were proved for the weighted Lorentz space $\Gamma^p(w)$, where w is an increasing weight function.

The aim of this paper is to treat similar problems in the context of Lorentz spaces of sequences. In particular, in this paper, we introduce the notion of level sequence, which corresponds to the notion of level function introduced by Halperin in [11] and Lorentz in [12]. We would like to pronounce that our results are not corollaries of the results proved in [9] although some of the techniques are similar (cf. our Example 3.5). One of the main applications of our results is that we obtain the best constant C_{ps} in the triangle inequality (1.4). This constant was found by a different approach in [13] where the best constants in q -convexity and q -concavity inequalities were found for Lorentz and Marcinkiewicz spaces of functions and sequences.

The paper is organized as follows. In Section 2, we state and discuss several technical lemmas, which will be used in the proofs of our theorems. In Section 3, we introduce and discuss one of the key tools used in this paper, the so-called level sequence. In Theorem 3.7 of Section 3, we prove optimal estimates between the quasinorm in $\ell^{p,s}$ of $x = (x_n)_n$ and of its level sequence $x^\circ = (x_n^\circ)_n$. Section 4 is dedicated to the study of the dual norm $\|x\|'_{p,s}$, $1 < p < s \leq \infty$, in relation to the other norms. For instance, one of the main results in Section 4 is that the dual norm of $x = (x_n)_n \in \ell^{p,s}$ coincides with the quasinorm of the level sequence $\|x^\circ\|_{p,s}$ (see Theorem 4.2). Moreover, $\|x\|_{p,s} \leq C_{ps}\|x\|'_{p,s}$, where C_{ps} is the optimal constant which appears in Theorem 4.4. The main result of Section 5, that is, that the dual norm coincides with the decomposition norm is given in Theorem 5.2. Finally, in the last section, we give some complementary results and remarks, as, for example, Theorem 6.1, which gives us the best constant in the “triangle inequality” for the quasinorm $\|x\|_{p,s}$. Moreover, we point out that our results can be given in a somewhat more general setting (see Remark 6.3 and Theorem 6.4).

Throughout this paper, given two sequences $x = (x_n)_n$, $y = (y_n)_n$, we denote that $x < y$ if

$$\sum_{i=1}^n x_i \leq \sum_{i=1}^n y_i, \quad \forall n \geq 1 \quad (1.9)$$

and we use the notation $x \leq y$ if

$$x_n \leq y_n, \quad \forall n \geq 1. \quad (1.10)$$

The number p' stands for the conjugate index of p ; that is, $1/p + 1/p' = 1$ and $\mathbb{N}$, $\mathbb{N}^*$ stand for the sets of nonnegative, respectively, positive integers.

2. Some Lemmas

First, we recall the following well-known majorization lemma (see, e.g., [14, page 9]).

Lemma 2.1. *Let u, v, w be nonnegative sequences and suppose that w is decreasing. If*

$$\sum_{k=1}^n u_k \leq \sum_{k=1}^n v_k \quad (n = 1, 2, \dots), \quad (2.1)$$

then

$$\sum_{k=1}^n u_k w_k \leq \sum_{k=1}^n v_k w_k \quad (n = 1, 2, \dots). \quad (2.2)$$

We also need the following auxiliary statement related to the dual norm.

Lemma 2.2. *Let $x = (x_n)_n \in \ell^{p,s}$, $1 < p < \infty$, $1 \leq s \leq \infty$. Then,*

$$\|x\|'_{p,s} = \|x^*\|'_{p,s}. \quad (2.3)$$

A proof can be found, for example, in [15, pages 45–49].

The next lemmas are crucial in the proofs of some of our main results and are also of independent interest.

Lemma 2.3. *Let $x = (x_n)_n \in \ell^{p,s}$, $1 < p < \infty$, $1 \leq s \leq \infty$. Then, the following statements hold.*

(a) *The equality*

$$\|x\|_{(p,s)} = \inf \left\{ \sum_k \|x^{(k)}\|_{p,s} \right\} \quad (2.4)$$

holds, where the infimum is taken over all finite nonnegative sequences $|x_n| = \sum_k x_n^{(k)}$.

(b) *If $0 \leq y \leq x$, then $\|y\|_{(p,s)} \leq \|x\|_{(p,s)}$.*

(c) *If $0 \leq y^{(k)} \leq x$ and $y^{(k)} \nearrow x$ as $k \rightarrow \infty$, then $\|y^{(k)}\|_{(p,s)} \rightarrow \|x\|_{(p,s)}$ as $k \rightarrow \infty$.*

Proof. The proof is similar to that of Lemma 2.7 in [9], so we omit the details. $\square$

3. Level Sequences

The notion of level function was introduced in the early 1950s by Halperin [11] and Lorentz [12] and generalized more recently by Sinnamon in, for example, [16–18] and Mastlyo and Sinnamon in [19]. Based on the extension given by Lorentz, the optimal constant in the triangle inequality in Lorentz spaces $L^{p,s}(R, \mu)$, where (R, μ) is a totally σ -finite nonatomic measure space and $1 < p < s \leq \infty$, was found.

Inspired by these results, we will use in this paper the concept of level sequence with respect to a given sequence and study Lorentz sequence spaces in this frame.

Definition 3.1. Let $\varphi = (\varphi_n)_n$ be a sequence of positive numbers and $\Phi_n = \sum_{i=1}^n \varphi_i$. A sequence $\tilde{X} = (\tilde{X}_n)_n$ is called φ -concave if, for arbitrary $n_1, n_2 \in \mathbb{N}$, we have that $\tilde{X}_n \geq L_n$, where $L_n = C\Phi_n + D$, C and D are constants, such that L "interpolate" $\tilde{X}$ in n_1, n_2 ; that is,

$$\begin{aligned} L_{n_1} &= \tilde{X}_{n_1}, \\ L_{n_2} &= \tilde{X}_{n_2}. \end{aligned} \quad (3.1)$$

Proposition 3.2. *The following statements are equivalent:*

$$\tilde{X} = (\tilde{X}_n)_n \text{ is } \varphi\text{-concave,} \quad (3.2)$$

$$\frac{\tilde{X}_n - \tilde{X}_{n_1}}{\Phi_n - \Phi_{n_1}} \geq \frac{\tilde{X}_{n_2} - \tilde{X}_{n_1}}{\Phi_{n_2} - \Phi_{n_1}}, \quad 1 \leq n_1 < n \leq n_2, \quad (3.3)$$

$$\frac{\tilde{X}_n - \tilde{X}_{n_1}}{\Phi_n - \Phi_{n_1}} \geq \frac{\tilde{X}_{n_2} - \tilde{X}_n}{\Phi_{n_2} - \Phi_n}, \quad 1 \leq n_1 \leq n < n_2, \quad (3.4)$$

$$\frac{\tilde{X}_{n_1} - \tilde{X}_{m_1}}{\Phi_{n_1} - \Phi_{m_1}} \geq \frac{\tilde{X}_{n_2} - \tilde{X}_{m_2}}{\Phi_{n_2} - \Phi_{m_2}}, \quad m_1 \leq m_2, \quad n_1 \leq n_2, \quad m_1 \neq n_1, \quad m_2 \neq n_2. \quad (3.5)$$

Proof. The proof of this proposition is literally the same as the proof of the equivalence of the corresponding statements in the case of functions, so we omit the details (see, e.g., [12]). $\square$

In the next theorem, we prove the existence of the level sequence and derive its properties. Since, as far as we know, this theorem has not been proved earlier in this context, we give the entire proof here.

Theorem 3.3. *Let $\varphi = (\varphi_n)_n$ be a positive sequence and $\Phi_n = \sum_{i=1}^n \varphi_i$. Let $x = (x_n)_n$ be a positive sequence and suppose that*

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n \varphi_i} = 0. \quad (3.6)$$

Then, there exists a unique nonnegative sequence $x^\circ = (x_n^\circ)_n$ satisfying the following conditions:

- (a) $(x_n^\circ / \varphi_n)_n$ is decreasing;
- (b) $x < x^\circ$;
- (c) the set $\{n : x_n^\circ \neq x_n\} = \cup_{k=1}^\infty I_k$, where $\{I_k\}$ are disjoint sets of positive integers, $I_k = \{n_k, \dots, n_k + m_k\}$ such that $\sum_{i \in I_k} x_i = \sum_{i \in I_k} x_i^\circ$ and $x_i^\circ / \varphi_i = \lambda_k$, for all $i \in I_k$.

Remark 3.4. The unique sequence $x^\circ = (x_n^\circ)_n$ in Theorem 3.3 is called the level sequence of $x = (x_n)_n$ with respect to $\varphi = (\varphi_n)_n$.

Proof. (a) Let $X_n = \sum_{i=1}^n x_i$, $n \in \mathbb{N}^*$. Condition (3.6) implies the existence of a sequence $\tilde{X} = (\tilde{X}_n)_n$, which is a φ -concave majorant of X . We denote $X_n^\circ := \inf \tilde{X}_n$ and from (3.5) we get that

$$\frac{\tilde{X}_{k_1}}{\tilde{X}_{k_2}} \leq \frac{\Phi_{k_1}}{\Phi_{k_2}} \leq 1, \quad \text{for } k_1 \leq k_2, \quad (3.7)$$

that is, $X^\circ = (X_k^\circ)_k$ is increasing. Thus, there exists a unique positive sequence $(x_n)_n$ such that $X_n^\circ = \sum_{i=1}^n x_i^\circ$, with $x_1^\circ = X_1^\circ$ and

$$x_n^\circ := X_n^\circ - X_{n-1}^\circ. \quad (3.8)$$

According to (3.5), we have that $(X_n^\circ - X_{n-1}^\circ)/(\Phi_n - \Phi_{n-1}) \geq (X_{n+1}^\circ - X_n^\circ)/(\Phi_{n+1} - \Phi_n)$, which means that $(x_n^\circ/\varphi_n)_n$ is decreasing and this proves (a).

(b) This statement follows directly from (3.8) and the definition of X_n° .

(c) Let us first note that two consecutive terms X_{k-1}° and X_k° are equal to X_{k-1} and X_k , respectively, if and only if $x_k = x_k^\circ$, $k \geq 1$.

Now, we assume that there exists $j \in \mathbb{N}^*$ such that $X_j^\circ > X_j$ and we denote $n := \max\{k < j : X_k^\circ = X_k \text{ and } X_{k-1}^\circ = X_{k-1}\} + 1$ and $m := \min\{k > j : X_k^\circ = X_k \text{ and } X_{k+1}^\circ = X_{k+1}\} - n + 1$. Now, we take L to "interpolate" X° in $n-1$ and $n+m-1$. Then, $L \geq X$ and $X^\circ \geq L$. It follows that $L_j = X_j^\circ = C\Phi_j + D$ for $j \in \{n-1, \dots, n+m-1\}$.

Thus, $x_j^\circ = C\varphi_j$, $j \in \{n, \dots, n-1+m\}$, $x_{n+m}^\circ = x_{n+m}$ and $x_{n-1}^\circ = x_{n-1}$. Since

$$\begin{aligned} X_{n-1}^\circ &= X_{n-1}, \\ X_{n+m-1}^\circ &= X_{n+m-1}, \end{aligned} \quad (3.9)$$

we obtain that

$$\sum_{i=n}^{n+m-1} x_i^\circ = \sum_{i=n}^{n+m-1} x_i \quad (3.10)$$

and the proof is complete. $\square$

The following example shows that the results for sequences do not follow from the results for function simply by taking step functions:

$$f(t) = \sum_{n=0}^{\infty} x_n \chi_{[n, n+1)}(t). \quad (3.11)$$

Example 3.5. For $f(t) = \chi_{[0,1]}(t)$, the level function is

$$f^\circ(t) = (1 - \alpha)t^{-\alpha} \chi_{[0,1]}(t), \quad (3.12)$$

where $\alpha = 1 - s'/p'$ so that

$$\|f^\circ\|_{p,s} = \left(\frac{s'}{p'}\right)^{1/s'}. \quad (3.13)$$

If $x = (x_n)_n$, where

$$x_n = \begin{cases} 1 & \text{if } n \leq k, \\ 0 & \text{if } n > k, \end{cases} \quad (3.14)$$

then

$$x_n^\circ = \begin{cases} \frac{k}{\sum_{i=1}^k i^{-\alpha}} n^{-\alpha} & \text{if } n \leq k, \\ 0, & \text{otherwise,} \end{cases} \quad (3.15)$$

$$\|x^\circ\|_{p,s} = k \left(\sum_{n=1}^k n^{-\alpha} \right)^{-1/s'}.$$

We also need the following technical lemma.

Lemma 3.6. *Let $1 \leq p < s < \infty$ and*

$$C_{ps} = \sup_{m,n \in \mathbb{N}^*} \left(\frac{1}{n} \sum_{k=m}^{m+n-1} k^{s/p-1} \right)^{1/s} \left(\frac{1}{n} \sum_{k=m}^{m+n-1} k^{s'/p'-1} \right)^{1/s'}. \quad (3.16)$$

Then,

$$C_{ps} = \left(\frac{p}{s}\right)^{1/s} \left(\frac{p'}{s'}\right)^{1/s'}. \quad (3.17)$$

Proof. We prove first that

$$C_{ps} = \sup_{n \in \mathbb{N}^*} \left(\frac{1}{n} \sum_{k=1}^n k^{s/p-1} \right)^{1/s} \left(\frac{1}{n} \sum_{k=1}^n k^{s'/p'-1} \right)^{1/s'}. \quad (3.18)$$

The inequality

$$C_{ps} \geq \sup_{n \in \mathbb{N}^*} \left(\frac{1}{n} \sum_{k=1}^n k^{s/p-1} \right)^{1/s} \left(\frac{1}{n} \sum_{k=1}^n k^{s'/p'-1} \right)^{1/s'} \quad (3.19)$$

is trivial. For the converse inequality, it is enough to prove that

$$\left(\sum_{k=1}^n k^{s/p-1} \right)^{1/s} \left(\sum_{k=1}^n k^{s'/p'-1} \right)^{1/s'} \geq \left(\sum_{k=m}^{m+n-1} k^{s/p-1} \right)^{1/s} \left(\sum_{k=m}^{m+n-1} k^{s'/p'-1} \right)^{1/s'} \quad (3.20)$$

for all $m \geq n$.

The above inequality can be obviously written in the form

$$\left(\sum_{k=1}^n k^{s/p-1} \right)^{s'/s} \left(\sum_{k=1}^n k^{s'/p'-1} \right) \geq \left(\sum_{k=1}^n (k+m-1)^{s/p-1} \right)^{s'/s} \left(\sum_{k=1}^n (k+m-1)^{s'/p'-1} \right). \quad (3.21)$$

Let us fix $m \geq 1$. Observe that for $m = 1$ we have equality, so we may suppose that $m > 1$. If we denote $x_k = k^{s'/p'-1}$, $z_k = (k+m-1)^{s'/p'-1}$, and $r = -s/s' < 0$, we have to prove that

$$\frac{\sum_{k=1}^n x_k}{\left(\sum_{k=1}^n x_k^r \right)^{1/r}} \geq \frac{\sum_{k=1}^n z_k}{\left(\sum_{k=1}^n z_k^r \right)^{1/r}}. \quad (3.22)$$

Observe that both $(x_k)_k$ and $(z_k)_k$ are decreasing sequences. Denote $X_n = \sum_{k=1}^n x_k$, $Z_n = \sum_{k=1}^n z_k$, and $C = X_n/Z_n$.

Since $(x_k/z_k)_k$ is decreasing, so is also $(X_k/Z_k)_k$, and hence $X_k/Z_k \geq X_n/Z_n = C$ for $k \leq n-1$. Applying Karamata's inequality (see, e.g., [20]) for the decreasing sequences $(x_k)_k$ and $y_k = Cz_k$ and for the convex function $\Phi(t) = t^r$, $r < 0$, we get that

$$\sum_{k=1}^n x_k^r \geq \sum_{k=1}^n (Cz_k)^r \quad (3.23)$$

or

$$\sum_{k=1}^n x_k^r \geq C^r \sum_{k=1}^n z_k^r. \quad (3.24)$$

The last inequality implies

$$\left(\sum_{k=1}^n x_k^r \right)^{1/r} \leq \frac{\sum_{k=1}^n x_k}{\sum_{k=1}^n z_k} \left(\sum_{k=1}^n z_k^r \right)^{1/r}, \quad \text{since } r < 0. \quad (3.25)$$

Since m was arbitrary, we get the desired inequality and (3.18) is proved. Moreover, it is known that with C_{ps} defined by (3.18) it yields that (3.17) holds (see Theorem 15 in [13]). The proof is complete. $\square$

The main result of this section contains optimal estimates between the norm of a sequence and of its level sequence with respect to the sequence $\varphi = (\varphi_n)_n$ with $\varphi_n = n^{-1+s'/p'}$, $n \geq 1$.

Theorem 3.7. Let $1 < p < \infty$, $p < s \leq \infty$, and let $x = (x_n) \in \ell^{p,s}$ be a nonnegative and decreasing sequence, and let $x^\circ = (x_n^\circ)_n$ be the level sequence with respect to the sequence $\varphi = (\varphi_n)$, where $\varphi_n = n^{-\alpha}$, $\alpha = 1 - s'/p'$. Then,

$$\|x^\circ\|_{p,s} \leq \|x\|_{p,s} \leq \left(\frac{p}{s}\right)^{1/s} \left(\frac{p'}{s'}\right)^{1/s'} \|x^\circ\|_{p,s}. \quad (3.26)$$

The constants in (3.26) are optimal.

Proof. Assume first that $s < \infty$ and consider the left-hand side inequality in (3.26). By applying Theorem 3.3, we have that $x_j^\circ/j^{-\alpha} = \lambda_k$, $j \in I_k$. Since $\alpha = (s/p - 1)/(s - 1)$, $(x_j^\circ)^{s-1} j^{s/p-1} = \lambda_k^{s-1}$, applying Hölder's inequality, we obtain that

$$\begin{aligned} \sum_{j \in I_k} (x_j^\circ)^s j^{s/p-1} &= \lambda_k^{s-1} \sum_{j \in I_k} x_j^\circ \\ &= \left(\sum_{j \in I_k} j^{-\alpha} \right)^{1-s} \left(\sum_{j \in I_k} x_j \right)^{s-1} \left(\sum_{j \in I_k} x_j^\circ \right) \\ &= \left(\sum_{j \in I_k} j^{-\alpha} \right)^{1-s} \left(\sum_{j \in I_k} x_j \right)^s \leq \sum_{j \in I_k} x_j^s j^{s/p-1}. \end{aligned} \quad (3.27)$$

By the above estimate and Theorem 3.3, we get the left-hand side inequality in (3.26). Let us now consider the sequence $\psi = (\psi_n)_n$, $\psi_n = x_n^{s-1} n^{s/p-1}$, $n \geq 1$, and let $\tilde{\psi} = (\tilde{\psi}_n)_n$ be the level sequence of $\psi = (\psi_n)_n$ with respect to $\varphi = (\varphi_n)_n$, $\varphi_n = 1$, $n \geq 1$. By applying Theorem 3.3, Lemma 2.1, and Hölder's inequality, we obtain that

$$\begin{aligned} \|x\|_{p,s}^s &= \sum_{n=1}^{\infty} x_n \psi_n \leq \sum_{n=1}^{\infty} x_n \tilde{\psi}_n \\ &\leq \sum_{n=1}^{\infty} x_n^\circ \tilde{\psi}_n \leq \|x^\circ\|_{p,s} \|\tilde{\psi}\|_{p',s'}. \end{aligned} \quad (3.28)$$

We note that to obtain the right-hand side inequality in (3.26) it is sufficient to prove that

$$\|\tilde{\psi}\|_{p',s'} \leq \left(\frac{p}{s}\right)^{1/s} \left(\frac{p'}{s'}\right)^{1/s'} \|x\|_{p,s}^{s-1}. \quad (3.29)$$

Let $E := \{n \geq 1 : \tilde{\psi}_n = \psi_n\}$. Then, we have that

$$\mathbb{N}^* - E = \bigcup_k I_k, \quad (3.30)$$

where $\{I_k\}$ are disjoint and such that

$$\tilde{\psi}_n = \frac{1}{|I_k| + 1} \sum_{j \in I_k} \psi_j, \quad \forall n \in I_k. \quad (3.31)$$

By Hölder's inequality, we obtain that

$$\sum_{j \in I_k} \psi_j \leq \left(\sum_{j \in I_k} j^{s/p-1} \right)^{1/s} \left(\sum_{j \in I_k} x_j^s j^{s/p-1} \right)^{1/s'}. \quad (3.32)$$

We also have that

$$\sum_{n \in E} (\tilde{\psi}_n)^{s'} n^{s'/p'-1} = \sum_{n \in E} (\psi_n)^{s'} n^{s'/p'-1} = \sum_{n \in E} x_n^s n^{s/p-1}. \quad (3.33)$$

Hence, by (3.31) and (3.32), it yields that

$$\begin{aligned} \sum_{n \in I_k} (\tilde{\psi}_n)^{s'} n^{s'/p'-1} &\leq \frac{1}{(|I_k| + 1)^{s'}} \left(\sum_{m \in I_k} m^{s/p-1} \right)^{s'/s} \left(\sum_{m \in I_k} x_m^s m^{s/p-1} \right), \\ \left(\sum_{n \in I_k} n^{s'/p'-1} \right) &= C_{ps,k}^{s'} \sum_{m \in I_k} x_m^s m^{s/p-1}, \end{aligned} \quad (3.34)$$

where

$$C_{ps,k} = \frac{1}{(|I_k| + 1)} \left(\sum_{n \in I_k} n^{s/p-1} \right)^{1/s} \left(\sum_{n \in I_k} n^{s'/p'-1} \right)^{1/s'}. \quad (3.35)$$

Hence, according to Lemma 3.6, it follows that (3.29) holds, which means that the right-hand side inequality in (3.26) is proved. It only remains to prove the sharpness of the obtained inequalities.

We note that the left-hand side inequality in (3.26) becomes equality if, for a fixed k_0 , we take $x = (x_n)_n$, where

$$x_n = \begin{cases} \frac{k_0}{\sum_{i=1}^{k_0} i^{-\alpha}} n^{-\alpha} & \text{if } n \leq k_0, \\ 0, & \text{otherwise.} \end{cases} \quad (3.36)$$

The right-hand side inequality (3.26) becomes equality for $x = (x_n)_n$, where

$$x_n = \begin{cases} 1 & \text{if } n \leq k_0, \\ 0, & \text{otherwise.} \end{cases} \quad (3.37)$$

We have that $\|x\|_{p,s} = (\sum_{n=1}^{k_0} n^{s/p-1})^{1/s}$. It is easy to verify that if $x^\circ = (x_n^\circ)_n$, where

$$x_n^\circ = \begin{cases} \frac{k_0}{\sum_{i=1}^{k_0} i^{-\alpha}} n^{-\alpha} & \text{if } n \leq k_0, \\ 0, & \text{otherwise,} \end{cases} \quad (3.38)$$

then

$$\|x^\circ\|_{p,s} = k_0 \left(\sum_{n=1}^{k_0} n^{-\alpha} \right)^{-1/s'}. \quad (3.39)$$

Since k_0 is arbitrary, we get that also the constant on the right-hand side inequality (3.26) is optimal.

Let us now consider the case $s = \infty$ and, hence, $\alpha = 1 - 1/p' = 1/p$. By using Hölder's inequality, we find that

$$\sum_{n \in I_k} \lambda_k n^{-1/p} = \sum_{n \in I_k} x_n^\circ = \sum_{n \in I_k} x_k \leq \|x\|_{p,\infty} \sum_{n \in I_k} n^{-1/p}. \quad (3.40)$$

Then, $\lambda_k \leq \|x\|_{p,\infty}$ for every k , which implies that $\|x^\circ\|_{p,\infty} \leq \|x\|_{p,\infty}$.

On the other hand, for any $n \in I_k$, and using again Hölder's inequality, we obtain that

$$\begin{aligned} x_n n^{1/p} &\leq \left(\sum_{i=1}^n x_i \right) n^{1/p-1} \leq \left(\sum_{i=1}^n x_i^\circ \right) n^{1/p-1} \\ &\leq n^{1/p} \left(\sum_{i=1}^n \frac{1}{n} i^{-1/p} \right) \sup_{1 \leq i \leq n} \{ i^{1/p} x_i^\circ \}. \end{aligned} \quad (3.41)$$

It follows that

$$\begin{aligned} \sup_{n \geq 1} \{ n^{1/p} x_n \} &\leq \sup_{n \geq 1} \left\{ \frac{n^{1/p} \sum_{i=1}^n i^{-1/p}}{n} \sup_{1 \leq i \leq n} (i^{1/p} x_i^\circ) \right\} \\ &= \|x^\circ\|_{p,\infty} C_p, \end{aligned} \quad (3.42)$$

where

$$\begin{aligned} C_p &= \sup_{n \geq 1} \frac{(\sup_{k=1, \dots, n} k^{1/p}) (\sum_{k=1}^n k^{-1/p})}{n} \\ &= \sup_{n \geq 1} \frac{n^{1/p} (\sum_{k=1}^n k^{-1/p})}{n} = p'. \end{aligned} \quad (3.43)$$

The constants are optimal also in this case. This can easily be proved if we consider the same sequences as in the case $s < \infty$. The proof is complete. $\square$

Remark 3.8. Let $1 < p < s \leq \infty$. Let $x = (x_n)_n \in \ell^{p,s}$ be a nonnegative and nonincreasing sequence, and let $x^\circ = (x_n^\circ)_n$ be the level sequence of x with respect to the sequence

$$\varphi_n = n^{-\alpha}, \quad \alpha = 1 - \frac{s'}{p'}, \quad n \geq 1. \quad (3.44)$$

Then, the equality

$$\|x^\circ\|_{p,s} = \|x\|_{p,s} \quad (3.45)$$

holds if and only if $x_n^\circ = x_n$ for all $n \geq 1$.

Indeed, inequality (3.27) becomes equality if and only if $(x_n n^\alpha)_n$ is constant on I_k . Therefore, (3.45) holds if and only if $x_n^\circ = x_n$, for all $n \geq 1$, hence if and only if $(x_n n^\alpha)_n$ is decreasing.

4. Level Sequences and the Dual Norm

This section is devoted to investigate the dual norm of a sequence defined in (1.7).

In the next proposition, we summarize some well-known properties of the dual norm of a sequence $x = (x_n)_n \in \ell^{p,s}$.

Proposition 4.1. *Let $1 < p < \infty$, $1 \leq s \leq \infty$, and $x = (x_n)_n \in \ell^{p,s}$. Then, the following statements hold.*

(a) *One has*

$$\|x\|'_{p,s} \leq \|x\|_{p,s}. \quad (4.1)$$

(b) *If $s \leq p$, then*

$$\|x\|'_{p,s} = \|x\|_{p,s}. \quad (4.2)$$

(c) *If $p < s \leq \infty$ and if the sequence $(x_n^* n^{1-s'/p'})_n$ is nonincreasing, then (4.2) holds.*

(d) *If $x = (x_n)_n \in \ell^{p,s}$, then one has that*

$$\|x\|'_{p,s} \leq \inf_{x \prec z} \|z\|_{p,s}. \quad (4.3)$$

Proof. Indeed, (a) follows from Hölder's inequality.

(b) Take $\psi = (\psi_n)_n$ with $\psi_n = (x_n^*)^{s-1} n^{s/p-1}$, $n \geq 1$. The sequence $\psi = (\psi_n)_n$ is nonincreasing and we have that

$$\begin{aligned} \|\psi\|_{p',s'}^{s'} &= \sum_{n=1}^{\infty} \psi_n^{s'} n^{s'/p'-1} = \|x\|_{p,s}^s, \\ \|x\|_{p,s}^s &= \sum_{n=1}^{\infty} x_n^* \psi_n. \end{aligned} \quad (4.4)$$

The last two equalities imply that $\|x\|_{p,s}' \geq \|x\|_{p,s}$ and, by (a), we get equality (4.2).

(c) The proof is similar to that of (b), so we omit the details.

(d) In view of Lemma 2.1, we have that

$$\sum_{n=1}^{\infty} x_n y_n \leq \inf_{x < z} \|z\|_{p,s} \|y\|_{p',s'}, \quad (4.5)$$

where the infimum is taken over all nonnegative sequences $z = (z_n)_n \in \ell^{p,s}$ such that $x < z$ and where $y = (y_n)_n \in \ell^{p',s'}$ is a nonnegative and nonincreasing sequence. This implies that (d) holds. $\square$

It was proved by Halperin in [11] (see also [12, Theorem 3.6.5]) that we have equality in (4.3) in the case of real functions defined on $(0, 1)$ and that the infimum is attained. For $p < s \leq \infty$, a complete proof for Lorentz spaces over σ -finite nonatomic measure spaces was given in the recent paper [9]. We remark here that the proofs given in [9, 11, 12] do not cover the result in the case when (R, μ) is a totally σ -finite measure space, completely atomic, with all atoms having the same measure. For completeness, we prove this result in the following theorem.

Theorem 4.2. *Let $1 < p < s \leq \infty$, and let $x = (x_n)_n \in \ell^{p,s}$ be a nonnegative and nonincreasing sequence. Let $\alpha = 1 - s'/p'$ and $\varphi^{(\alpha)} = (\varphi_n^{(\alpha)})_n$, where $\varphi_n^{(\alpha)} = n^{-\alpha}$. Then, one has that*

$$\|x\|_{p,s}' = \inf_{x < z} \|z\|_{p,s} = \|x^\circ\|_{p,s}, \quad (4.6)$$

where $x^\circ = (x_n^\circ)_n$ is the level sequence of $x = (x_n)_n$ with respect to the sequence $\varphi^{(\alpha)} = (\varphi_n^{(\alpha)})_n$.

Proof. By Proposition 4.1 and Theorem 3.3(b), it is sufficient to prove that

$$\|x\|_{p,s}' \geq \|x^\circ\|_{p,s}. \quad (4.7)$$

We denote $E = \{n \geq 1 : x_n = x_n^\circ\}$. According to Theorem 3.3 it yields that $\mathbb{N}^* - E = \cup_k I_k$, where I_k are disjoint finite subsets of positive integers such that

$$\sum_{n \in I_k} x_n = \sum_{n \in I_k} x_n^\circ. \quad (4.8)$$

Assume first that $s < \infty$ and let $\psi = (\psi_n)_n$, where $\psi_n = (x_n^\circ)^{s-1} n^{s/p-1}$, $n \geq 1$. As before, we have that

$$\|\psi\|_{p',s}^{s'} = \sum_{n=1}^{\infty} \psi_n^{s'} n^{s'/p'-1} = \|x^\circ\|_{p,s}. \quad (4.9)$$

Choose $y = (y_n)_n$, where $y_n = \psi_n / \|x^\circ\|_{p,s}^{s-1}$. Then, $\|y\|_{p',s'} = 1$.

From Theorem 3.3, it follows that, for each k , $x_n^\circ = \lambda_k n^{-\alpha}$, $n \in I_k$. Thus, $\psi_n = \lambda_k^{s-1}$ and

$$\|x^\circ\|_{p,s}^{s-1} \sum_{n \in I_k} x_n y_n = \lambda_k^{s-1} \sum_{n \in I_k} x_n = \lambda_k^{s-1} \sum_{n \in I_k} x_n^\circ = \sum_{n \in I_k} (x_n^\circ)^s n^{s/p-1}. \quad (4.10)$$

Moreover, we have that

$$\|x^\circ\|_{p,s}^{s-1} \sum_{n \in E} x_n y_n = \sum_{n \in E} (x_n^\circ)^s n^{s/p-1}. \quad (4.11)$$

Thus, we obtain that $\sum_{n=1}^{\infty} x_n y_n = \|x^\circ\|_{p,s}$. This implies (4.7).

Let now $s = \infty$. We consider only the nontrivial case $I_1 = \{1, 2, \dots, m\}$. Let $y = (y_n)_n$, where

$$y_n = \begin{cases} \frac{1}{\sum_{i=1}^m i^{-1/p}} & \text{if } n \leq m, \\ 0, & \text{otherwise} \end{cases} \quad (4.12)$$

and $\|y\|_{p',1} = 1$. Then,

$$\begin{aligned} \sum_{n=1}^{\infty} x_n y_n &= \frac{1}{\sum_{i=1}^m i^{-1/p}} \sum_{n=1}^m x_n \\ &= \frac{1}{\sum_{i=1}^m i^{-1/p}} \sum_{n=1}^m x_n^\circ = \lambda_1 \\ &= \|x^\circ\|_{p,\infty}. \end{aligned} \quad (4.13)$$

This implies (4.7), and the proof is complete. $\square$

Remark 4.3. Let $1 < p < s \leq \infty$ and let $x = (x_n)_n \in \ell^{p,s}$ be a nonnegative and nonincreasing sequence. Then, by Remark 3.8, the equality (see (4.2))

$$\|x\|_{p,s}' = \|x\|_{p,s} \quad (4.14)$$

holds if and only if $(x_n n^\alpha)_n$ is decreasing. Note also that (4.2) holds in general if $1 \leq s < p$ (see Proposition 4.1(b)).

The final result in this section gives the sharp estimate of the standard norm via the dual norm.

Theorem 4.4. Let $1 < p < \infty$ and $p < s \leq \infty$. Then, for any sequence $x = (x_n)_n \in \ell^{p,s}$, it yields that

$$\|x\|_{p,s} \leq C_{ps} \|x\|'_{p,s}, \quad (4.15)$$

where C_{ps} is defined by (3.17). The constant is optimal.

Proof. The proof follows immediately from Theorems 4.2 and 3.7. $\square$

5. The Decomposition Norm

In this section, we prove that the dual norm and the decomposition norm coincide. The following lemma plays an important role in the proof of the main result, and it was proved in the recent paper [9].

Lemma 5.1. Let $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_v$ be positive numbers, and let $\{\eta_{jk}\}$ be a $(N \times v)$ -matrix of positive numbers, $1 \leq j \leq N$, $1 \leq k \leq v$. Set

$$\beta_k = \sum_{j=1}^N \eta_{jk}, \quad k = 1, \dots, v. \quad (5.1)$$

Assume that

$$\beta_1 + \dots + \beta_k \geq \alpha_1 + \dots + \alpha_k \quad (5.2)$$

for any $k = 1, \dots, v$. Let $\eta = \max \eta_{jk}$. Then, for any $j = 1, \dots, N$, there exists a permutation $\{\tilde{\eta}_{jk}\}_{k=1}^v$ of the v -tuple $\{\eta_{jk}\}_{k=1}^v$ such that

$$\alpha_k \leq \tilde{\beta}_k + \eta, \quad \tilde{\beta}_k = \sum_{j=1}^N \tilde{\eta}_{jk}, \quad (5.3)$$

for any $k = 1, \dots, v$.

Our main result in this section reads the following.

Theorem 5.2. Let $1 < p < \infty$ and $1 \leq s \leq \infty$. Then, for any sequence $x = (x_n)_n \in \ell^{p,s}$,

$$\|x\|'_{p,s} = \|x\|_{(p,s)}. \quad (5.4)$$

Proof. If $s \leq p$, then it is well known that (see, e.g., [7, 15])

$$\|x\|'_{p,s} = \|x\|_{p,s} = \|x\|_{(p,s)}. \quad (5.5)$$

Let $1 < p < s \leq \infty$. We prove first that

$$\|x\|'_{p,s} \leq \|x\|_{(p,s)}. \quad (5.6)$$

Let $y = (y_n)_n \in \ell^{p',s'}$, and let

$$x = \sum_k x^{(k)}. \quad (5.7)$$

Then, by Hölder's inequality, we have that

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \sum_k \sum_{n=1}^{\infty} |x_n^{(k)} y_n| \leq \sum_k \|x^{(k)}\|_{p,s} \|y\|_{p',s'}. \quad (5.8)$$

By taking the infimum over all representation (5.7), we obtain (5.6). It remains only to prove that

$$\|x\|_{(p,s)} \leq \|x\|'_{p,s}. \quad (5.9)$$

In view of Lemma 2.3, it is sufficient to prove (5.9) in the case of positive nonincreasing sequences. We can assume that there exists n_0 such that $x_i = 0$, for $i > n_0$, $n_0 \in \mathbb{N}^*$. From Theorem 3.3, we have also that $x_i^\circ = 0$ for $i > n_1$, $n_1 \in \mathbb{N}^*$. Denote $k_0 = \max\{n_0, n_1\}$. According to Theorem 4.2, it yields that

$$\|x\|'_{p,s} = \|x^\circ\|_{p,s'}, \quad (5.10)$$

where $x^\circ = (x_n^\circ)_n$ is the level sequence of $x = (x_n)_n$ with respect to the sequence $\varphi = (\varphi_n)_n$, $\varphi_n = n^{-\alpha}$, $\alpha = 1 - s'/p'$.

Let $\varepsilon > 0$. Choose $\delta = \delta(\varepsilon, k_0) > 0$ such that $\delta < \varepsilon(1/\|z\|_{p,s})$, with $z = (z_n)_n$, where

$$z_n = \begin{cases} 1 & \text{if } n \leq k_0, \\ 0, & \text{otherwise.} \end{cases} \quad (5.11)$$

Choose $N \in \mathbb{N}^*$ such that

$$\max_i \{x_i^\circ\} < N \cdot \delta. \quad (5.12)$$

By using Lemma 5.1 with $\alpha_k := x_k$, $\beta_k := x_k^\circ$, $\eta_{jk} := x_k^\circ/N$, $1 \leq k \leq k_0$, $1 \leq j \leq N$, we have that there exists a permutation of $\{\eta_{jk}\}$, $\{\tilde{\eta}_{jk}\}$ such that

$$x_k \leq \sum_{j=1}^N \tilde{\eta}_{jk} + \delta, \quad \forall k \leq k_0. \quad (5.13)$$

We define now $t^{(j)} = (t_k^{(j)})_k$, where $t_k^{(j)} := \tilde{\eta}_{jk}$. Since $t^{(j)}$ is equimeasurable with x°/N , we have that

$$\|t^{(j)}\|_{p,s} = \frac{\|x^\circ\|_{p,s}}{N}, \quad j \leq N, \quad (5.14)$$

which, in its turn, implies that

$$\|x\|_{(p,s)} \leq \sum_{j=1}^N \|t^{(j)}\|_{p,s} + \delta \cdot \|z\|_{p,s} \leq \|x^\circ\|_{p,s} + \varepsilon. \quad (5.15)$$

Since ε was arbitrary, it follows that (5.9) holds. The proof is complete. $\square$

Corollary 5.3. *Let $x = (x_n)_n \in \ell^{p,s}$, $1 \leq p < \infty$, $1 \leq s \leq \infty$. Then,*

$$\|x\|_{(p,s)} = \|x^*\|_{(p,s)}. \quad (5.16)$$

Proof. Equality (5.16) follows immediately from (5.4) and Lemma 2.2. $\square$

6. Further Results and Remarks

From Theorems 4.4 and 5.2, we can obtain the following sharp version of the “triangle inequality.”

Theorem 6.1. *Let $1 < p < s \leq \infty$, and suppose that $x^{(k)} = (x_n^{(k)})_n \in \ell^{p,s}$, $k = 1, \dots, N$. Then, one has that*

$$\left\| \sum_{k=1}^N x^{(k)} \right\|_{p,s} \leq C_{ps} \sum_{k=1}^N \|x^{(k)}\|_{p,s}, \quad (6.1)$$

where C_{ps} is given by (3.17) and the constant is optimal.

Remark 6.2. This result can also be derived from a recent result of Kamińska and Parrish [13]. They solved the problem in a completely different way.

Proof. We note that (6.1) is equivalent to the inequality

$$\|x\|_{p,s} \leq C_{ps} \|x\|_{(p,s)}, \quad (6.2)$$

where $x = (x_n)_n$ is any sequence from $\ell^{p,s}$. Inequality (6.2) follows directly from Theorems 4.4 and 5.2. Moreover, for $x = (x_n)_n$ with

$$x_n = \begin{cases} 1 & \text{if } n \leq k_0, \\ 0, & \text{otherwise,} \end{cases} \quad (6.3)$$

for a fixed $k_0 \in \mathbb{N}^*$, we obtain that

$$\|x\|_{p,s} = \left(\sum_{n=1}^{k_0} n^{s/p-1} \right)^{1/s}. \quad (6.4)$$

From Theorems 4.2 and 5.2, we have that

$$\|x\|_{(p,s)} = \frac{k_0}{\left(\sum_{n=1}^{k_0} n^{s'/p'-1}\right)^{1/s'}} \quad (6.5)$$

and therefore we obtain equality in (6.2), and the proof is complete. $\square$

Remark 6.3. We want to pronounce that in this paper we have formulated our results only for $\ell^{p,s}$ spaces, but our proofs show that some results are true for much more general spaces. For example, Theorems 3.7, 4.4, 5.2, and 6.1 are true in the more general case of resonant measure space (R, μ) . For example, Theorem 5.2 can be generalized as follows.

Theorem 6.4. *Let $1 < p < \infty$ and $1 \leq s \leq \infty$. Then, for any function $f \in L^{p,s}(R, \mu)$, where (R, μ) is a totally σ -finite resonant measure space (see, e.g., [15, Definition 2.3, page 45]), one has that*

$$\|f\|'_{p,s} = \|f\|_{(p,s)}. \quad (6.6)$$

Proof. Indeed, according to Theorem 5.2, we conclude that equality (6.6) holds for $f \in L^{p,s}(R, \mu)$, where (R, μ) is a totally σ -finite measure space, completely atomic, with all atoms having the same measure. Hence, together with Theorem 5.2 in [9] and by applying [15, Theorem 2.7, page 51], we obtain the desired result. $\square$

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